CK-12 FlexBook

Pateros Geometry FlexBook

CK-12
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CHAPTER 1 Basics of Geometry

Chapter Outline

1.1 The Three Dimensions

Learning Objectives

Here you will review the three dimensions.

You live in a three-dimensional world. Solid objects, such as yourself, are three dimensional. In order to better understand why your world is three dimensional, consider zero, one, and two dimensions.

Zero Dimensions

A point has a dimension of zero. In math, a point is assumed to be a dot with no size (no length or width).

FIGURE 1.1

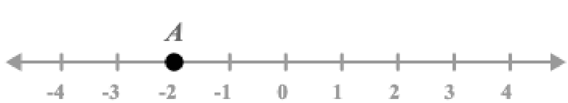
One Dimension

A line segment has a dimension of one. It has a length.

FIGURE 1.2

A number line is an example of a **line**, which is like an infinitely long line segment without any endpoints. Like a line segment, a line is one dimensional, so you only need a single number to describe a point on a number line (in the interactive below, the location of point A is -2). Remember that by definition, a line is straight.

FIGURE 1.3



Two Dimensions

A shape has a dimension of two. It can be compared to a line segment because it has a specific maximum length and width.

FIGURE 1.4

The rectangular coordinate system used for graphing is an example of a **plane**, which is like an infinitely large shape without measurable sides. To describe a point on the rectangular coordinate system you need two numbers, the x-coordinate and the y-coordinate (in the interactive below, the point P is located at (3, -2).

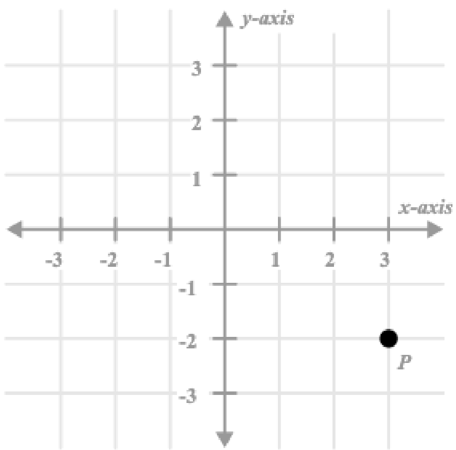


FIGURE 1.5

Basic Geometric Definitions: Alternate Dimensions

Play with the interactive below to explore and understand Alternate Dimensions.

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/geometric-definitions/plix/Basic-Geometric-Definitions-Alternate-Dimensions-528695525aa413715cbb1b2e

Three Dimensions

A solid has a dimension of three. It has length, width and height.

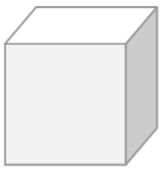


FIGURE 1.6

You can turn the rectangular coordinate system into a three dimensional coordinate system by creating a third axis, the z-axis, that is perpendicular to both the x-axis and y-axis.

To describe a point on the Cartesian coordinate system, you need three numbers, the x-coordinate, the y-coordinate and the z-coordinate.

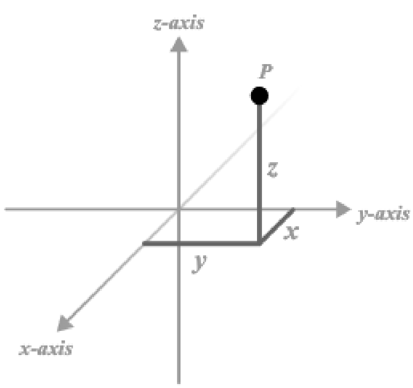


FIGURE 1.7

Because paper and screens have dimensions of two, it is hard to represent three-dimensional objects on them. Artists use perspective techniques to allow the viewer to imagine the three dimensions.

MEDIA

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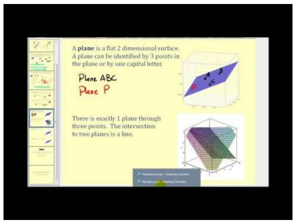
URL: http://www.ck12.org/flx/render/embeddedobject/222476

The number of values needed to describe a location tells you what dimensional space the location is in.



MEDIA

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Determining Number of Dimensions

You graph a line on a rectangular coordinate system. How many dimensions does that line have?

Even though the rectangular coordinate system has two dimensions, the line itself has only one dimension.

You plot a point on a line. How many dimensions does the point have?

Even though the line has one dimension, the point is simply a location *on* the line, and the point itself has zero dimensions.

Determining Number of Points

How many points make up a line?

FIGURE 1.8

A line is made up of _____ points.

Geometry Terms: Collinear Constellations

Play with the interactive below to explore and understand Geometric Terms.

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/geometric-definitions/plix/Basic-Geometric-Definitions-Collinear-Constellations-533de3445aa4137fcf4052fc

Describing Objects

Is the edge of a desk best described as a point, a line, a plane, or solid?

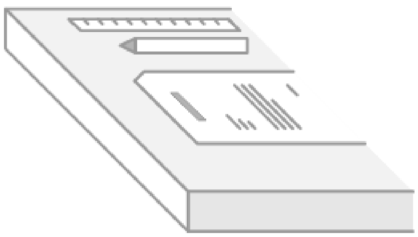


FIGURE 1.9

The edge of a desk is best described as a _____. It has one dimension.

Examples

Example 1

How might you imagine a four-dimensional figure?

Answer: The world we experience is three-dimensional, so we can visualize three dimensions with real examples. Four dimensions don't exist in our world, so it is very hard to imagine an object with dimension four. It helps to think about how two dimensions become three dimensions. A shape with dimension two moves up and down to create a solid with dimension three. Similarly, you can imagine a solid (such as a cube) with dimension three, moving within itself to create a tesseract, with dimension four.

Example 2

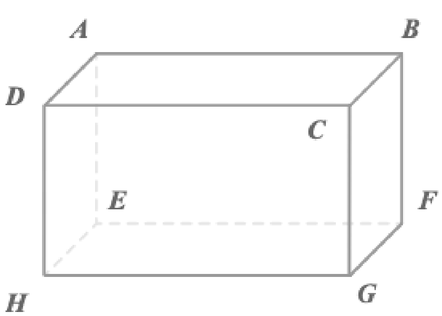


FIGURE 1.11

a. Name three points from the three-dimensional figure above. How many dimensions does each *point* have?

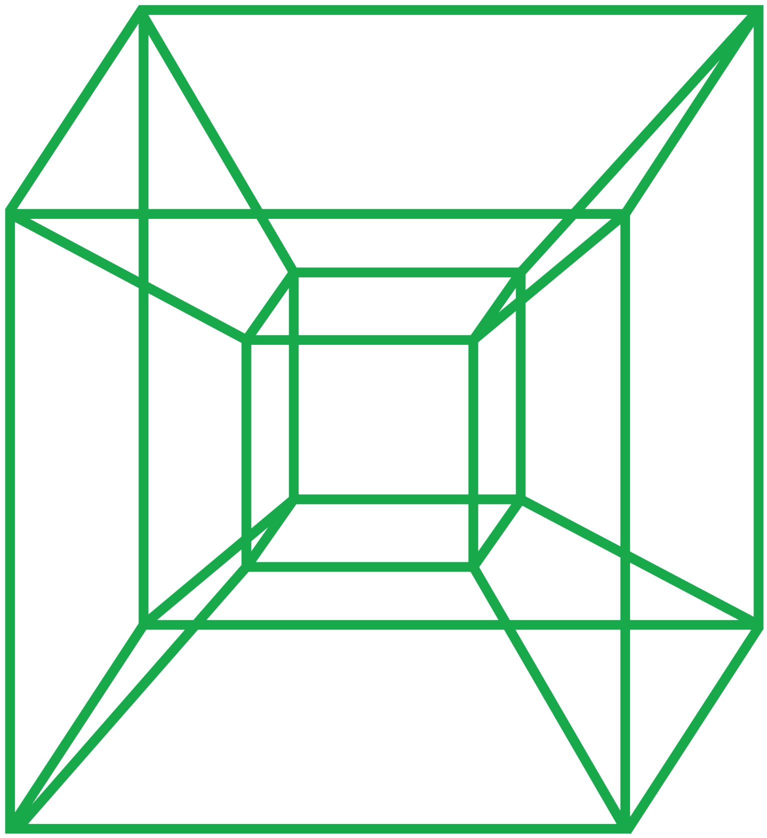


FIGURE 1.10
An impression of a four-dimensional cube.

Answer: Any points that make up this prism will work. These points are called vertices. For example, point A, point B, point C. Although the rectangular prism is three-dimensional, each *point* is simply a location in space, and as such has zero dimensions.

b. Name three line segments from the figure above. How many dimensions does each line segment have?

Answer: Any line segments that make up this prism will work. These line segments are called edges. For example, $\overline{AB}$, $\overline{BC}$, $\overline{CD}$. Although the figure is three-dimensional, each edge is only a line segment, and so is one-dimensional.

c. Name three sides from the figure above. How many dimensions does each side have?

Answer: Any sides that make up this prism will work. As part of a three-dimensional figure, they are two-dimensional shapes called faces. For example, $ABCD$ (the top face), $BCGF$ (the right side), and $CGHD$ (the front face).

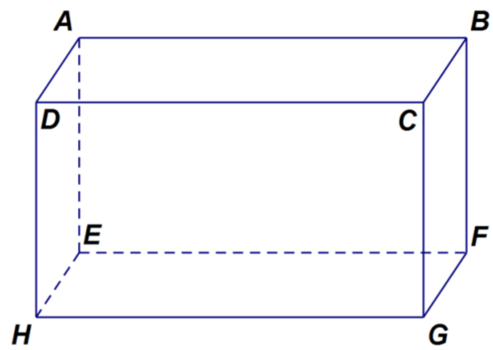
Review

1. In your own words, explain why a line has a dimension of one and a plane has a dimension of two.
2. Give a real-world example of something with a dimension of one.

3. Give a real-world example of something with a dimension of two.

4. Give a real-world example of something with a dimension of three.

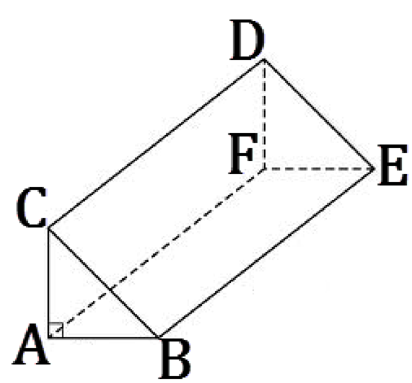
Use the figure below for #5-#6.



5. Points are considered coplanar if they lie on the same plane. What's an example of a point that is coplanar with points H and E?

6. What's an example of a point that is coplanar with points D and E?

Use the figure below for #7-#12.



7. Name three points from the figure above.

8. Name three line segments from the figure above.

9. Name three planes from the figure above.

10. Name a point that is coplanar with points A and B.

11. Name another point that is coplanar with points A and B, but not also coplanar with your answer to #10 such that all four points are on the same plane.

12. Name a point that is coplanar with C and E.

13. A plane has a dimension of ____.

14. A line segment has a dimension of ____.

15. A cube has a dimension of ____.

16. If two lines are perpendicular, how many right angles are formed? Why?

17. If two lines in a plane are parallel and a third line in the plane is perpendicular to one of them, what can we conclude? Why?

18. How many lines can be perpendicular to a line through a given point? Why?

19. Is it possible for two lines in different planes to intersect? Why or why not? Is it appropriate to call these lines parallel? Why or why not?

20. Bethany drives north for 2 miles, west for 3 miles, north for 4 miles, then east for 1 mile. Which pairs of segments are perpendicular, which are parallel?

21. We can think of space in terms of dimensions. For example, A 2-dimensional space is called a plane. A 3-dimensional space is, well, a 3-dimensional space! What's a 1-dimensional space called? How about a 0-dimensional space? The intersection of two distinct 2-dimensional spaces is a space of how many dimensions? The intersection of two distinct 1-dimensional spaces is a space of how many dimensions? Explain your answers. Extend this idea to 3-dimensions and beyond.

22. Give examples of real-world scenarios that can be modeled by objects of different dimensions. Do you think that there are any real-world objects that are 2-dimensional? Why or why not? Do you think it is possible to model any real-world scenario with more than 3-dimensions? Why or why not?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.1.

1.2 Angles and Lines

Learning Objectives

Here you will review different types of angles and lines.

Line Segment and Ray

A line segment is a portion of a line with two endpoints. A ray is a portion of a line with one endpoint. Line segments are named by their endpoints and rays are named by their endpoint and another point. In each case, a segment or ray symbol is written above the points. Below, the line segment is $\overline{AB}$ and the ray is $\overrightarrow{AB}$.

FIGURE 1.12

Angles

- When two rays meet at their endpoints, they form an angle.
- Depending on the situation, an angle can be named with an angle symbol $\angle$ and its vertex, or by three letters.

If three letters are used, the middle letter should be the vertex. The angle below could be called $\angle B$ or $\angle ABC$ or $\angle CBA$. Use three letters to name an angle if using one letter would not make it clear what angle you are talking about.

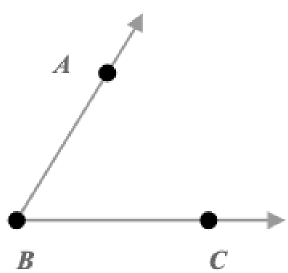


FIGURE 1.13

The size of an angle is measured in degrees. Therefore, 'the measure of the angle ABC' refers to the size, or measure, of the angle in degrees, often written $m\angle ABC$. You can use a protractor or geometry software to measure angles. Remember that a full circle has 360°.

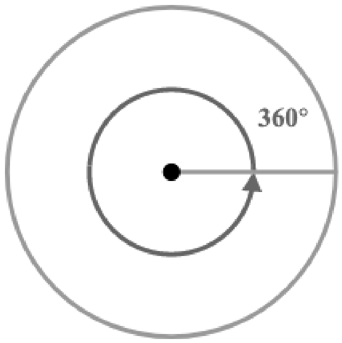


FIGURE 1.14

Types of Angles

- An angle that is exactly 0°is called a zero angle.
- An angle that is less than 90°is called an acute angle.
- An angle that is exactly 90°(one quarter of a circle) is called a right angle. A right angle is noted with a little square at its vertex.
- An angle that is more than 90°but less than 180°is called an obtuse angle. An angle that is exactly 180°(one half of a circle) is called a straight angle.

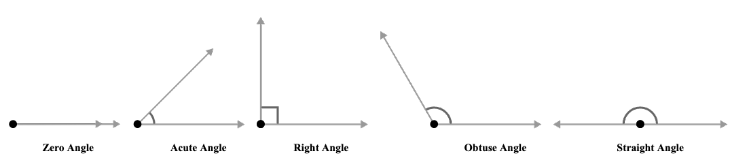


FIGURE 1.15

Identifying the Type of Angles

Name the angle below and classify it by its measure.

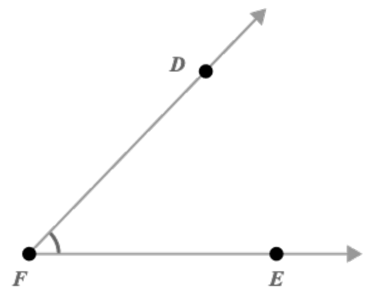


FIGURE 1.16

$\angle F$ or $\angle DFE$ or $\angle EFD$. It is _____ angle.

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PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/classifying-angles/plix/Angles-in-the-Game-of-Pool-5331d1915aa413234ceb02af

Complementary and Supplementary Angles

Two angles are complementary if the sum of their measures is 90°.

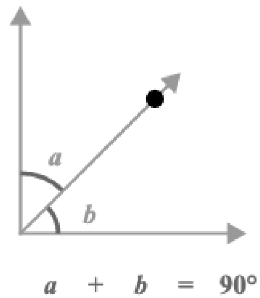


FIGURE 1.17

Two angles are supplementary if the sum of their measures is 180°.

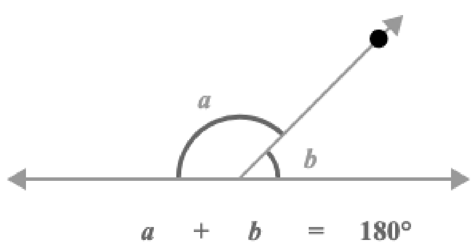


FIGURE 1.18

Two angles that together form a straight angle will always be supplementary.

Finding an Unknown Angle

x and y are complementary angles with measure of $y = 20^\circ$. What is the measure of x?

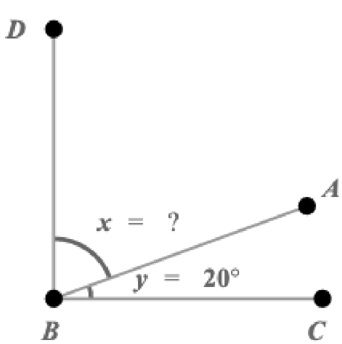


FIGURE 1.19

The measure of x is _____.

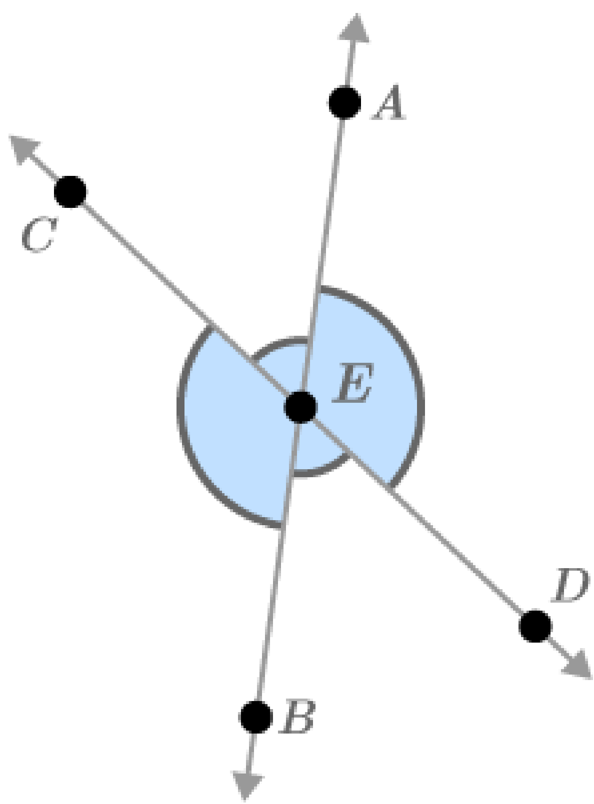
Adjacent and Vertical Angles

When two lines intersect, many angles are formed, as shown below.

In the diagram above, $\angle AEC$ and $\angle AED$ are adjacent angles because they are next to each other and share a ray. They are also supplementary, because together they form a straight angle. $\angle AEC$ and $\angle DEB$ are called vertical angles. Vertical angles will always have the same measure.

Consider this example problem:

Let $m\angle AEC = x^\circ$. Show that $m\angle DEB$ must also equal x°.



$$
\begin{aligned}
\angle AEC &= x^\circ \\
\angle AED &= 180 - \angle AEC \\
\angle AED &= 180 - x \\
\hline
\angle DEB &= 180 - \angle AED \\
\angle DEB &= 180 - (180 - x) \\
\angle DEB &= 180 - 180 + x \\
\angle DEB &= x \\
\angle DEB &= \angle AEC
\end{aligned}
$$

FIGURE 1.20

This shows that vertical angles will always have the same measure.

Identifying Angles

Explain why you must use three letters to identify any of the angles in the diagram below.

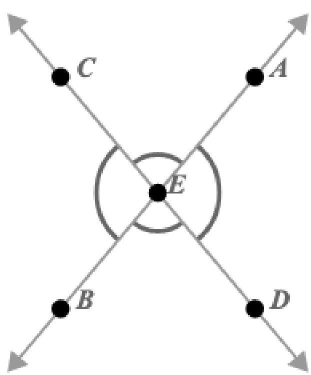
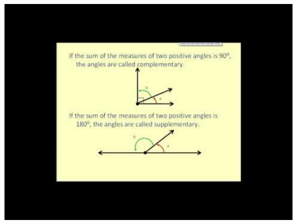


FIGURE 1.21

All angles in this diagram have a vertex of E. Therefore $\angle E$ is ambiguous because it could refer to many different angles. Use three letters with E as the middle letter to be clear about which angle you are referring to $\angle AEC$, $\angle AED$, $\angle DEB$, $\angle BEC$.



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Examples

Example 1

Angles are formed by intersecting lines or rays. If you take any two lines or rays, will you form at least one angle?

As long as the lines or rays intersect, at least one angle will be formed. If the lines (or rays) are parallel, and therefore don't intersect, then no angles will be formed.

FIGURE 1.22

Example 2

Estimate the measure of angle $\angle DFE$. Use a protractor to confirm your answer.

Remember that exactly half of a right angle is 45°. This angle looks to be more than half of a right angle. You might guess that it is approximately 55°. Using a protractor, you can see that it is about 60°.

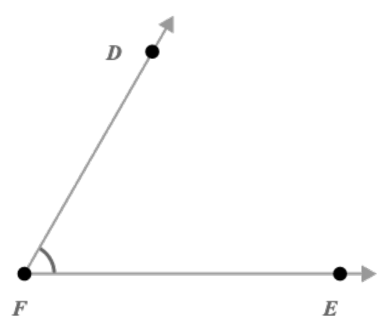


FIGURE 1.23

Example 3

What are two lines that form a right angle called?

Perpendicular lines.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/identify-line-types/plix/Identify-Types-of-Lines-5461664c8e0e081d8411f800

Review

1. What's the difference between a line segment, a line, and a ray?
2. Draw an example of a right angle.
3. Draw an example of an obtuse angle.
4. Draw an example of an acute angle.
5. Why are two angles that make a straight angle always supplementary?
6. If $m\angle ABC = (2x+4)^\circ$, $m\angle DEF = (3x-5)^\circ$, and $\angle ABC$ and $\angle DEF$ are complementary, what are the measures of the angles?
7. If $m\angle ABC = (2x+4)^\circ$, $m\angle DEF = (3x-5)^\circ$, and $\angle ABC$ and $\angle DEF$ are supplementary, what are the measures of the angles?

Use the diagram below for #8-#12.

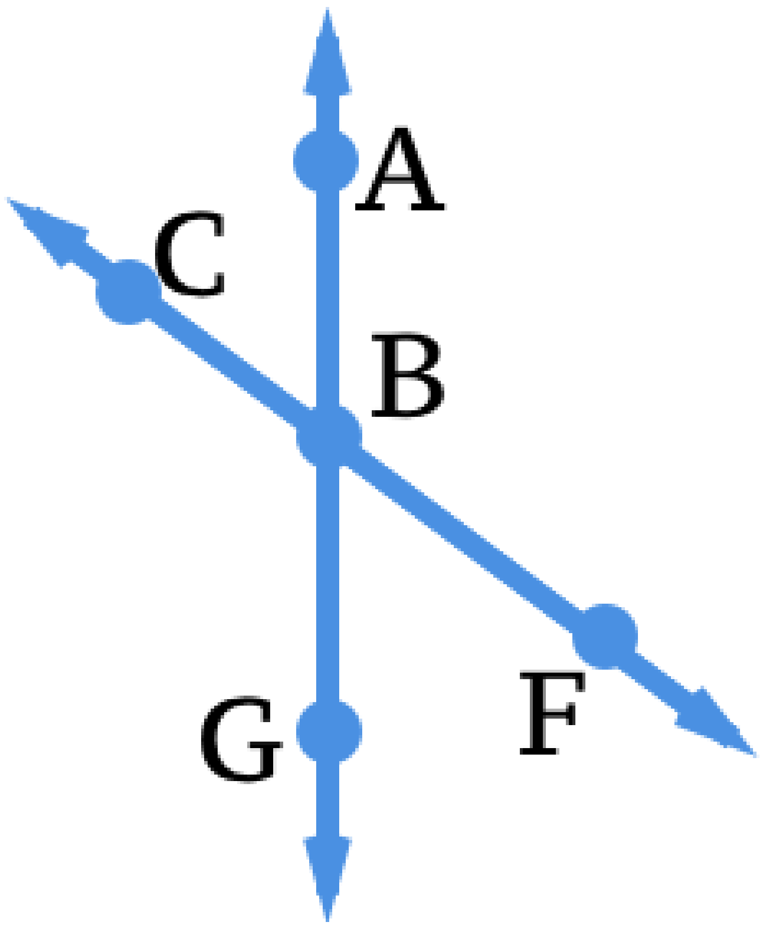


FIGURE 1.24

8. Give an example of vertical angles.

9. Give an example of a straight angle.

10. Give an example of supplementary angles.

11. If $m\angle ABC = 70^\circ$, find $m\angle ABF$.

12. If $m\angle ABC = 70^\circ$, find $m\angle FBG$.

13. What do you remember about perpendicular lines?

Use the angle in the image for #14-#15.

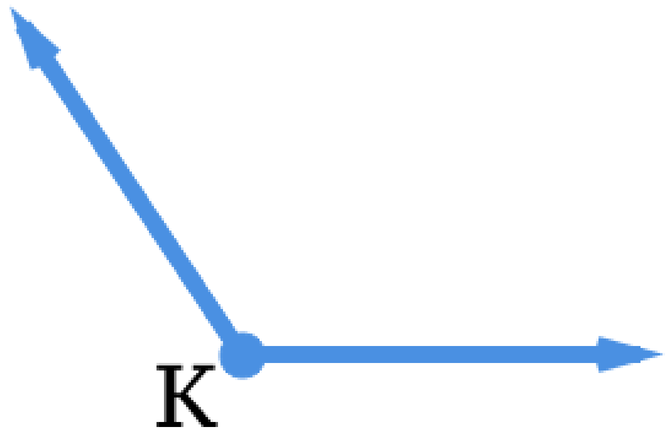


FIGURE 1.25

14. Name the angle and classify it based on its size.

15. Estimate the measure of the angle. Use a protractor to confirm your answer.

16. Draw a diagram in which two angles are supplementary to the same angle. What must be true about the original two angles? Explain.

17. We use the term *complementary* to describe angles that sum to ___ and *supplementary* to describe angles that sum to ___. What about angles that sum to 360°? Invent a name for such angles and justify your choice.

18. Draw two vertical angles. How much must each ray of one of the vertical angles be rotated in order to match up with the other vertical angle? Explain.

19. Draw two angles of the same measure that are not vertical. Draw two angles that are supplementary but not adjacent. Draw two angles that are adjacent and have the same measure. What is the measure of each angle in the last drawing? Why?

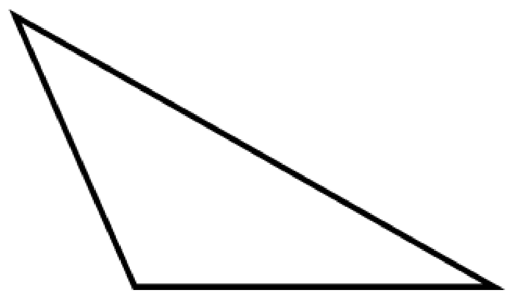
Review (Answers)

To see the Review answers, open this PDF file and look for section 1.2.

1.3 Copies of Line Segments and Angles

Here you will learn the difference between a construction and a drawing. You will also learn how to create copies of line segments, angles, and triangles.

Use a straightedge to draw a triangle like the one below on your paper. Describe at least two ways to use a compass and straightedge to copy the triangle.



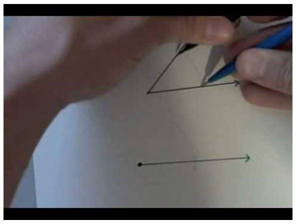
Line Segments and Angles

As you have studied math, you have often created **drawings**. Drawings are a great way to help communicate a visual idea. A **construction** is similar to a drawing in that it produces a visual outcome. However, while drawings are often just rough sketches that help to convey an idea, constructions are step-by-step processes used to create accurate geometric figures.

Constructions take us back over 2000 years to the ancient Greeks, before computers or other advanced technology. Using only the tools of a **compass** and a **straightedge**, they discovered how to copy segments, angles and shapes, how to create perfect regular polygons, and how to create perfect parallel and perpendicular lines. Today, learning constructions is a way to apply your knowledge of geometric principles. You can do constructions by hand, or with dynamic geometry software. In this concept, the focus is on hand constructions and making copies of segments, angles, and triangles.

To create a construction by hand, there are a few tools that you can use:

1. **Compass:** A device that allows you to create a circle with a given radius. Not only can compasses help you to create circles, but also they can help you to copy distances.
2. **Straightedge:** Anything that allows you to produce a straight line. A straightedge should not be able to measure distances. An index card works well as a straightedge. You can also use a ruler as a straightedge, as long as you only use it to draw straight lines and not to measure.
3. **Paper:** When a geometric figure is on a piece of paper, the paper itself can be folded in order to construct new lines.



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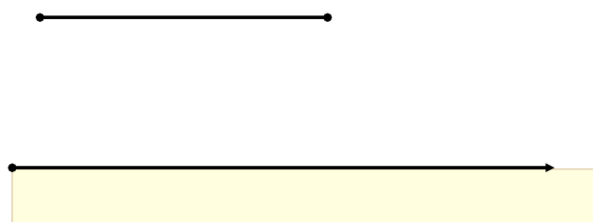
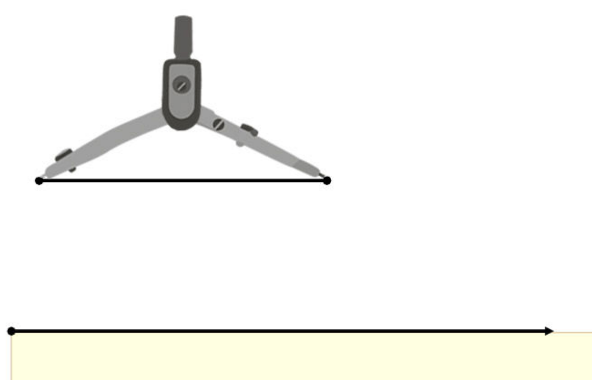
Let's take a look at some example problems.

1. Use a straightedge to draw a line segment on your paper like the one shown below. Then, use your straightedge and compass to copy the line segment exactly.

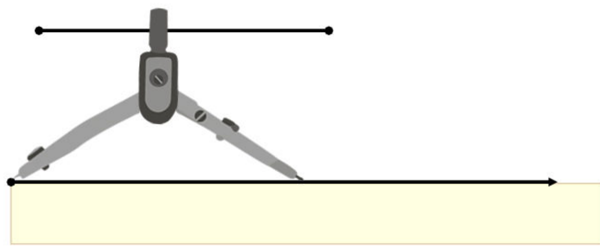
First use your straightedge and pencil to create a new ray.



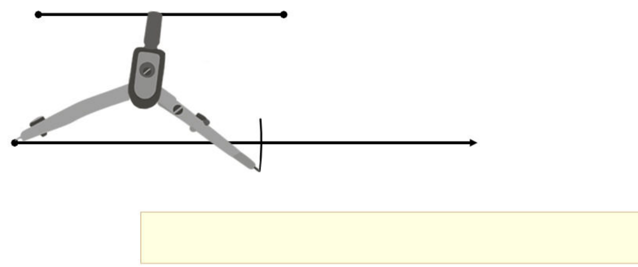
Now, you have one endpoint of your line segment. Your job is to figure out where the other endpoint should go on the ray. Use your compass to measure the width of the original line segment.



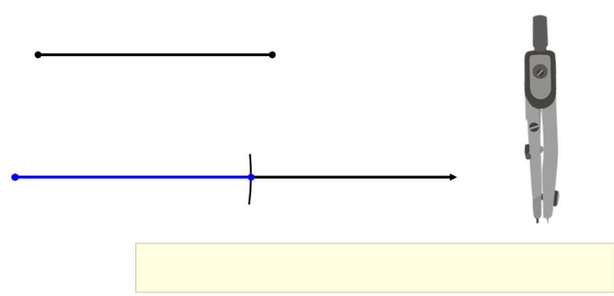
Now, move the compass so that the tip is on the endpoint of the ray.



You can now see where the endpoint of the segment should lie on the ray. Draw a little arc with the compass to mark where the endpoint should go.



You can use your straightedge to draw the copied line segment in a different color if you wish.



Note that in this construction, the compass was used to copy a distance. This is one of the primary uses of a compass in constructions.

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Click image to the left or use the URL below.

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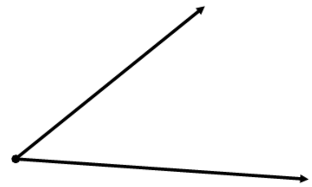
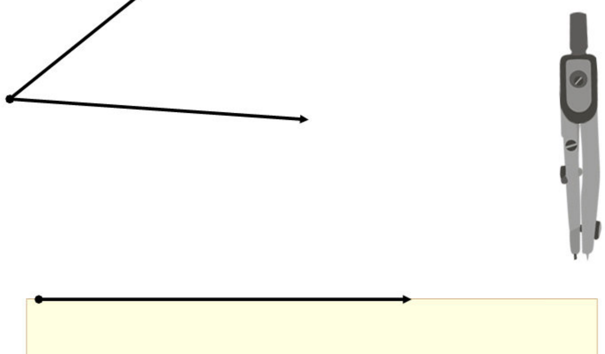
2. Use a straightedge to draw an angle on your paper like the one shown below. Then, use your straightedge and compass to copy the angle exactly.

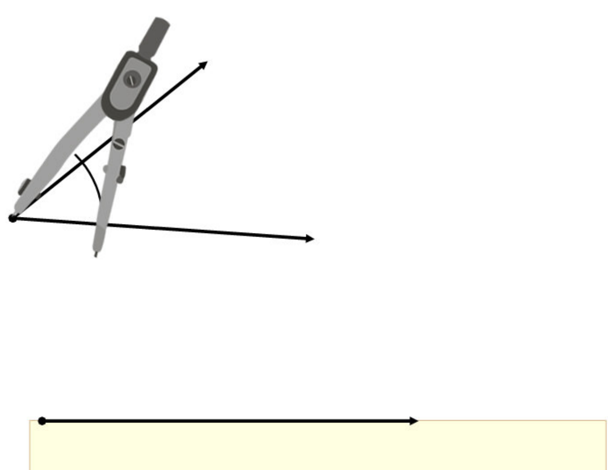


Keep in mind that what defines the angle is the opening between the two rays. The lengths of the rays are not relevant.

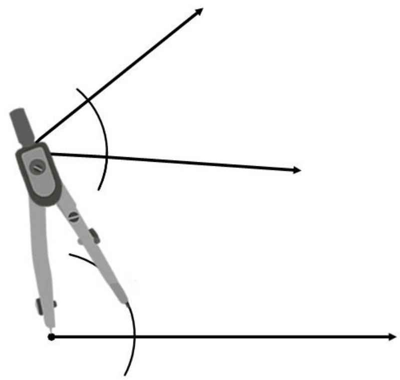
Start by using your straightedge and pencil to draw a new ray. This will be the bottom of the two rays used to create the angle.



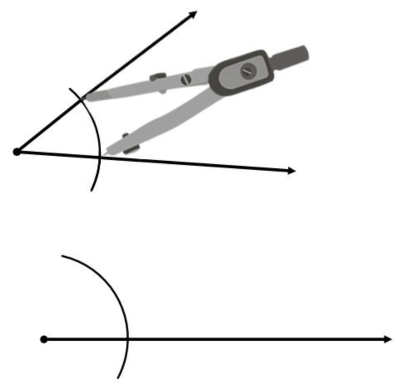
Next, use your compass to make an arc through the original angle. It does not matter how wide you open your compass for this.



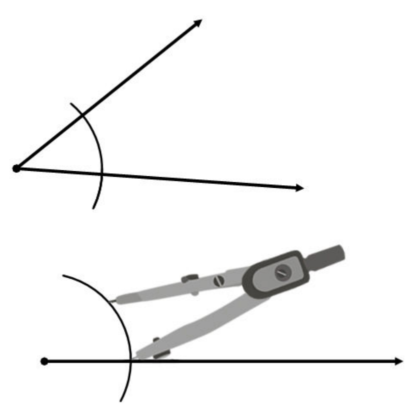
Next, leave your compass open to the same width, and make a similar arc through the new ray.



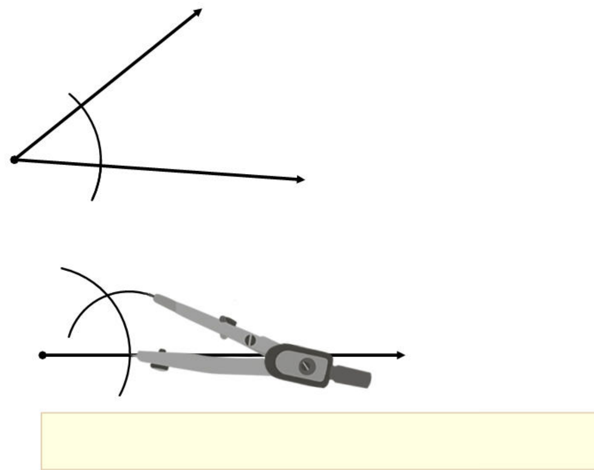
3. Now, you know that the second ray necessary to create the new angle will go somewhere through that arc. Measure the width of the arc on the original angle using the compass.



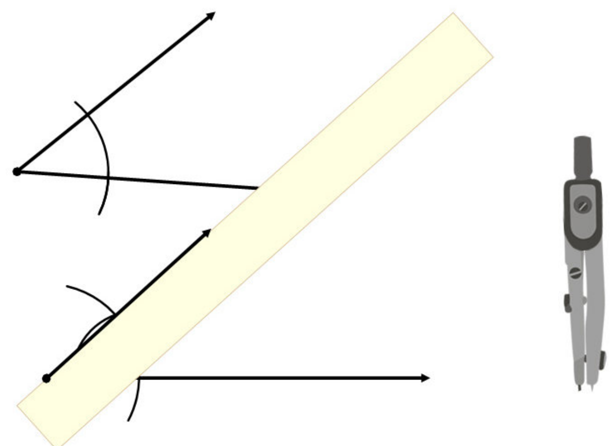
Leave the compass open to the same width, and move it to the new angle.



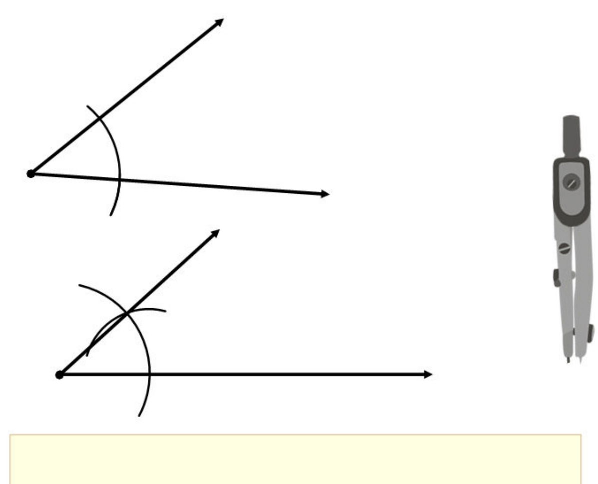
Make a mark to show where the pencil on the compass intersects the arc.



4. Use a straightedge to draw another ray that passes through the point of intersection of the two compass markings.



You have now copied the angle exactly.



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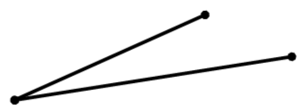
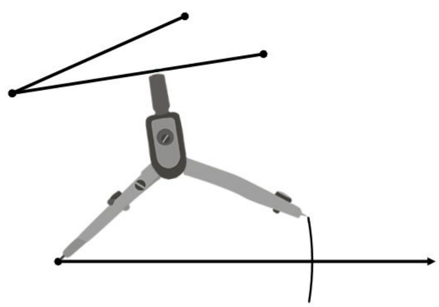
5. An angle is created from two line segments. Use a straightedge to draw a similar figure on your paper. Then, use the straightedge and compass to copy the figure exactly.

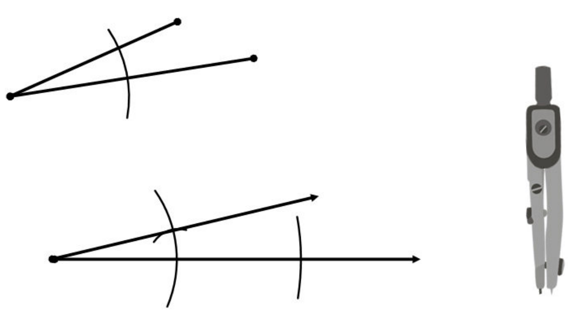


To copy this figure, you will need to copy both the line segments and the angle.

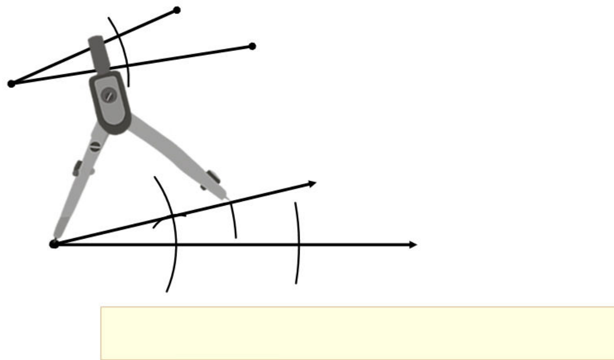
Start by copying the line segment on the bottom using the process outlined in #1 (draw a ray, use the compass to measure the width of the line segment, mark off the endpoint on the ray).



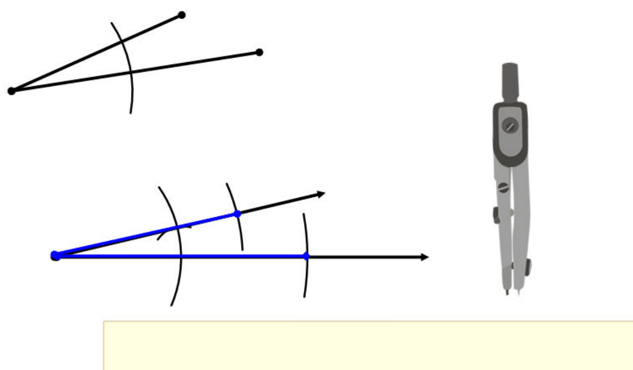
Next, copy the angle using the process outlined in #2 (draw an arc through the angle and draw the same arc through the new ray, measure the width of the arc, draw a new ray through the intersection of the two markings).



Finally, copy the second line segment by measuring its length using the compass and marking off the correct spot for the endpoint.



You can now draw the copied segments in a different color for emphasis.



Examples

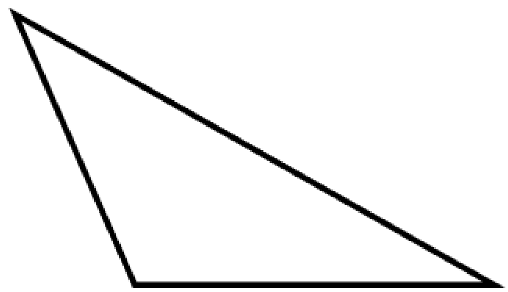
Example 1

Earlier, you were asked to describe two ways how to use a compass and a straightedge to copy a triangle.

To copy a triangle means to create a congruent triangle. There are four triangle congruence criteria that work for any type of triangle: SSS, SAS, AAS, ASA. You can use SSS, SAS, or ASA combined with copying angles and line segments to copy a triangle.

1. SSS: Copy one line segment. Copy the other two line segments so that their endpoints intersect. **See Example 2**.
2. SAS: Copy one line segment. Copy an angle from one of the endpoints of the line segment. Copy a second line segment onto the ray created by the copied angle. Connect the endpoints to form the triangle. **See Example 3**.
3. ASA: Copy one line segment. Copy two angles, one from each endpoint. The intersection of the angles will produce the third vertex of the triangle. **See Example 4**.

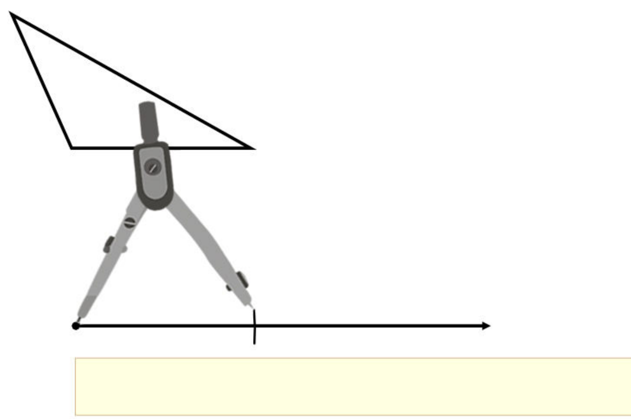
You drew a triangle similar to the one below for the concept problem.



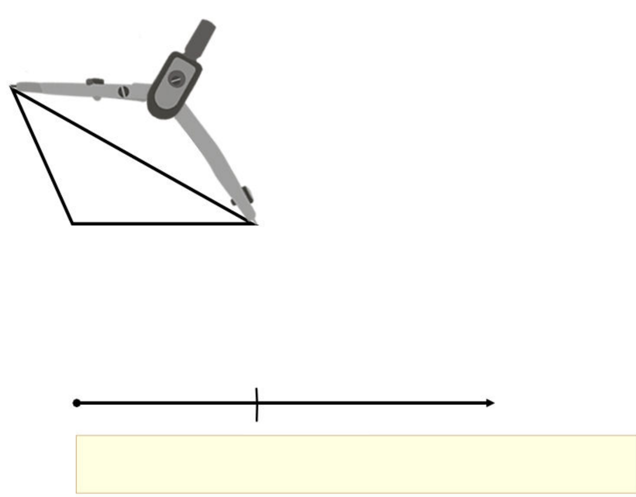
Example 2

Copy your triangle using SSS.

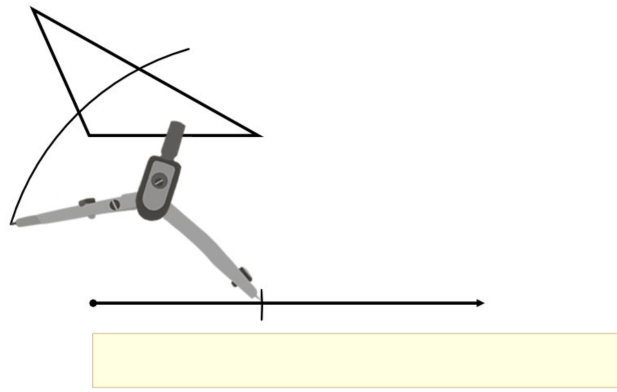
Start by copying one line segment. Here, the base line segment is copied.



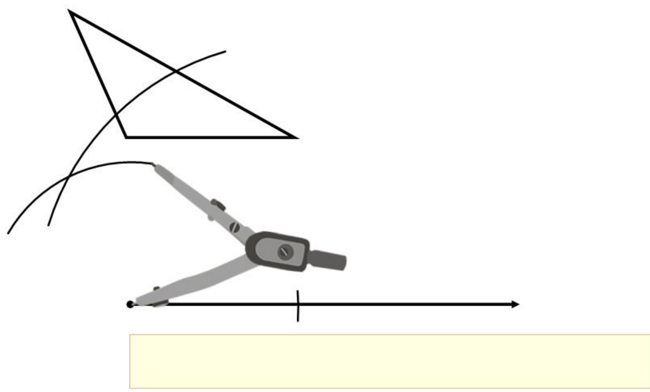
Next, use the compass to measure the length of one of the other sides of the triangle.



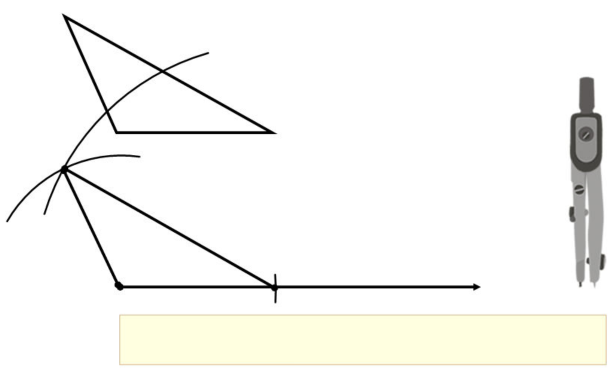
Move the compass to the location of the new triangle and make an arc to mark the length of the second side of the triangle from the correct endpoint.



Repeat with the third side of the triangle.



The point where the arcs intersect is the third vertex of the triangle. Connect to form the triangle.

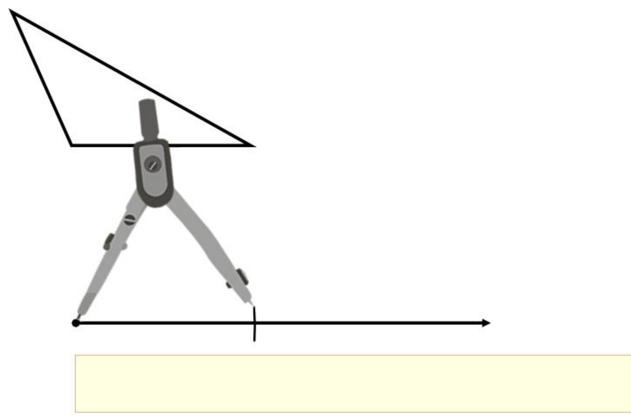


Note that with this method, you have only used the lengths of the sides of the triangle (as opposed to any angles) to construct the new triangle.

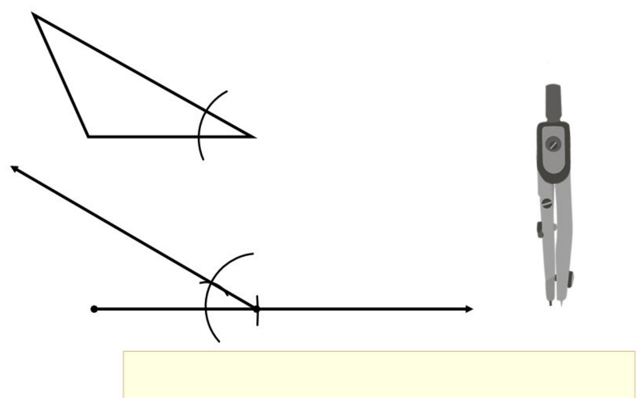
Example 3

Copy your triangle using SAS.

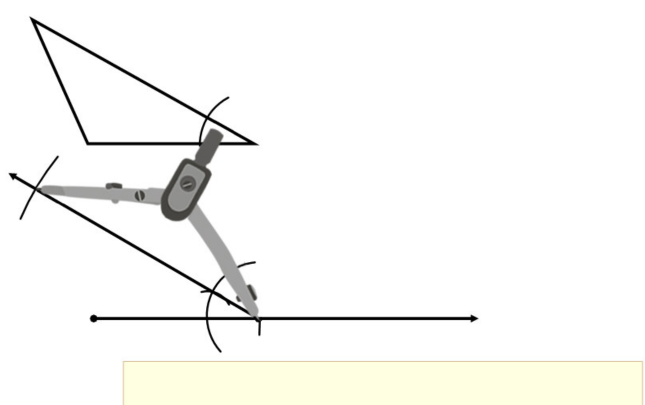
Start by copying one line segment. Here, the base of the triangle is copied.



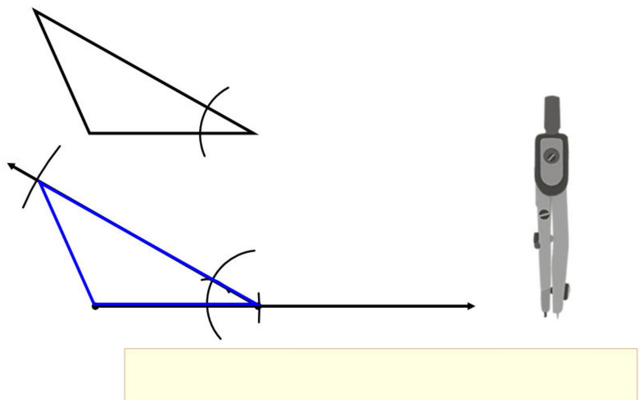
Next, copy the angle at one of the endpoints of the line segment.



Copy the second side of the triangle (that creates the angle you copied) onto the ray that you just drew.



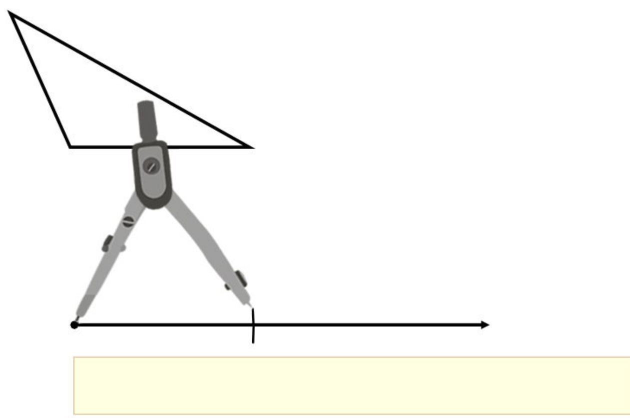
Connect to form the triangle.



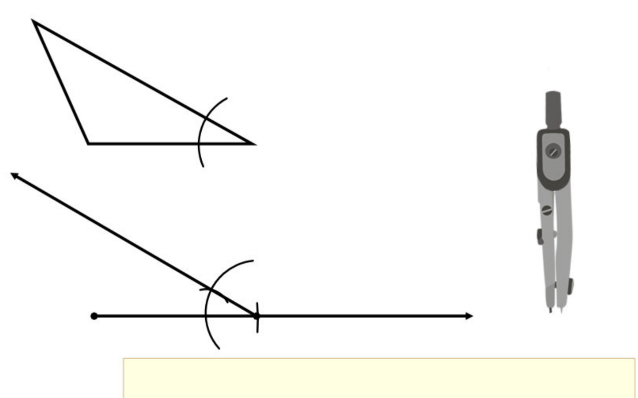
Example 4

Copy your triangle using ASA.

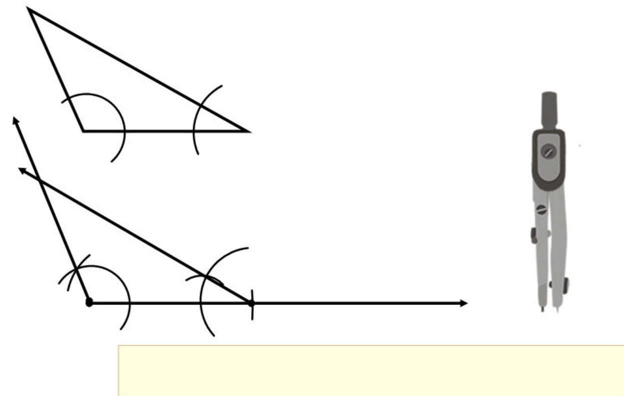
Start by copying one line segment.



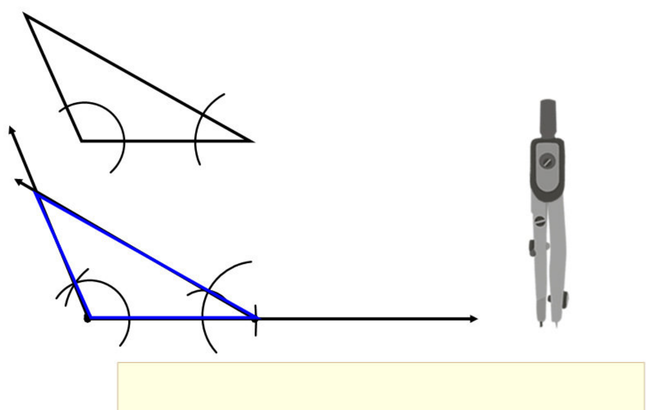
Next, copy the angle at one of the endpoints.



Copy the angle at the other endpoint of the line segment.



Connect to form the triangle.



Review

1. What is the difference between a drawing and a construction?
2. What is the difference between a straightedge and a ruler?
3. Describe the steps for copying a line segment.
4. Describe the steps for copying an angle.
5. When copying an angle, do the lengths of the lines matter? Explain.
6. Explain the connections between copying a triangle and the triangle congruence criteria.
7. Draw a line segment and copy it with a compass and straightedge.
8. Draw another line segment and copy it with a compass and straightedge.
9. Draw an angle and copy it with a compass and straightedge.
10. Draw another angle and copy it with a compass and straightedge.
11. Use your straightedge to draw a triangle. Copy the triangle using $SSS \cong$. Describe your steps.
12. Copy the triangle from #11 using $SAS \cong$. Describe your steps.
13. Copy the triangle from #11 using $ASA \cong$. Describe your steps.
14. Can you copy the triangle from #11 using $AAS \cong$? Explain.
15. Compare the methods for copying the triangle. Is one method easier than the others? Explain.
16. Graph the points A(1, 4), B(4, 2) and C(1, 5). Join the points to create line segment AC. Do you think it is possible to find the slope of this line? Explain.
17. Draw a triangle. Describe the differences in your drawing if you were to show ASA, SAS, or SSS.
18. Do you think a compass is more or less accurate than a ruler? What are the advantages of using a compass when constructing a triangle?

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.1.

1.4 Bisectors of Line Segments and Angles

Learning Objectives

Here you will learn how to construct bisectors of line segments and angles.

Describe how to construct a 45° angle using a compass and a straightedge.

Bisectors

A **construction** is similar to a **drawing** in that it produces a visual outcome. However, while drawings are often just rough sketches that help to convey an idea, constructions are step-by-step processes used to create accurate geometric figures. To create a construction by hand, there are a few tools that you can use:

1. **Compass:** A device that allows you to create a circle with a given radius. Not only can compasses help you to create circles, but also they can help you to copy distances.
2. **Straightedge:** Anything that allows you to produce a straight line. A straightedge should not be able to measure distances. An index card works well as a straightedge. You can also use a ruler as a straightedge, as long as you only use it to draw straight lines and not to measure.
3. **Paper:** When a geometric figure is on a piece of paper, the paper itself can be folded in order to construct new lines.

To **bisect** a segment or an angle means to divide it into two congruent parts. A bisector of a line segment will pass through the midpoint of the line segment. A **perpendicular bisector** of a segment passes through the midpoint of the line segment and is perpendicular to the line segment.

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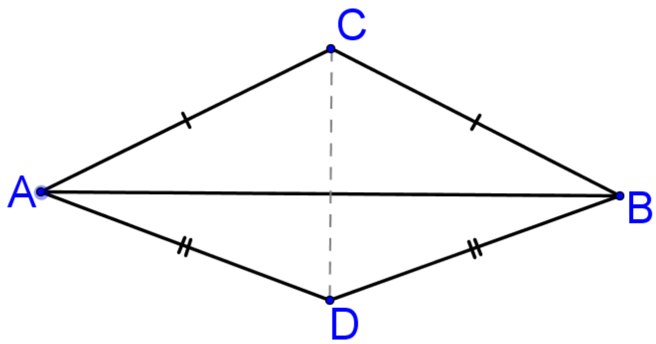
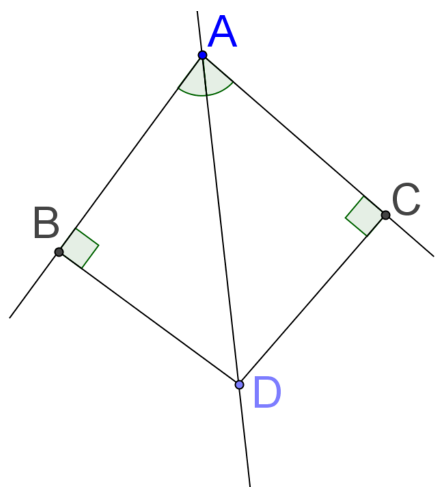
$\overleftrightarrow{DE}$ is the perpendicular bisector of $\overline{AC}$, so $\overline{AB} \cong \overline{BC}$ and $\overline{AC} \perp \overleftrightarrow{DE}$ when $\angle DBC = 90^\circ$.

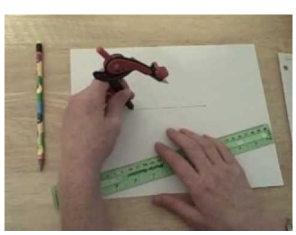
In order to construct bisectors of segments and angles, it's helpful to remember some relevant theorems:

Any point on the perpendicular bisector of a line segment will be equidistant from the endpoints of the line segment. *This means that one way to find the perpendicular bisector of a segment (such as $\overline{AB}$ below) is to find two points that are equidistant from the endpoints of the line segment (such as C and D below) and connect them.*



Any point on the angle bisector of an angle will be equidistant from the rays that create the angle. *This means that one way to find the angle bisector of an angle (such as $\angle BAC$ below) is to find two points that are equidistant from the rays that create the angle (such as points A and D below).*





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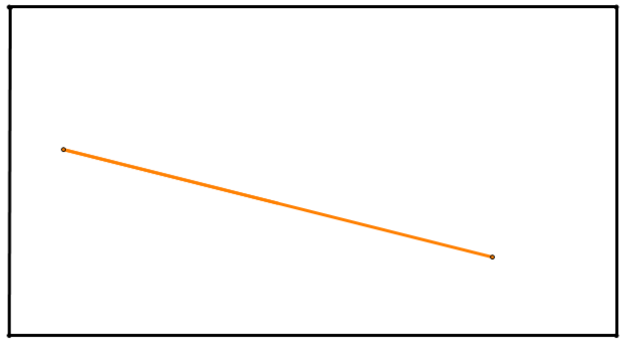
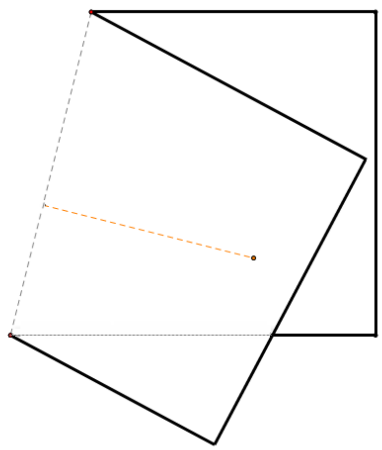
Let's take a look at different ways to find the perpendicular bisector of a line segment.

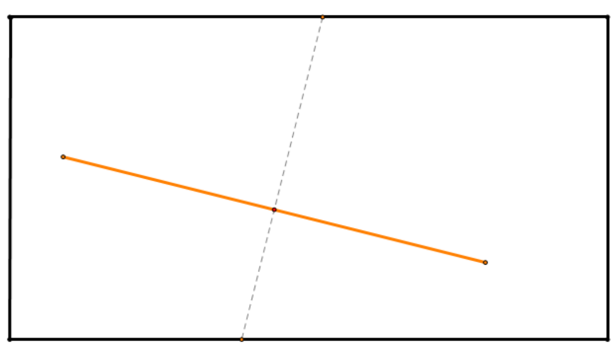
1. Draw a line segment like the orange one below on a piece of paper (the rectangle below represents the paper). Fold the paper in order to find the perpendicular bisector of the line segment.



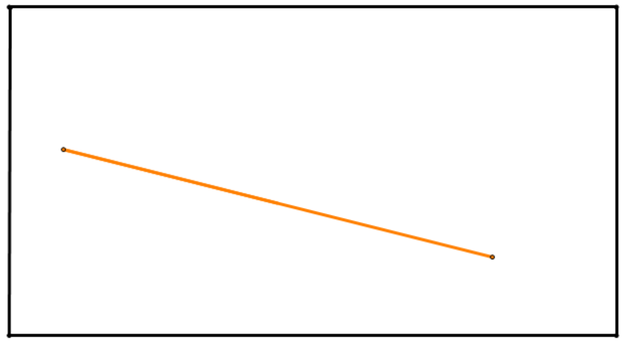
To find the perpendicular bisector, fold the paper so that the endpoints lie on top of each other. By doing this, you have matched the two halves of the line segment exactly.

Any point on the fold is now equidistant from both endpoints, because the endpoints are now in the same place! This means that the crease made by the fold will be the perpendicular bisector of the line segment.

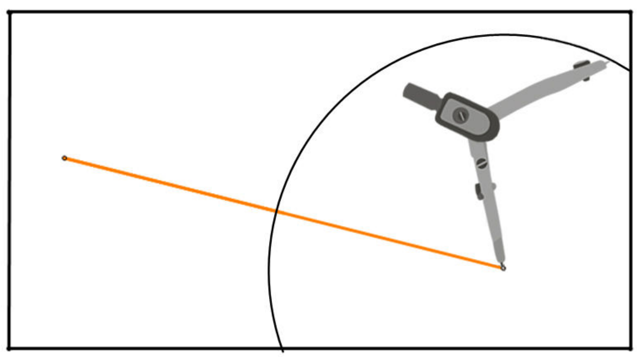




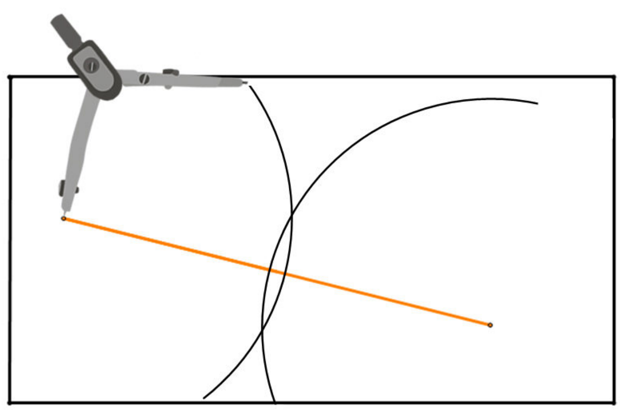
2. Draw a line segment like the orange one below on a piece of paper (the rectangle below represents the paper). Use a compass and straightedge to find the perpendicular bisector of the line segment.



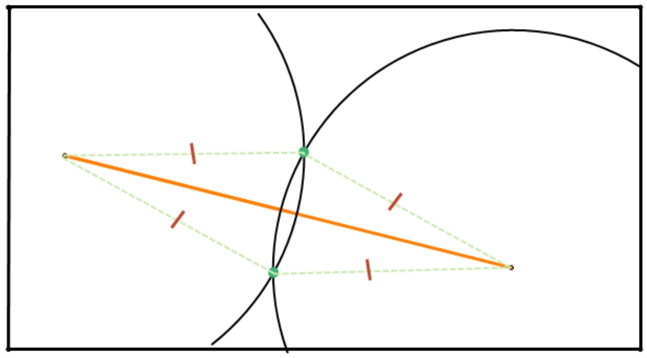
Use your compass to draw a circle centered at each endpoint with the **same radius**. Make sure the radius of the circles is large enough so that the circles will intersect. *If the circles are too big to fit on the paper, draw the portion of the circle that fits on the paper.* Here is the first circle:



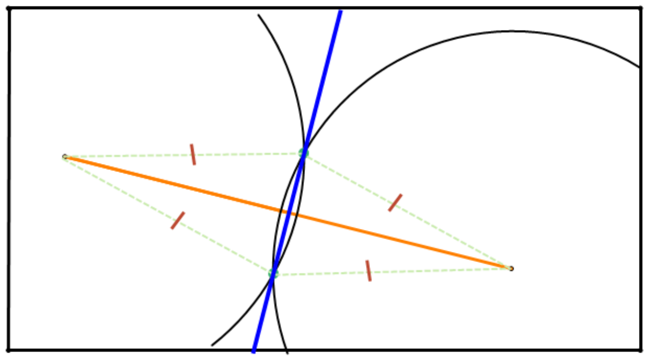
Here is the second circle:



The points of interest are the two points where the circles intersect. Because the radius of each circle is the same, each of these points are equidistant from the endpoints of the line segment.



Therefore, each of these points lies on the perpendicular bisector of the line segment. Use your straightedge to draw a line connecting the two intersection points. This is the perpendicular bisector of the line segment.



How can you use a compass and straightedge to find the midpoint of a line segment?

One way is to use the process from Example B to construct the perpendicular bisector. The midpoint of the line segment will be the point where the perpendicular bisector intersects the line segment.

Examples

Example 1

Earlier, you were asked how to construct a 45° angle using a compass and a straightedge.

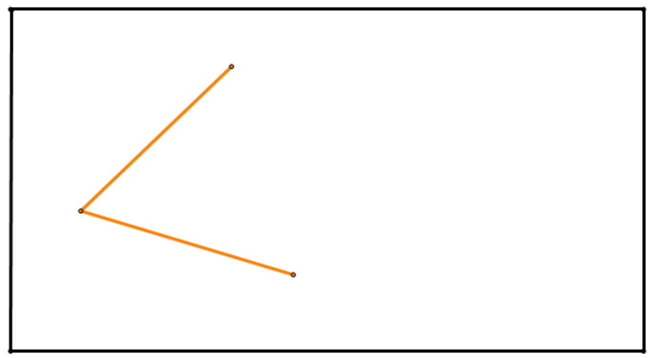
One way to construct a 45° angle is to:

- Draw a segment and construct its perpendicular bisector. This will give you 90° angles.
- Construct the angle bisector of one of those 90° angles. This will produce two 45° angles. *Note: Steps for constructing an angle bisector are explored in the guided practice questions.*

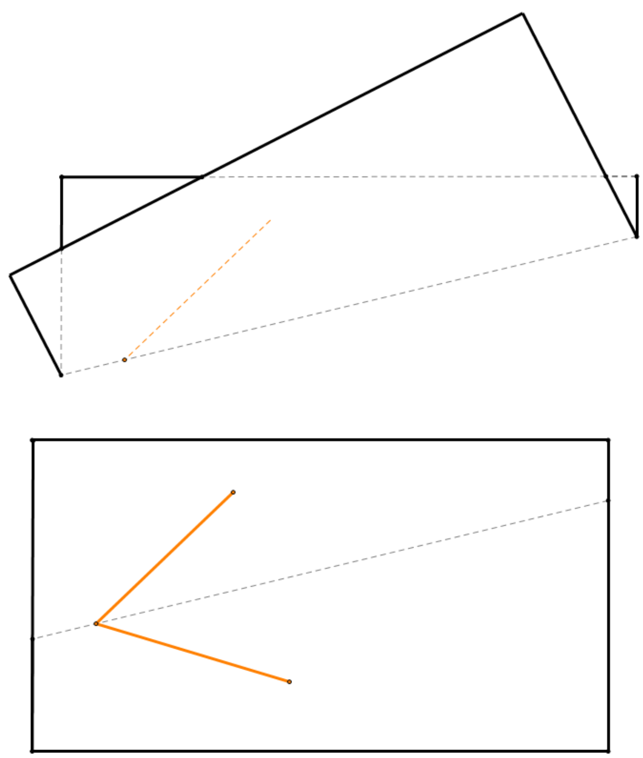
Can you think of another way to construct a perfect 45° angle with just a compass and a straightedge?

Example 2

Draw an angle like the one below on a piece of paper (the rectangle below represents the paper). Note that the two segments creating the angle do NOT need to be the same length. Fold the paper to create the bisector of the angle.

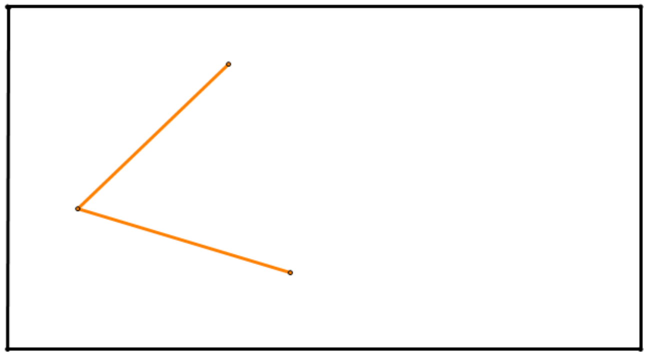


Fold the paper so that one segment overlaps the other segment. The crease will be the angle bisector.

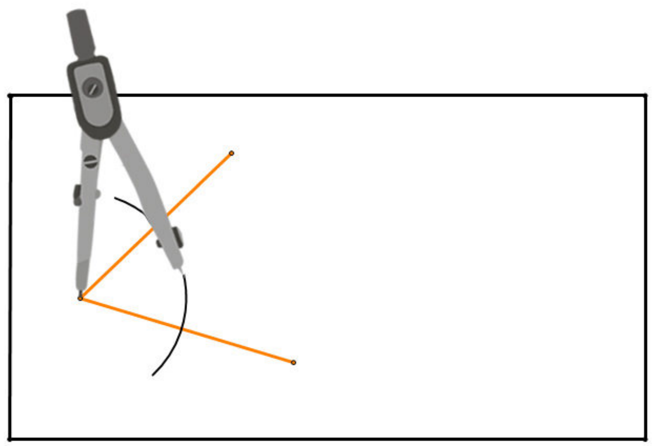


Example 3

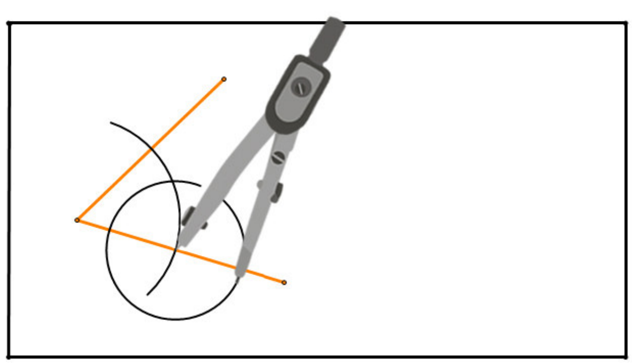
Draw an angle like the one below on a piece of paper (the rectangle below represents the paper). Note that the two segments creating the angle do NOT need to be the same length. Use a compass and straightedge to find the bisector of the angle.



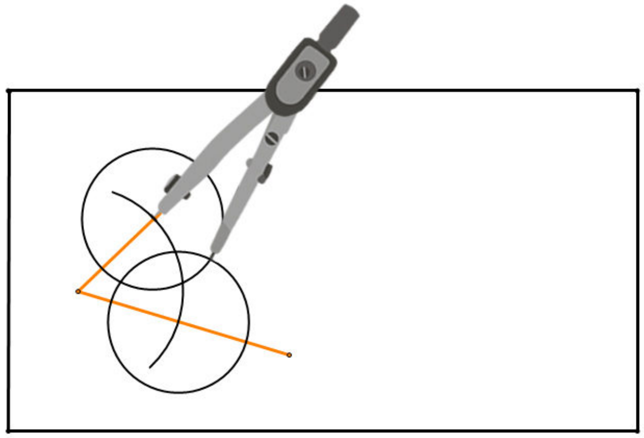
Use your compass to create a portion of a circle centered at the vertex of the angle that passes through both segments creating the angle:



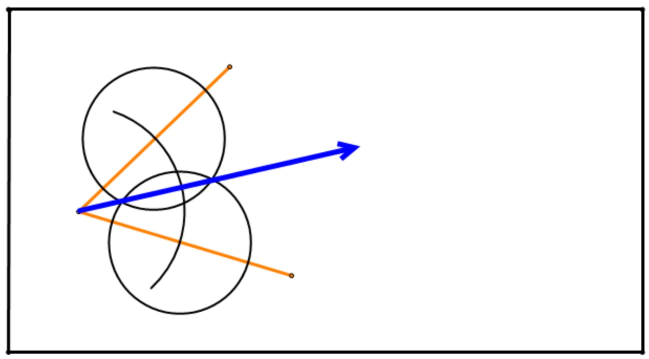
Create two circles with the same radius centered at each intersection point of the arc and the two segments. Make sure the radius is big enough so that the circles will overlap. Here is the first circle:



Here is the second circle:



The points where the circles intersect are equidistant from the segments creating the angle. Therefore, they define the bisector of the angle. Connect those intersection points to create the angle bisector:



CK-12 PLIX Interactive

PLIX

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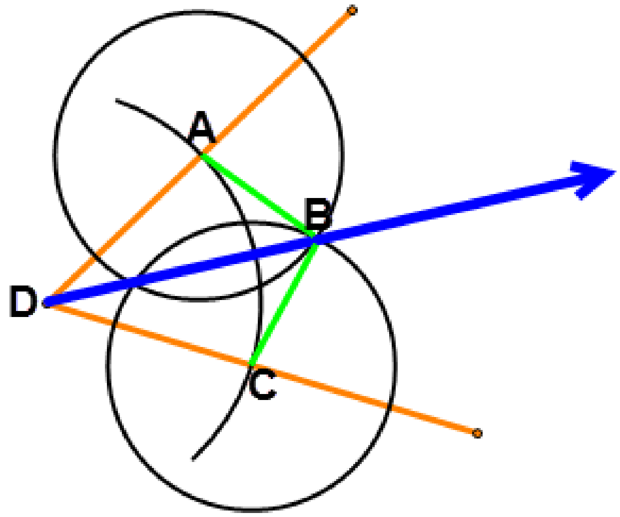
URL: http://www.ck12.org/geometry/construct-bisectors-of-line-segments-and-angles/plix/Cut-an-Angle-56a969f2da2cfe67f70285c3

Example 4

Prove that the angle bisector created using the method in Example 3 is actually the angle bisector.

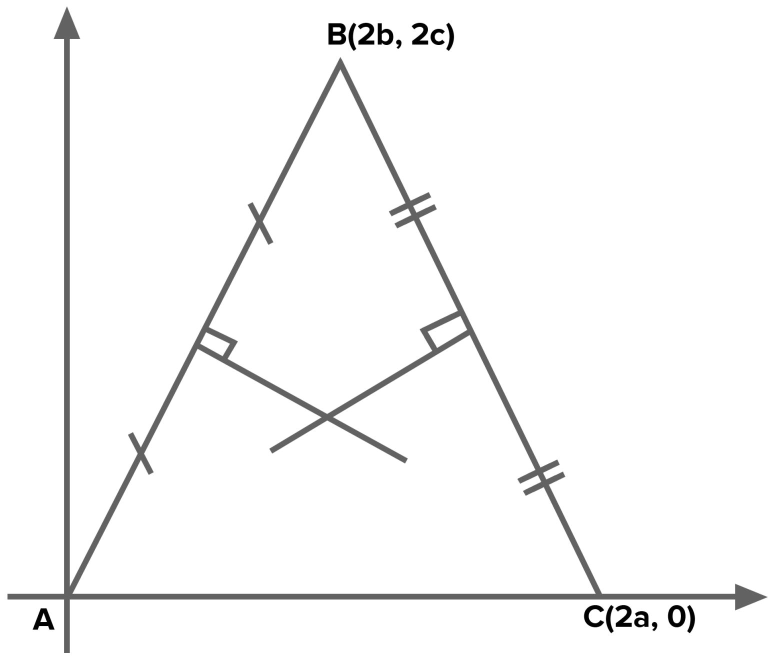
Consider the construction in the image. Points have been labeled and two additional segments have been drawn in (shown in green):



In this picture, $\overline{AD} \cong \overline{CD}$ because they are both radiuses of the same partial circle centered at point D. $\overline{AB} \cong \overline{CB}$ because they are both radiuses of congruent circles centered at A and C respectively. $\overline{BD} \cong \overline{BD}$ by the reflexive property. Therefore, $\Delta ABD \cong \Delta CBD$ by $SSS \cong$. $\angle ADB \cong \angle BDC$ because they are corresponding parts of congruent triangles. Therefore, $\overrightarrow{BD}$ must be the angle bisector of $\angle ADC$.

Review

1. What does it mean to *bisect* a segment or an angle?
2. Describe the steps for finding the perpendicular bisector of a line segment.
3. Describe how to use the perpendicular bisector of a line segment to find the midpoint of the line segment.
4. What's the difference between a bisector and a perpendicular bisector? How can you construct a non-perpendicular bisector of a line segment?
5. Draw a line segment on your paper and construct the perpendicular bisector of the segment.
6. Draw another line segment on your paper and construct the perpendicular bisector of that segment using another method.
7. Draw an angle on your paper and construct the bisector of the angle.
8. Draw another angle on your paper and construct the bisector of that angle using another method.
9. Use your straightedge to draw a triangle on your paper. Construct the angle bisector of each angle. What point have you found?
10. Use your straightedge to draw a triangle on your paper. Construct the perpendicular bisector of each side of the triangle. What point have you found?
11. Use the triangle and your work from #10. Construct the medians of the triangle. What point have you found?
12. Compare and contrast the two methods for finding a bisector-paper folding vs. compass and straightedge.
13. Construct a 45° angle (look at the concept problem for help). Then, construct a 22.5° angle.
14. Construct an isosceles right triangle. *Hint: Start by creating a right angle by constructing a perpendicular bisector*.
15. If possible, extend your construction of an isosceles right triangle to construct a square. Describe your steps.
16. Draw a triangle on grid paper. Construct perpendicular bisectors for each of the lines of the triangles. Point X is the intersection point of the three bisectors. What does this suggest about the perpendicular bisectors of the sides of a triangle?
17. In the diagram below, prove that the perpendicular bisector of AC passes through the points of intersection of the other two perpendicular bisectors of triangle ABC.



Review (Answers)

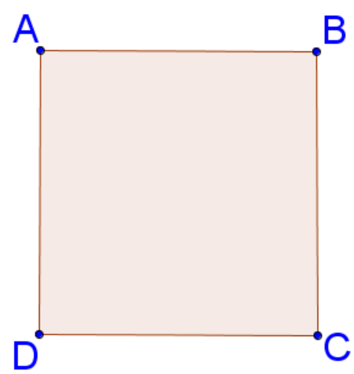
To see the Review answers, open this PDF file and look for section 5.2.

1.5 Geometry Software for Constructions

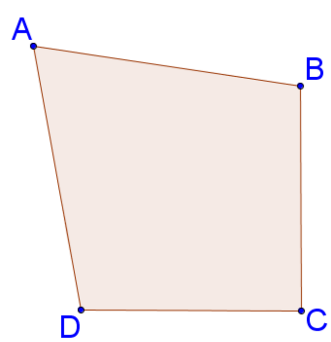
Learning Objectives

Here you will practice constructing shapes using geometry software.

Michael is working in an interactive geometry software tool, and uses the polygon tool to plot four points to make a square:



However, when Michael moves point A to try to resize his square, the shape is no longer a square!



How can Michael use an interactive graphing software tool to make a real square that will stay a square even if he moves one of the points?

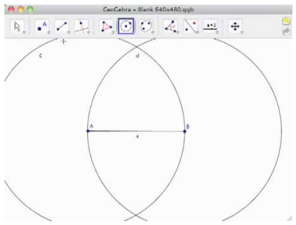
Geometry Software

When using interactive geometry software, you can make shapes that **look** like a particular shape, but don't **stay looking** like that shape if any of the points are moved. This is the problem that Michael faced in the Concept.

In order to create a true specific shape, the shape needs to be *constructed* to have the properties of that particular shape. When points are moved, these properties will be maintained and so the shape will remain the desired shape.

For example, to construct a parallelogram you cannot just make a polygon that looks like a parallelogram. You need to create parallel lines and form a parallelogram from the intersections of those lines.

In the examples and practice, you will explore how to create different shapes using interactive geometry software. Keep in mind that there are often many methods for creating a particular shape. As you work through the problems, see if you can come up with other methods that would create each shape.



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URL: http://www.ck12.org/flx/render/embeddedobject/73886

Let's do a few problems using interactive geometry software.

In the interactive below, follow the steps.

MEDIA

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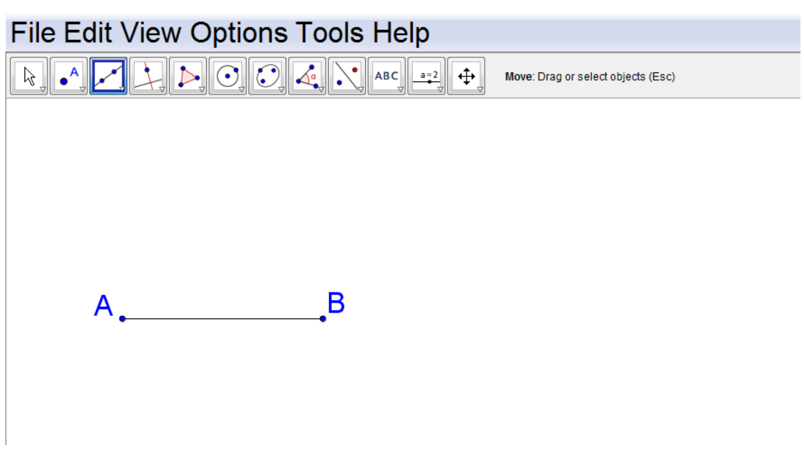
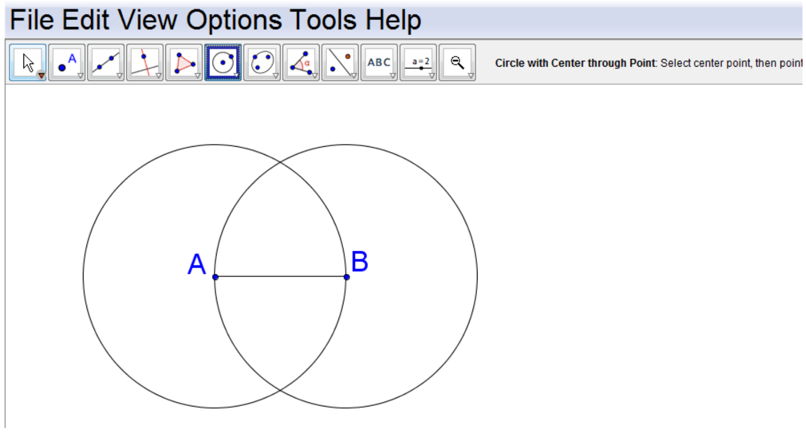
URL: http://www.ck12.org/flx/render/embeddedobject/224315

1. Use interactive geometry software to construct an equilateral triangle that stays an equilateral triangle even if you move its points.

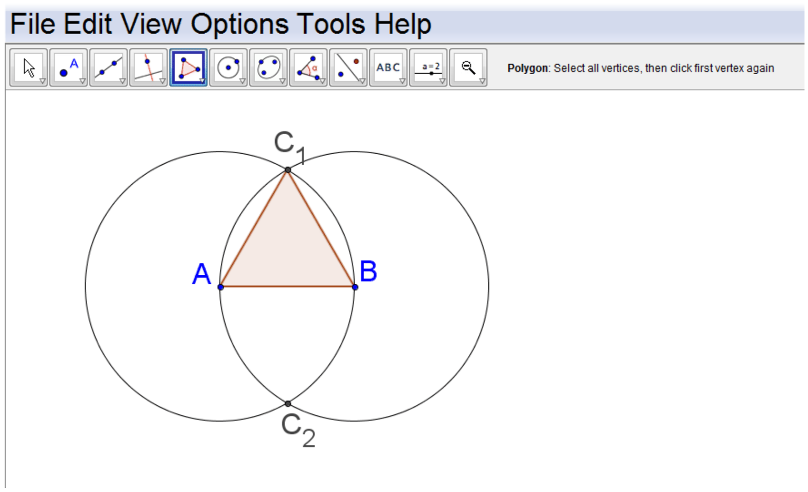
Start by making a segment between two points.



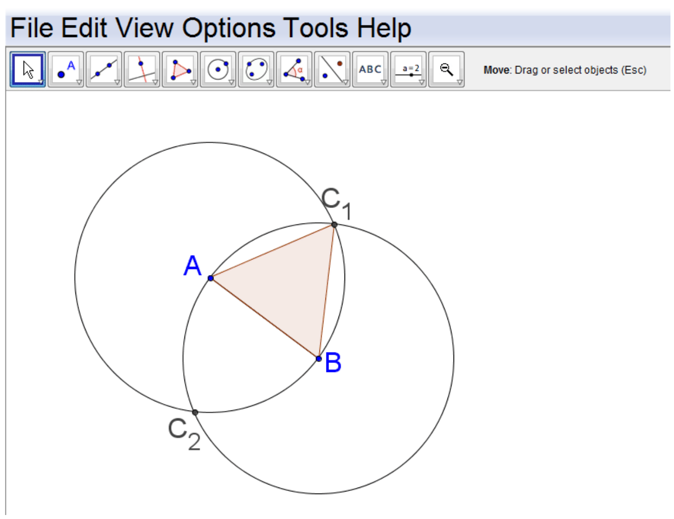
Now you must find the correct location for point C. Make two circles, one centered at point A and one centered at point B, with a radius the length of $\overline{AB}$.



The circles intersect at two points. Either of them will work for point C. Connect to form an equilateral triangle.



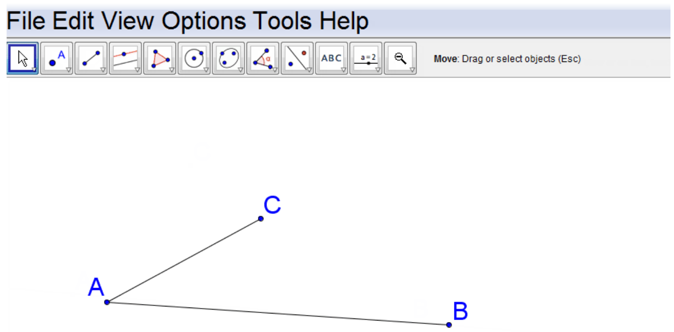
Verify that your construction stays an equilateral triangle even if you move any of the points in the image.



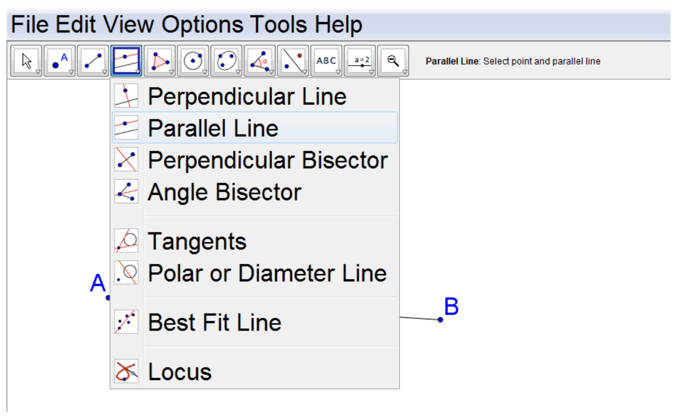
Note that at this point you could select the circles and hide them (right click and de-select "show object") in order to just see the equilateral triangle.

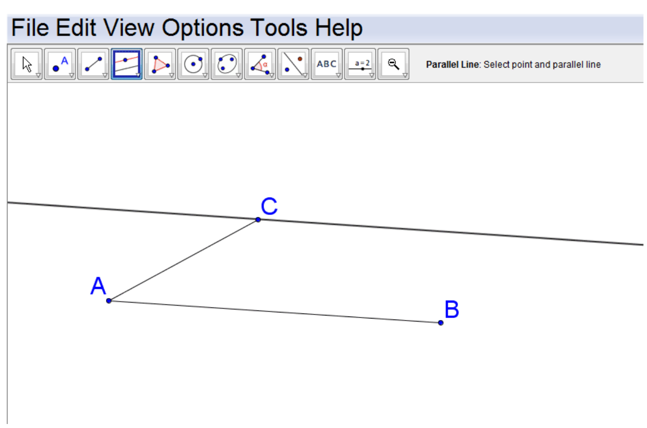
2. Use interactive geometry software to construct a parallelogram that stays a parallelogram even if you move its points.

Start by creating two line segments for two adjacent sides of the parallelogram.

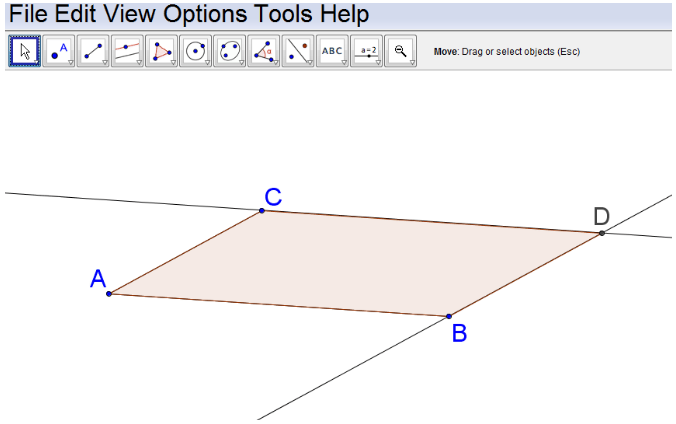


Construct a line parallel to each line segment that passes through the third point by selecting the correct button, then the line segment, and then the third point.

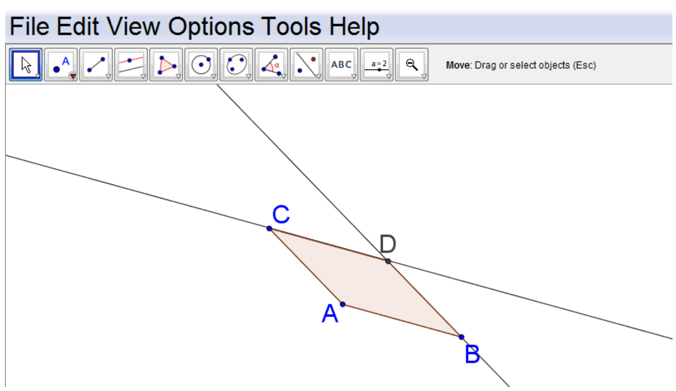




The intersection of the parallel lines is the fourth vertex of the parallelogram. Create the parallelogram from points A, B, C and D.

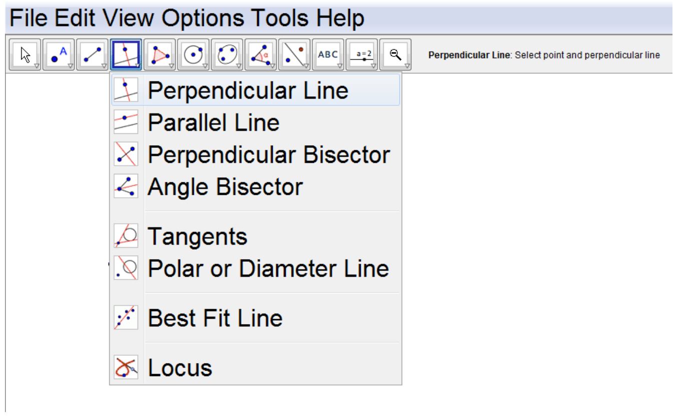


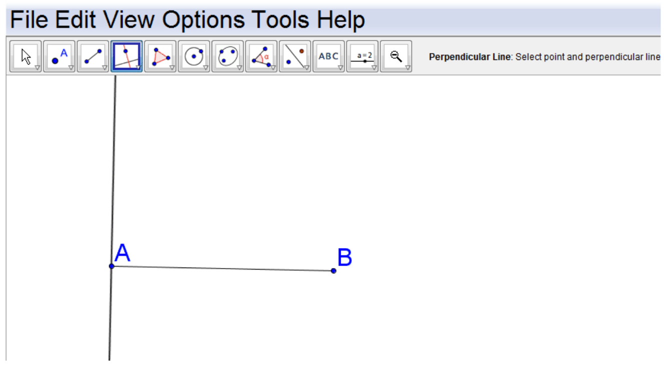
Verify that your construction stays a parallelogram even if you move any of the points in the image.



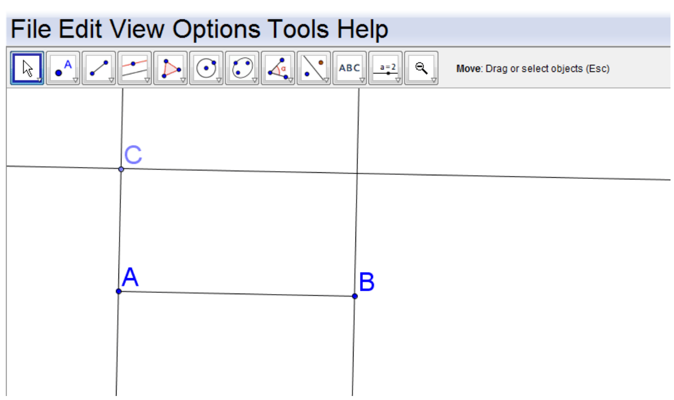
3. How could you modify the steps for constructing a parallelogram as shown in the second example above to construct a rectangle?

Start by constructing a line segment. Then, instead of randomly creating a second line segment, construct a line perpendicular to your first line segment that passes through one of the endpoints.

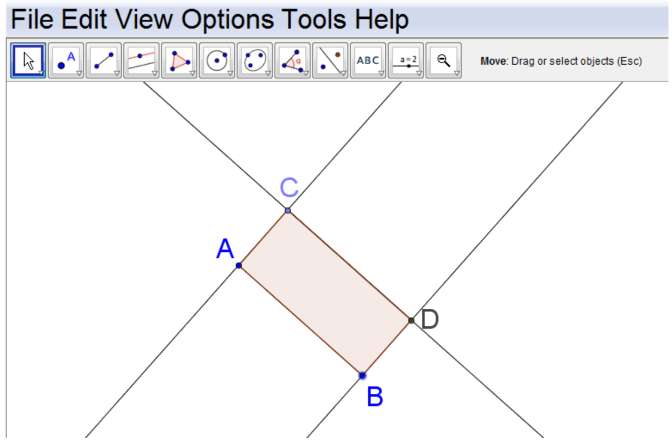




Create point C somewhere on the perpendicular line. Then, construct parallel lines through the two sides of the rectangle as was done in the second example above.



The intersection of the two parallel lines is point D. Create the polygon and verify that it stays a rectangle even if you move any of its points.



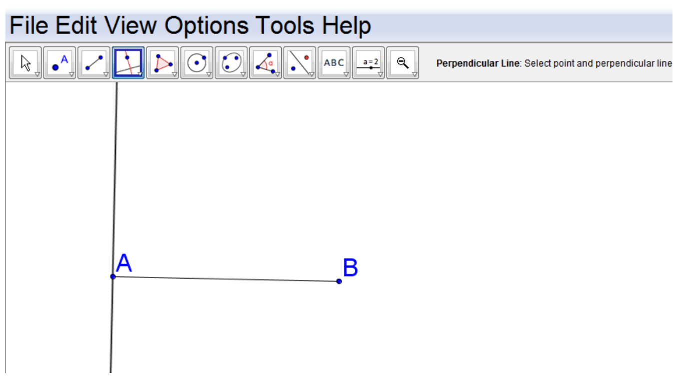
Examples

Example 1

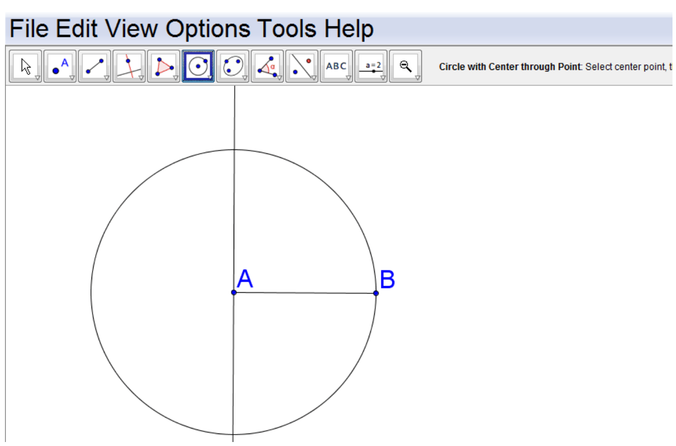
Earlier, you were asked how can Michael use interactive geometry software to make a real square that will stay a square even if he moves one of the points.

To construct a square, you can use a method that is similar to the method for constructing a rectangle in the previous problems. The only difference is you have to make sure that point C is constructed on the perpendicular line so that $AC = AB$. You can verify that these distances are the same by using a circle.

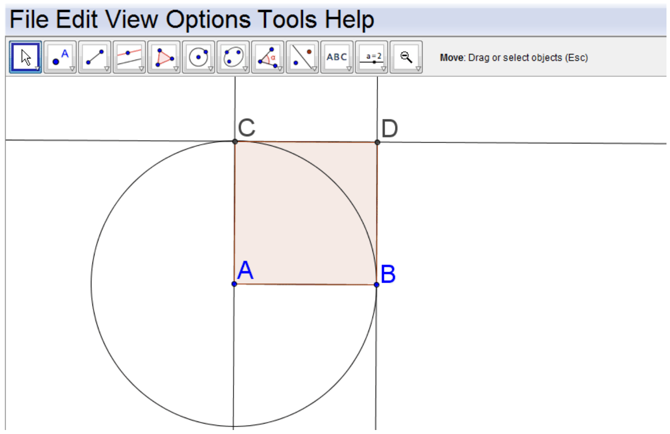
Start by constructing a line segment $\overline{AB}$ and a perpendicular line through point A.



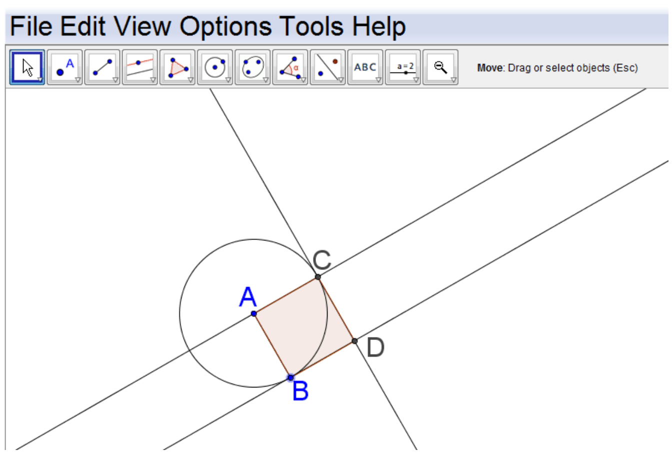
Next construct a circle centered at point A that passes through point B.



The perpendicular line intersects the circle in two points. Either of these points can be point C. Continue by constructing parallel lines and forming the quadrilateral.



Verify that it stays a square even if you move any of the points.

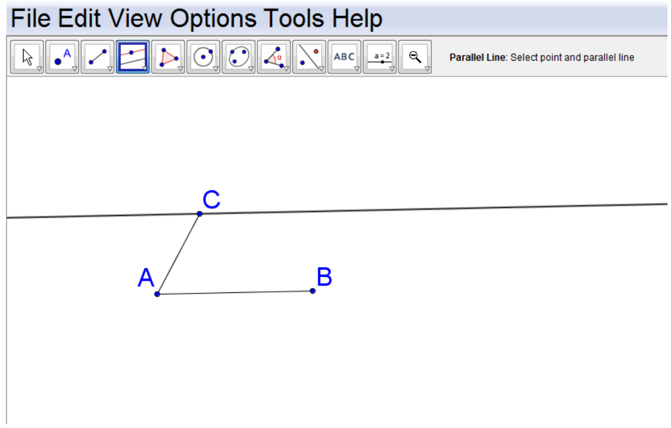


Remember that you can hide the circle and extra lines if desired in order to just see the square. Also remember that this is only one of many methods for constructing a true square with interactive geometry software.

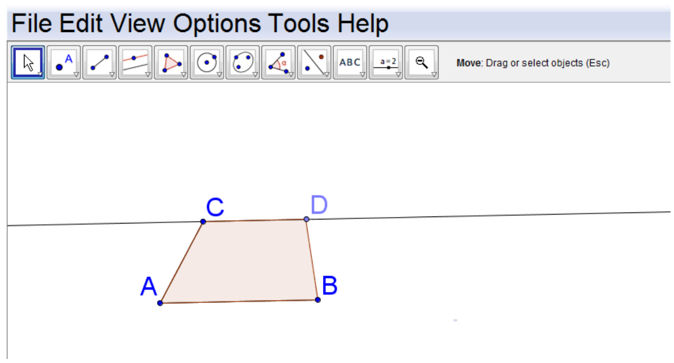
Example 2

Use interactive geometry software to construct a trapezoid that stays a trapezoid even if you move its points.

Construct two line segments for two sides of the trapezoid. Then, construct a line parallel to one of those sides.



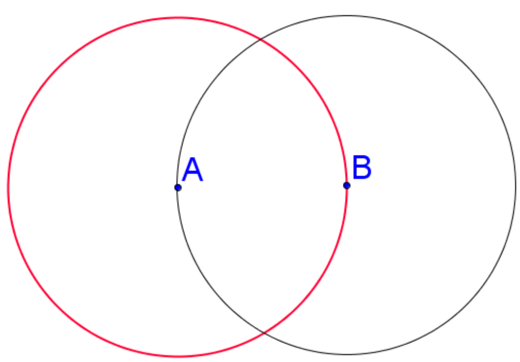
Construct point D somewhere on the parallel line. Connect the points to form the trapezoid. Move the points to verify that the quadrilateral remains a trapezoid.



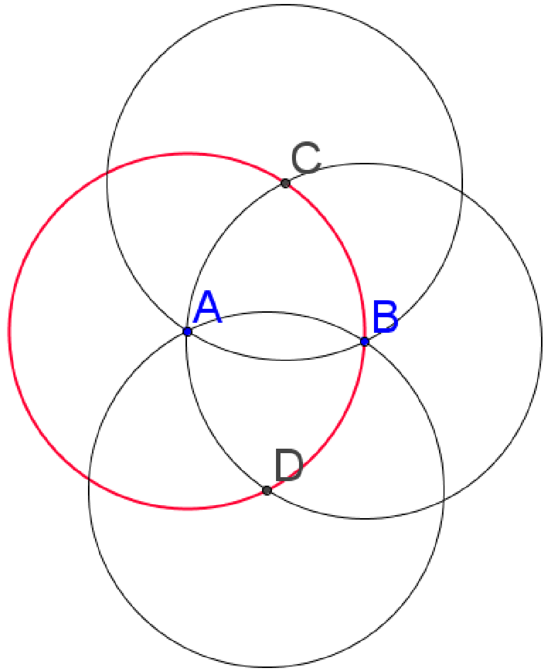
Example 3

Use interactive geometry software to construct a regular hexagon that stays a regular hexagon even if you move its points.

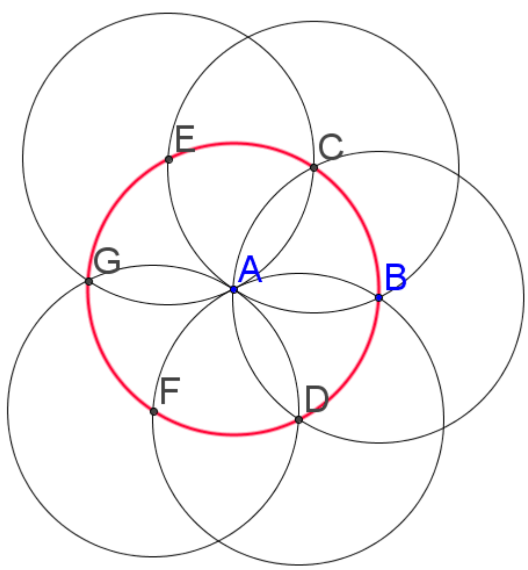
Construct a circle centered at A that passes through point B. *This will be the circle that circumscribes the hexagon*. Then, construct a circle centered at B that passes through A.



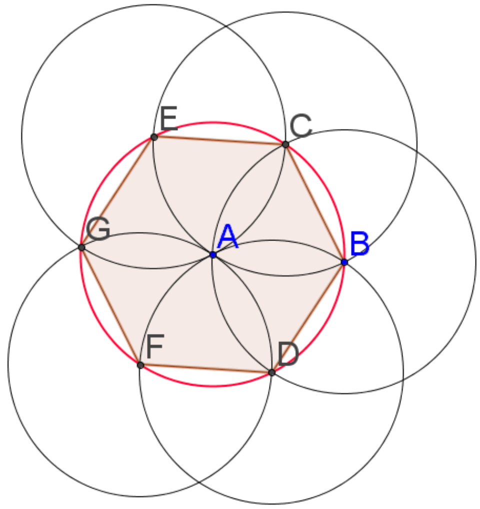
There are two points of intersection. Create these two points. Then, construct circles centered at these points that pass through A (and B).



There are two new points of intersection on the circle centered at point A. Construct these points. Then, construct circles centered at these points that pass through point A.



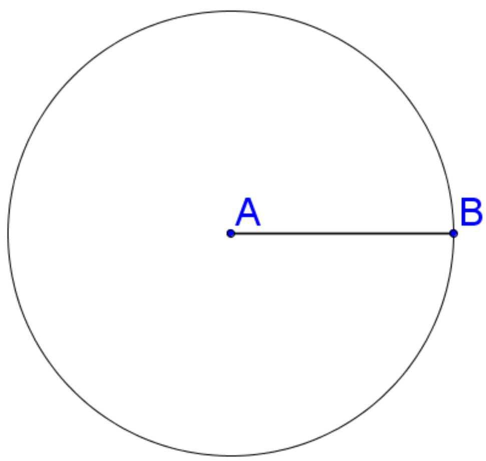
You have now found six points that are evenly spaced around the circle centered at A. These six points define the regular hexagon.



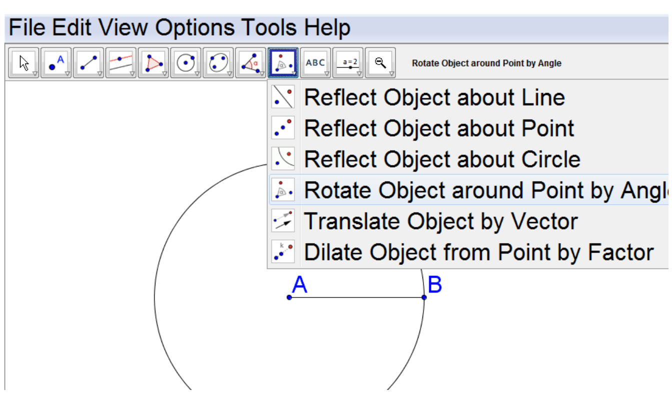
Example 4

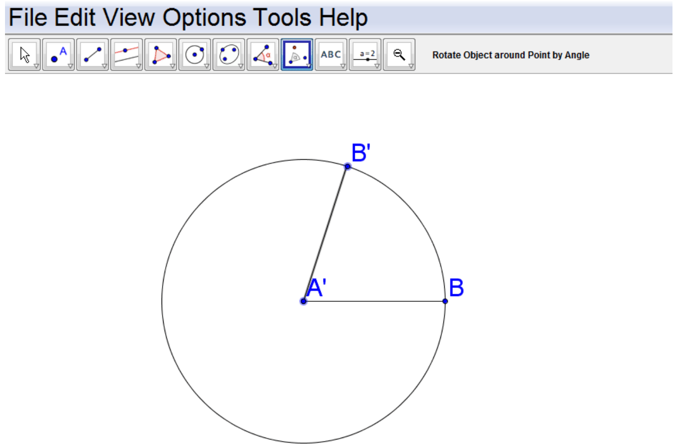
Use interactive geometry software to construct a regular pentagon by repeatedly rotating the radius of a circle 72°

Construct a circle and its radius.

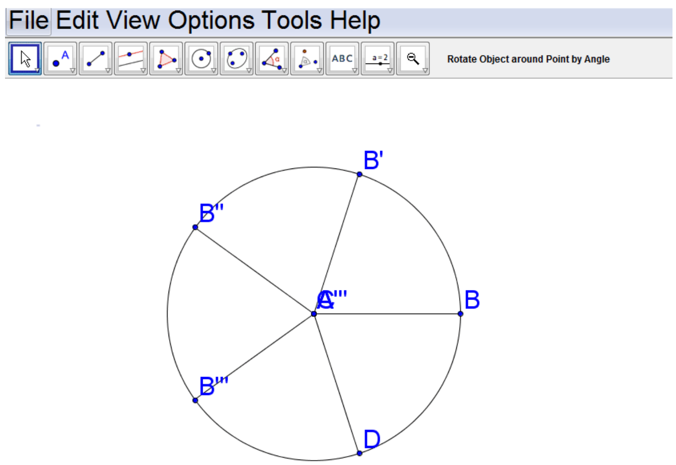


Select "Rotate Object around Point by Angle", then the radius, then point *A* (the center of the circle). Input 72° as the angle of rotation.

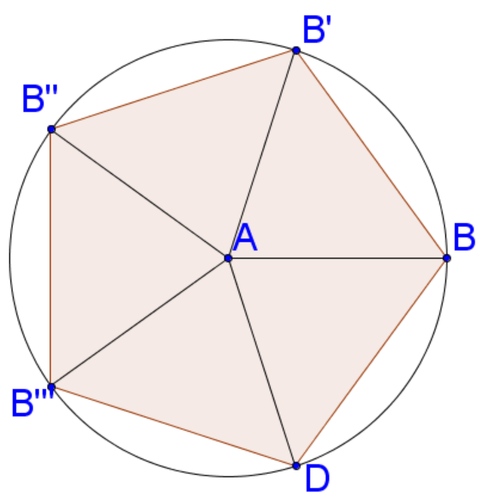




Repeat the rotation with the new radius. Continue repeating until there are five radii, evenly spaced around the circle.



Connect the five points on the circle to form a regular pentagon.



Review

1. Use the method from the first practice problem in the section to construct an equilateral triangle.

2. Use the method from the second practice problem in the section to construct a parallelogram.

3. Use the method from the third practice problem in the section to construct a rectangle.

4. Find a new way to construct a rectangle using interactive geometry software. Describe your method and justify why your method must produce a rectangle.

5. Use the method from Example 2 to construct a trapezoid.

6. Use the method from Example 3 to construct a hexagon.

7. Use a method similar to the one in Example 3 to construct a hexagon.

8. Compare and contrast the two methods for constructing hexagons. Which do you prefer and why?

9. Explain how to use a method similar to the one in Example 4 to construct a regular decagon.

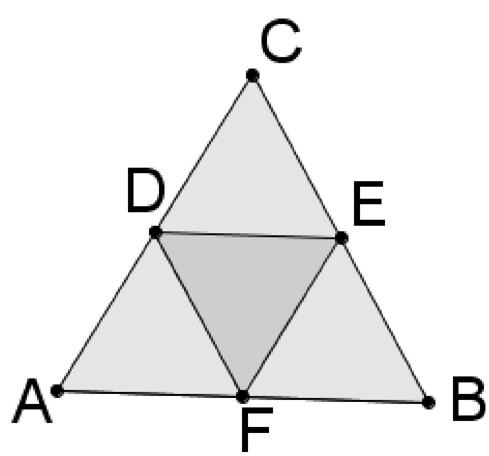
10. Use the method from the Example 1 to construct a square.

11. Find a new way to construct a square with interactive geometry software. Describe your method and justify why your method must produce a square.

12. Construct a rhombus that is not a square with interactive geometry software. Describe your method and justify why your method must produce a rhombus.

13. Find a new way to construct a rhombus that is not a square using interactive geometry software. Describe your method and justify why your method must produce a rhombus.

14. Construct the picture below using interactive geometry software. Describe your steps.



15. How does dynamic geometry software help to illustrate the difference between drawings and constructions?

16. Draw a complex shape using interactive geometry software. Describe all of your steps as you draw the shape.

17. Create instructions to help your classmate construct a triangle with three hemispheres, one hemisphere on each side of the triangle.

18. Design a logo using interactive geometry software for a new company. List your steps as you design the logo.

Review (Answers)

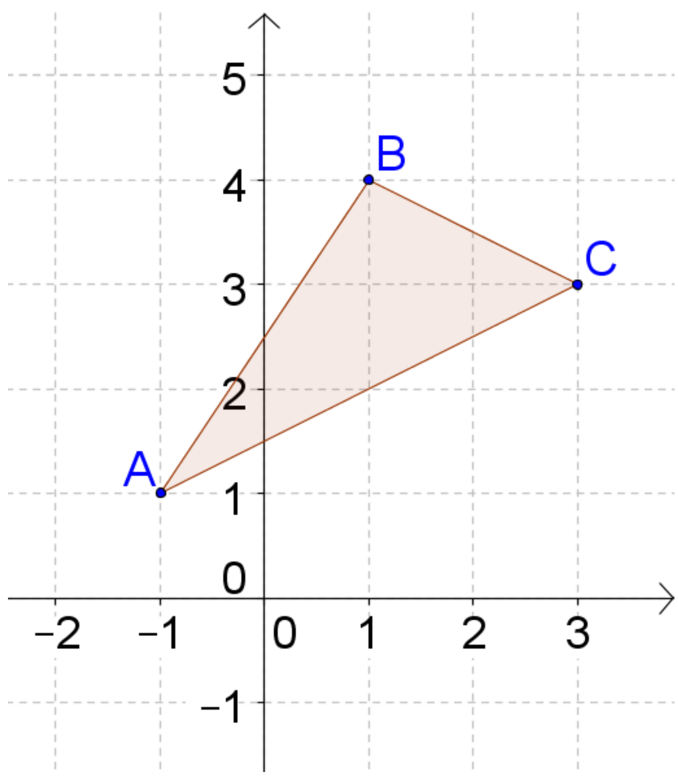
To see the Review answers, open this PDF file and look for section 5.5.

1.6 The Distance Formula

Learning Objectives

Here you will derive a formula for finding the distance between two points and use this formula to find the area and perimeter of polygons.

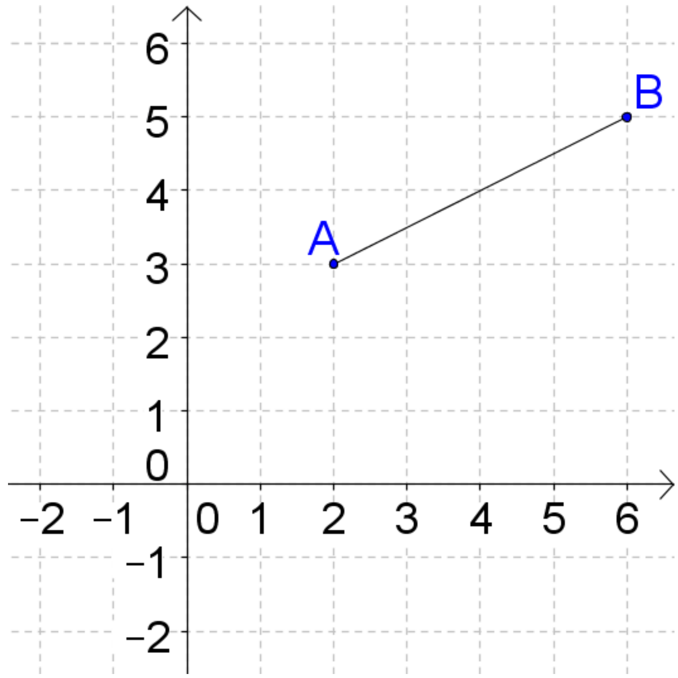
Find the area of the triangle below.



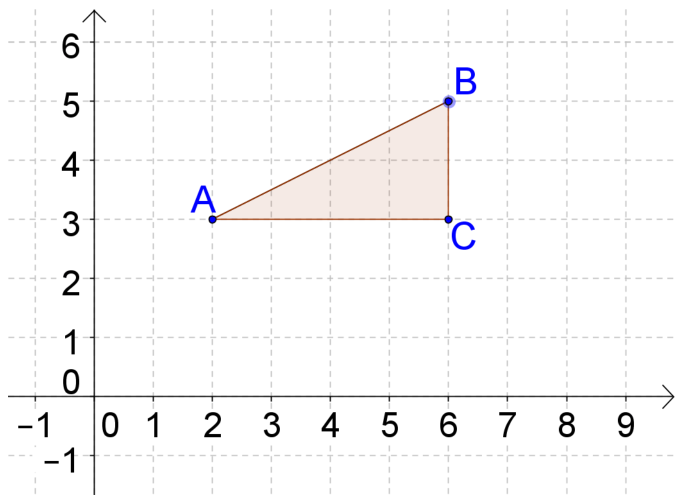
Distance Formula

When working on a coordinate plane, you can always find the distance between two points (or the length of a line segment) by creating a right triangle and using the Pythagorean Theorem.

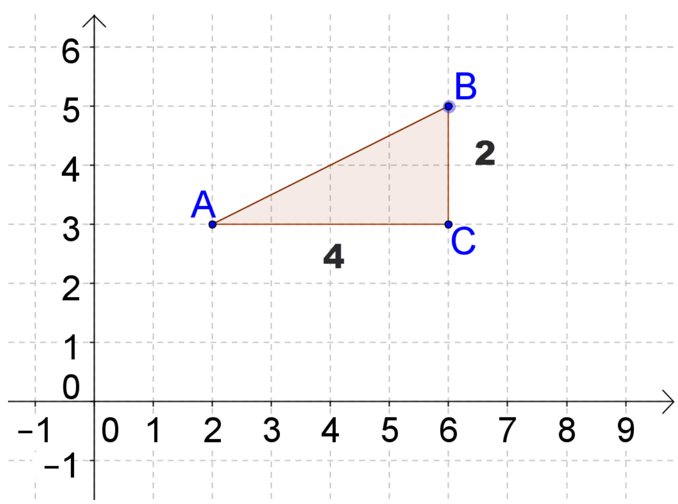
For example, suppose you want to find the length of $\overline{AB}$.



Draw a right triangle such that $\overline{AB}$ is the hypotenuse.



You can find the lengths of $\overline{BC}$ and $\overline{AC}$ by counting because they are vertical and horizontal line segments.

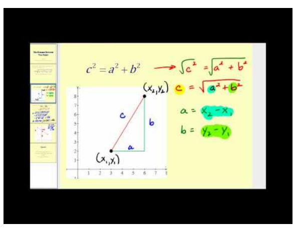


Use the Pythagorean Theorem to find the length of $\overline{AB}$.

$$\begin{aligned}(AB)^2 &= 2^2 + 4^2\\ AB &= \sqrt{20} = 2\sqrt{5}\ un\end{aligned}$$

Notice that $AB = \sqrt{BC^2 + AC^2}$. This generalizes to a formula known as the distance formula. The **distance formula** states that the distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. You will prove this generalized distance formula in Example A.

With the help of the distance formula, you can find the perimeter of polygons on a coordinate plane by finding the sum of the lengths of all the sides. You can also find the area of polygons by finding the lengths of key line segments like bases and heights.



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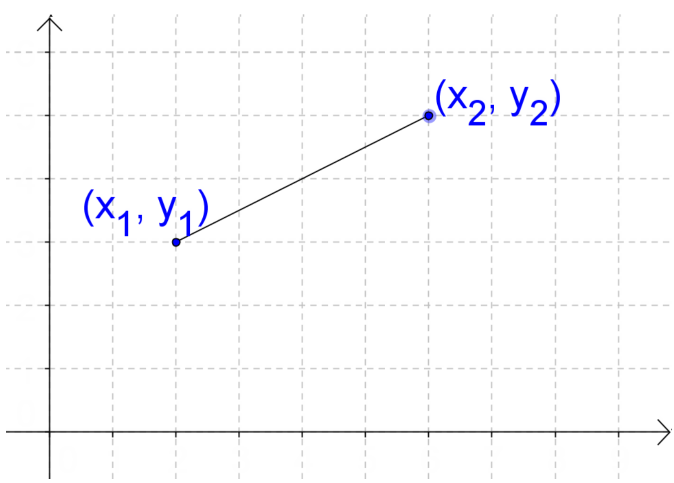
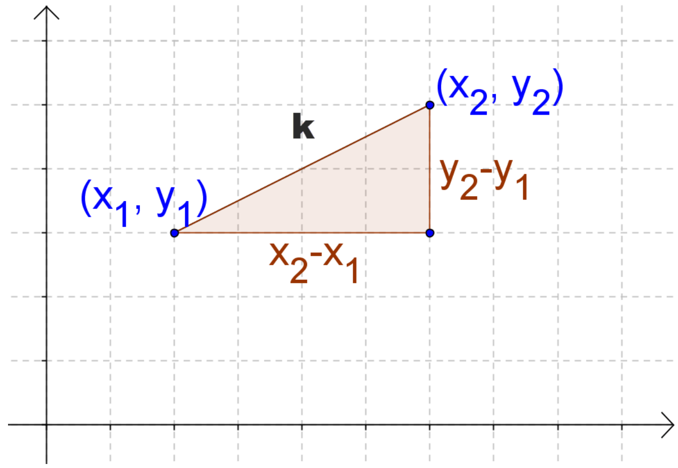
$\sqrt{(x_2 - x_1)^2 (y_2 - y_1)^2}$.

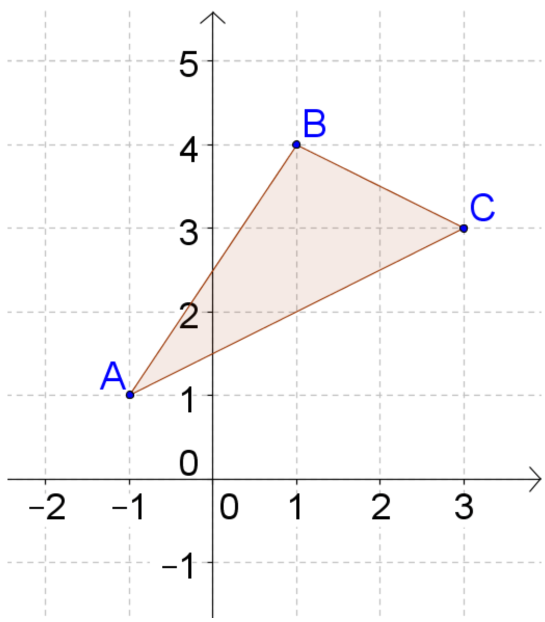


Draw in a right triangle. The lengths of the legs are $(x_2 - x_1)$ and $(y_2 - y_1)$. For now, let the length of the hypotenuse of the triangle (the distance you are trying to find) be "k".



By the Pythagorean Theorem, $(x_2-x_1)^2+(y_2-y_1)^2=k^2$. Therefore, $k=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. The distance between the two points is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

Find the perimeter of the triangle below.



Use the distance formula to find the length of each side.

$$AB=\sqrt{2^2+3^2}=\sqrt{13}\approx 3.61\ un$$
$$BC=\sqrt{1^2+2^2}=\sqrt{5}\approx 2.24\ un$$
$$AC=\sqrt{2^2+4^2}=\sqrt{20}=2\sqrt{5}\approx 4.47\ un$$

The perimeter of the triangle is the distance around the triangle, which is the sum of the lengths of the three sides. The perimeter is $\sqrt{13}+\sqrt{5}+2\sqrt{5}=\sqrt{13}+3\sqrt{5}\approx 10.31$ units.

Find the distance between $(2,7)$ **and** $(-1,5)$.

You can use the distance formula without plotting the points. Let $(x_1,y_1)=(2,7)$ and $(x_2,y_2)=(-1,5)$.

$$\text{Distance} = \sqrt{(-1-2)^2+(5-7)^2} = \sqrt{(-3)^2+(-2)^2} = \sqrt{9+4} = \sqrt{13}\ un$$

Note that it doesn't matter which point you choose for (x_1,y_1). You could have also calculated the distance the other way, with $(x_1,y_1)=(-1,5)$:

$\text{Distance} = \sqrt{(-2-1)^2+(7-5)^2} = \sqrt{(-3)^2+(2)^2} = \sqrt{9+4} = \sqrt{13}\ un$

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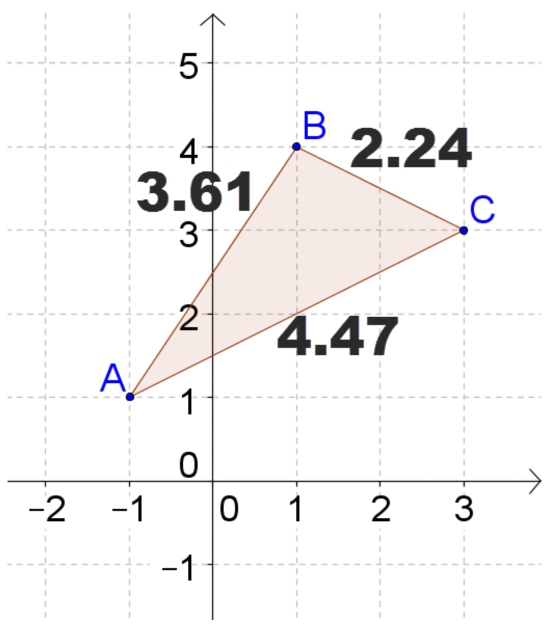
Examples

Example 1

Earlier, you were asked to find the area of the triangle.

Consider the triangle with its three side lengths labeled (lengths found in the second problem).

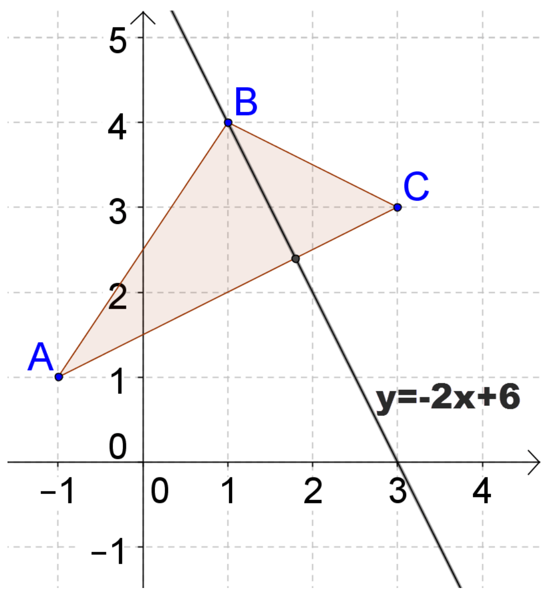


To find the area of the triangle, you can use the formula $A=\frac{bh}{2}$. Any of the three sides can be the base, but you need a height that is perpendicular to the base you choose. Here, let $\overline{AC}$ be the base, so $b=4.47\ un$

Now, you need to find a line perpendicular to $\overline{AC}$ through point B. $\overline{AC}$ has a slope of $\frac{1}{2}$. A line perpendicular to $\overline{AC}$ will have a slope of -2. **You need to find a line with a slope of -2 that passes through the point** $(1,4)$**.**

$$\begin{aligned}4 &= -2(1)+b\\6 &= b\end{aligned}$$

The equation is $y=-2x+6$.

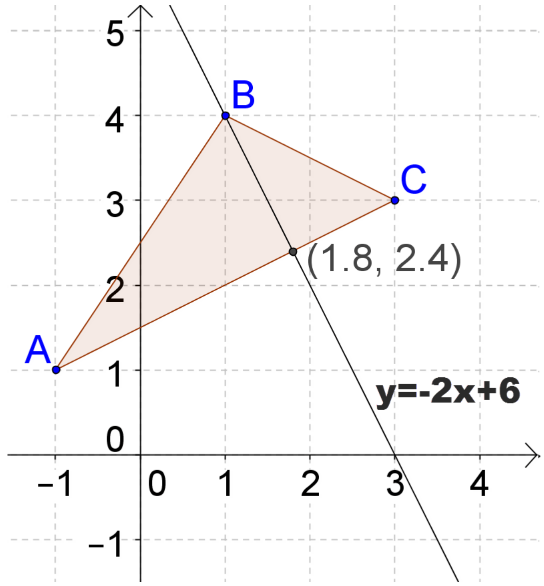


Next, you need to find where this line intersects $\overline{AC}$. The equation of the line that contains $\overline{AC}$ is $y=\frac{1}{2}x+\frac{3}{2}$. Solve the system of equations to find the point of intersection.

$y=-2x+6$ and $y=\frac{1}{2}x+\frac{3}{2}$

$$\begin{aligned}-2x+6 &= \frac{1}{2}x+\frac{3}{2}\\\frac{9}{2} &= \frac{5}{2}x\\x &= 1.8\\y &= -2(1.8)+6\\y &= 2.4\end{aligned}$$

The point of intersection is $(1.8,2.4)$.



Next, you need to find the length of the height, which is the distance from $(1.8, 2.4)$ to $(1, 4)$.

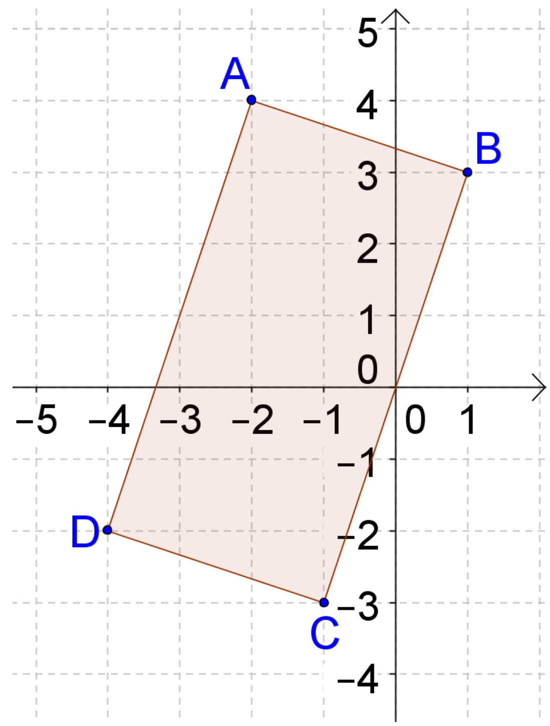
$h = \sqrt{(1.8-1)^2+(2.4-4)^2} = 1.79\ un$

Now that you know the lengths of the base and the height, you can find the area of the triangle. The area of the triangle is:

$$A = \frac{bh}{2}$$
$$A = \frac{(4.47)(1.79)}{2}$$
$$A \approx 4\ un^2$$

Example 2

Find the perimeter of the rectangle below.



Because the polygon is a rectangle, its opposite sides are the same length. Use the distance formula to find the length of two of the sides.

$$AB = \sqrt{3^2+1^2} = \sqrt{10}$$
$$BC = \sqrt{6^2+2^2} = \sqrt{40} = 2\sqrt{10}$$

The perimeter of the rectangle is:

$$P = \sqrt{10}+2\sqrt{10}+\sqrt{10}+2\sqrt{10} = 6\sqrt{10}\ un$$

Example 3

Find the area of the rectangle from #2.

To find the area of the rectangle, use the formula $A = bh$. The base and the height lengths were found in #1.

$$A = \left(\sqrt{10}\right)\left(2\sqrt{10}\right)$$
$$A = 20\ un^2$$

Example 4

Find the distance between $(-16,4)$ and $(312,211)$. How does this calculation help to show why having the distance formula is helpful?

Let $(x_1,y_1) = (-16,4)$ and $(x_2,y_2) = (312,211)$.

$$d=\sqrt{(312-(-16))^2+(211-4)^2}\approx 387.01\ un$$

Because these two points are so far apart, it would have been unrealistic to plot them and try to find the distance between them by drawing in a right triangle and using the Pythagorean Theorem. The distance formula makes these types of calculations much quicker.

CK-12 PLIX Interactive

PLIX

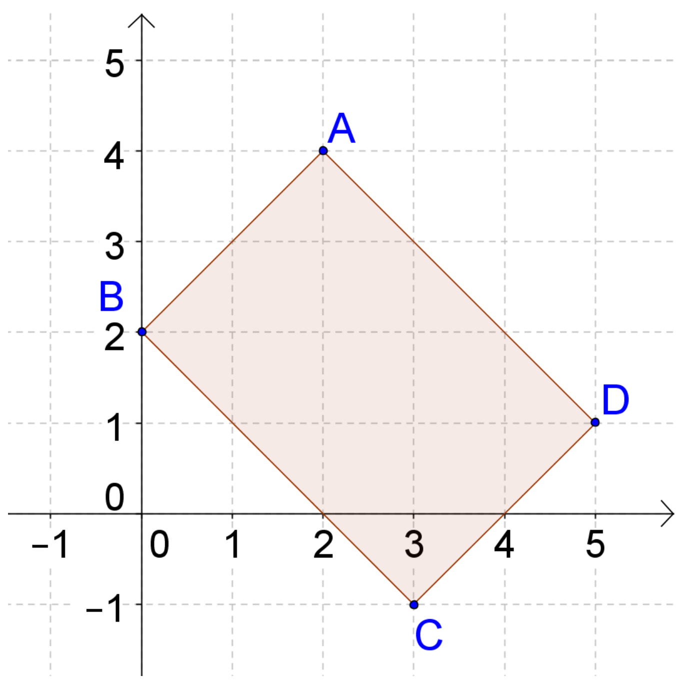
Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/line-segments/plix/Distance-Between-Two-Points-Taxicab-Distance-55832d41da2cfe54a17e9297

Find the distance between each pair of points.

1. $(7,11)$ and $(4,23)$.
2. $(19,56)$ and $(-21,45)$.
3. $(-11,91)$ and $(89,16)$.

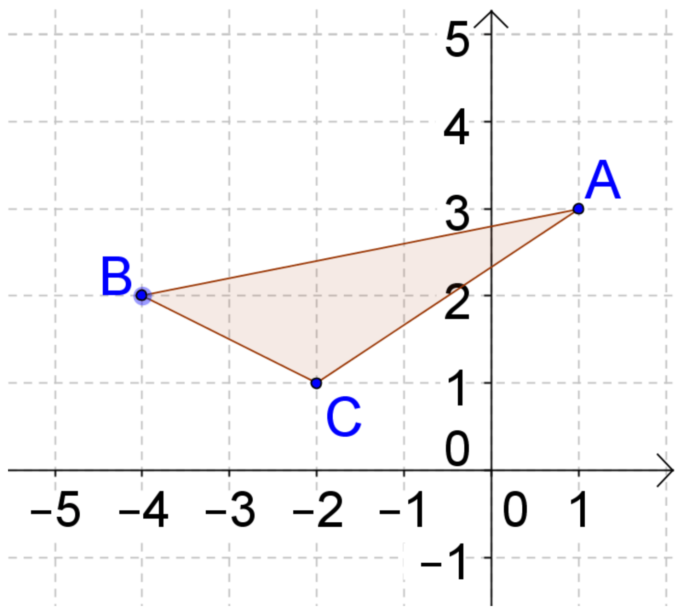
For #4-#5, use the rectangle below.



4. Find the perimeter of the rectangle.

5. Find the area of the rectangle.

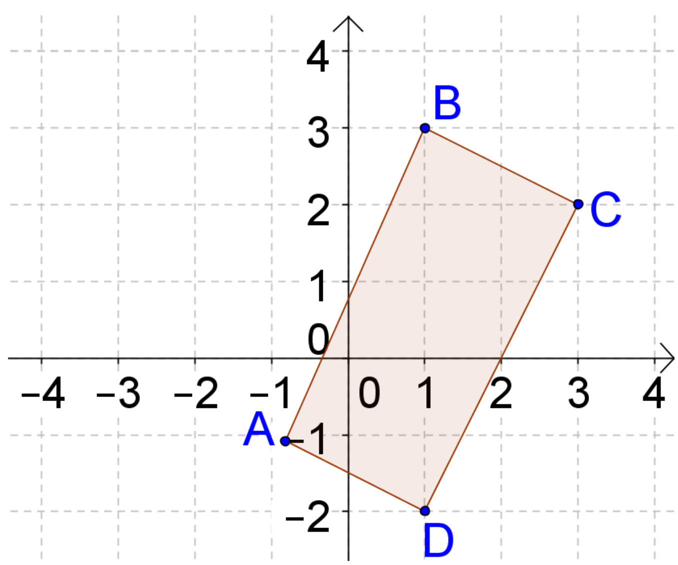
For #6-#7, use the triangle below.



6. Find the perimeter of the triangle.

7. Find the area of the triangle.

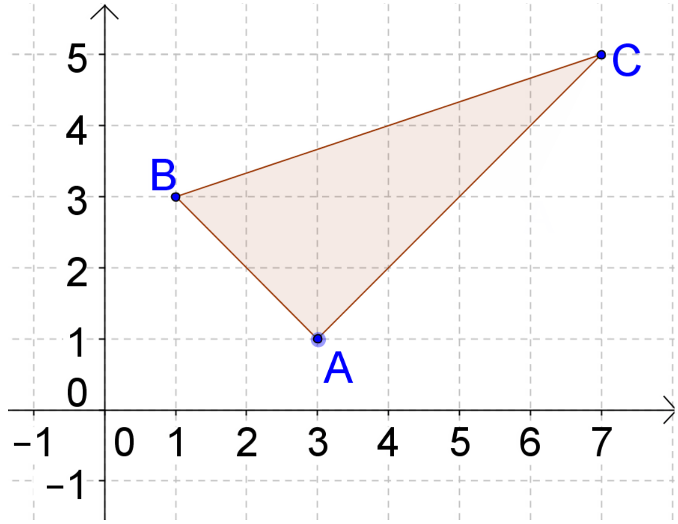
For #8-#9, use the rectangle below.



8. Find the perimeter of the rectangle.

9. Find the area of the rectangle.

For #10-#11, use the triangle below.

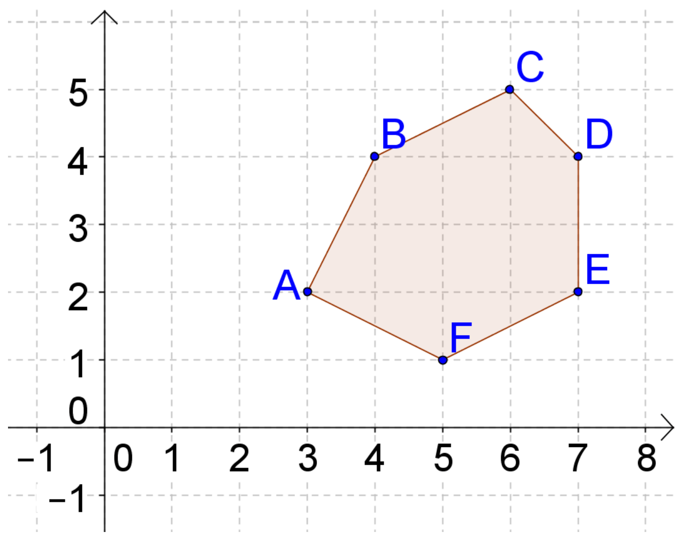


10. Find the perimeter of the triangle.

11. Find the area of the triangle.

For #12-#14, find the perimeter of each polygon.

12.



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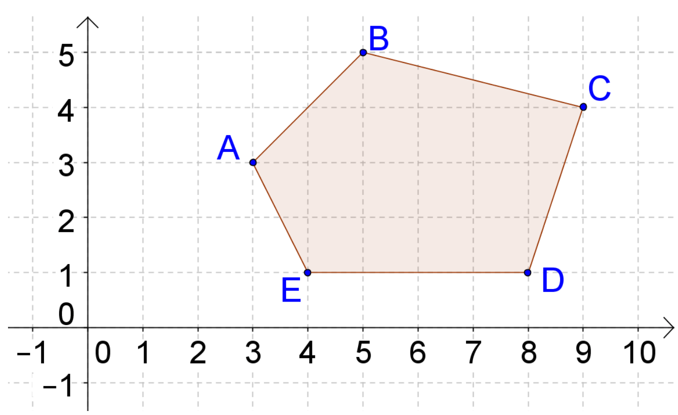

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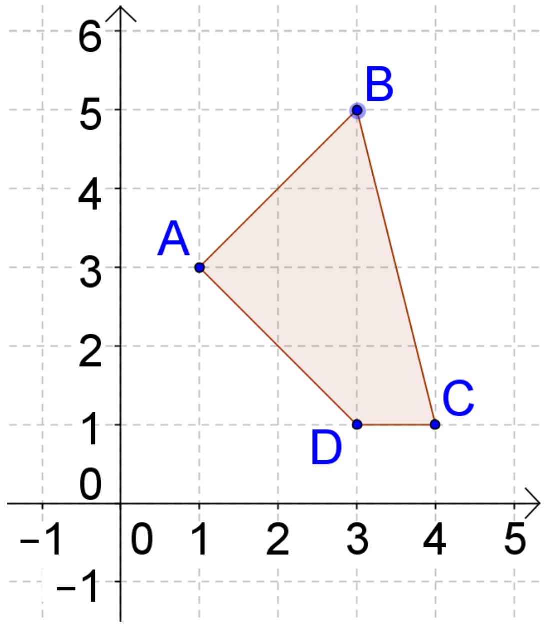


15. What does the distance formula have to do with the Pythagorean Theorem?

16. A triangle has vertices A(-3, -6), B(-1, 5), and C(2, 3). Determine if the triangle is equilateral, scalene, or isosceles.

17. Four points create a shape *ABCD*. You are given the points A(6, 8), B(14, 6), C(-1, -3), and D(-9,-1). Can you determine what kind of figure this is? Justify your answer.

Review (Answers)

To see the Review answers, open this PDF file and look for section 10.5.

Vocabulary

Here you will derive a formula for finding the distance between two points and use this formula to find the area and perimeter of polygons.

1.7 Slope of Parallel and Perpendicular Lines

Learning Objectives

Here you will prove that parallel lines have slopes that are equal and perpendicular lines have slopes that are opposite reciprocals. You will also practice solving problems involving parallel and perpendicular lines.

Find the equation of the line parallel to $y = 2x - 4$ that passes through the point $(2, -3)$. Then, find the equation of the line perpendicular to $y = 2x - 4$ that passes through the point $(2, -3)$. How are the two lines that you found related?

Slope of Parallel and Perpendicular Lines

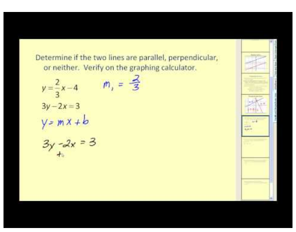
Consider two lines. There are three ways that the two lines can interact:

1. They are parallel and so they never intersect.
2. They are perpendicular and so they intersect at a right angle.
3. They intersect, but they are not perpendicular.

Recall that the **slope** of a line is a measure of its **steepness**. For a line written in the form $y = mx + b$, "m" is the slope. Given two lines, their slopes can help you to determine whether the lines are parallel, perpendicular, or neither.

In the past you learned that **two lines are parallel if and only if they have the same slope**.

In the past you also learned that **two lines are perpendicular if and only if they have slopes that are opposite reciprocals**. This means that if the slope of one line is m, the slope of a line perpendicular to it will be $-\frac{1}{m}$. Another way of thinking about this is that the product of the slopes of perpendicular lines will always be -1. (Note that $(m)(-\frac{1}{m}) = -\frac{m}{m} = -1$).



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MEDIA

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Determining the Distinction Between Two Lines

Consider two lines $y = ax + b$ and $y = ax + c$ with $b \neq c$. Note that these two lines have the same slope, a .

How do you know that the two lines are distinct (not the same line?)

The two lines are distinct because they have different y-intercepts. The first line has a y-intercept at $(0, b)$and the second line has a y-intercept at $(0, c)$.

Use algebra to find the point of intersection of the lines. What happens?

You can use substitution to attempt to find the point of intersection.

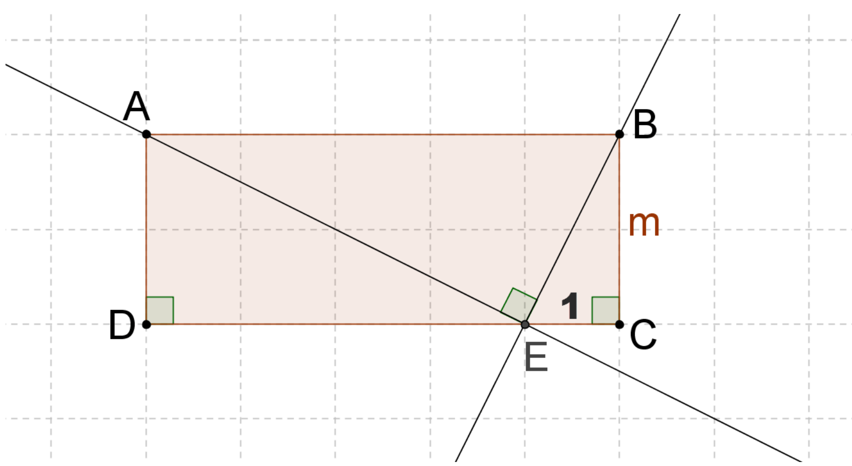
$y = ax + b$ and $y = ax + c$

Therefore:

$$\begin{aligned} ax + b &= ax + c \\ b &= c \end{aligned}$$

This is a contradiction because it was stated that $b \neq c$. Therefore, these two lines do not have a point of intersection. This means the lines must be parallel. *This proves that if two lines have the same slope, then they are parallel.*

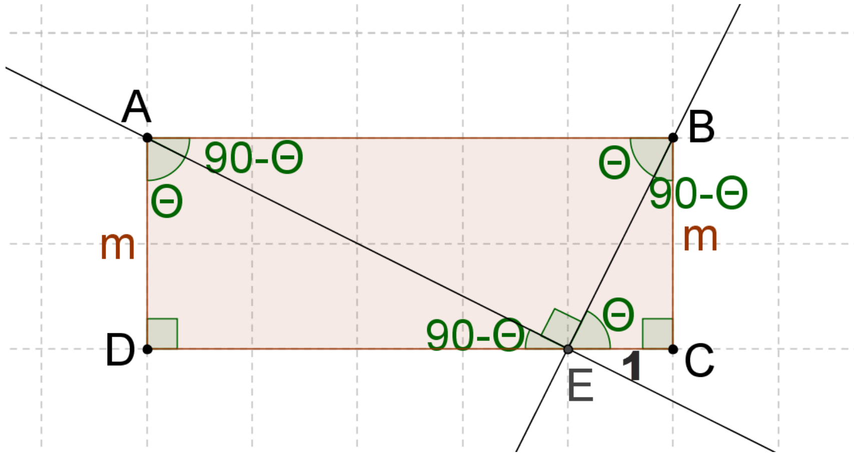
Consider rectangle $ABCD$ with $BC = m$, $EC = 1$ and perpendicular lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{BE}$.



Determining Similarity

Find the length of AD. Then, show that $\triangle ADE$ is similar to $\triangle ECB$.

Because it is a rectangle, $AD = BC = m$. The two triangles are similar because they have congruent angles. Let $\angle BEC = \theta$ and label all angles in the picture in terms of θ.



You can see that each of the three triangles in the picture have the same angle measures, so they must all be similar. In particular, $\triangle ADE$ is similar to $\triangle ECB$.

Now, Use the fact that $\triangle ADE$ is similar to $\triangle ECB$ to find the length of DE. Then, find the slopes of lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{BE}$ and show that their product is -1.

Because $\triangle ADE$ is similar to $\triangle ECB$, the following proportion is true:

$$\frac{m}{1} = \frac{DE}{m}$$

Solving this proportion you have that $DE = m^2$.

The slopes of the lines can be found using $\frac{rise}{run}$. The slope of line $\overleftrightarrow{AE}$ is $-\frac{m}{m^2} = -\frac{1}{m}$ and the slope of $\overleftrightarrow{BE}$ is $\frac{m}{1} = m$.

The product of the slopes is $\left(-\frac{1}{m}\right)(m) = -\frac{m}{m} = -1$.

This proves that if two lines are perpendicular, then their slopes will be opposite reciprocals (the product of the slopes will be -1).

Examples

Example 1

Earlier, you were asked how two lines that you found are related.

To find the equation of the line parallel to $y = 2x - 4$ that passes through the point $(2, -3)$, remember that parallel lines must have equal slopes. This means that the new line must have a slope of 2 and pass through the point $(2, -3)$. All you need to do is solve for the y-intercept.

$$\begin{aligned} -3 &= 2(2) + b \\ -3 &= 4 + b \\ b &= -7 \end{aligned}$$

The equation of the line is $y = 2x - 7$.

To find the equation of the line perpendicular to $y=2x-4$ that passes through the point $(2,-3)$, remember that perpendicular lines will have opposite reciprocal slopes. This means that the new line must have a slope of $-\frac{1}{2}$ and pass through the point $(2,-3)$. Again, all you need to do is solve for the y-intercept.

$$\begin{aligned}-3 &= -\frac{1}{2}(2)+b\\ -3 &= -1+b\\ -4 &= b\end{aligned}$$

The equation of the line is $y=-\frac{1}{2}x-4$

The two lines that were found $\left(y=2x-7 \text{ and } -\frac{1}{2}x-4\right)$ are also perpendicular. Note that they have opposite reciprocal slopes.

Example 2

Consider two parallel lines $y=ax+b$ and $y=cx+d$ with $b\neq d$. Show that $a=c$.

Suppose $a\neq c$. You can solve a system of equations to find the point of intersection of the two lines.

$y=ax+b$ and $y=cx+d$

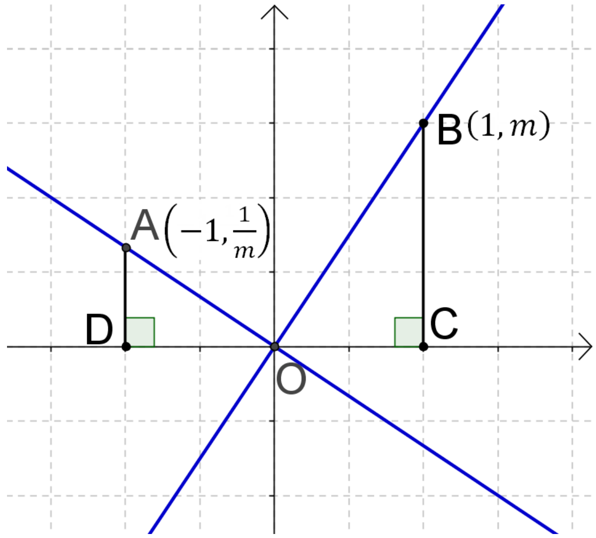
Therefore:

$$\begin{aligned}ax+b &= cx+d\\ x(a-c) &= d-b\\ x &= \frac{d-b}{a-c}\end{aligned}$$

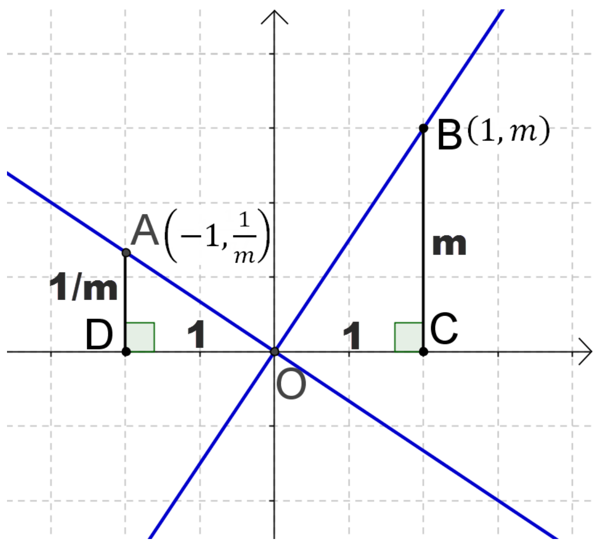
If $a\neq c$, then this point exists so the lines intersect. This is a contradiction because it was stated that the lines were parallel. Therefore, a must be equal to c. *This proves that if two lines are parallel then they must have the same slope.*

Example 3

Consider two lines intersecting at the origin as shown below. Find the lengths of the legs of each triangle. Then, show that $\triangle BCO$ is similar to $\triangle ODA$.



The lengths of the legs of the triangles are shown below.



$\triangle BCO \sim \triangle ODA$ with a ratio of $m:1$ by $SAS \sim$. $\frac{BC}{OD} = \frac{m}{1}$ and $\frac{CO}{DA} = \frac{1}{\frac{1}{m}} = \frac{m}{1}$. Also $\angle C \cong \angle D$.

Example 4

Using the picture from #3, find the slopes of lines $\overleftrightarrow{AO}$ and $\overleftrightarrow{BO}$ and verify that their product is -1. Then use the fact that $\triangle BCO$ is similar to $\triangle ODA$ to show that $\overleftrightarrow{AO}$ and $\overleftrightarrow{BO}$ must be perpendicular.

The slope of line $\overleftrightarrow{AO}$ is $-\frac{1}{m}$ and the slope of $\overleftrightarrow{BO}$ is $\frac{m}{1}$. The product of the slopes is $\left(-\frac{1}{m}\right)(m) = -\frac{m}{m} = -1$.

Because $\triangle BCO$ is similar to $\triangle ODA$, their corresponding angles must be congruent. This means that:

- $m\angle OCB = m\angle DOA$
- $m\angle BOC = m\angle DAO$

Also, because they are right triangles:

- $m\angle OCB + m\angle BOC = 90^\circ$

- $m\angle DOA + m\angle DAO = 90^\circ$

By substitution, $m\angle DOA + m\angle BOC = 90^\circ$. Because $\angle DOA, \angle BOC$ and $\angle AOB$ form a straight line, the sum of their measures must be 180°. Therefore, $m\angle AOB$ must be 90°.

Because $m\angle AOB = 90^\circ$, $\overleftrightarrow{AO}$ and $\overleftrightarrow{BO}$ must be perpendicular. *This proves that if two lines have opposite reciprocal slopes, then they are perpendicular.*

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PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/algebra/comparing-equations-of-parallel-and-perpendicular-lines/plix/Parallel-and-Perpendicular-Lines-56ce2ffa8e0e080bd41b6119

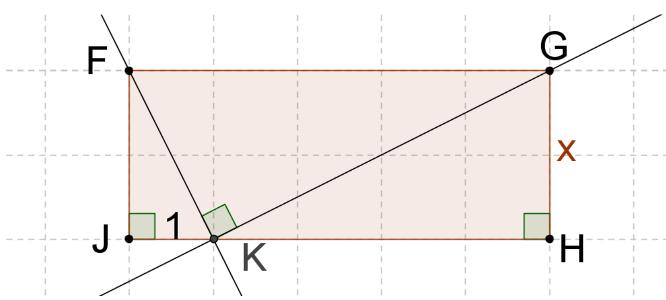
1. Describe the three ways that two lines could interact. Draw a picture of each.

2. What does it mean for two lines to be parallel? How are the slopes of parallel lines related?

3. What does it mean for two lines to be perpendicular? How are the slopes of perpendicular lines related?

4. Use algebra to show why the lines $y = 3x - 4$ and $y = 3x + 7$ (lines with the same slope) must be parallel.

5. Use the method from Example B and Example C to show why the slopes of lines $\overleftrightarrow{FK}$ and $\overleftrightarrow{KG}$ must be opposite reciprocals. *Assume that $FGHJ$ is a rectangle.*



6. Find the line parallel to $y = 3x - 5$ that passes through $(2, 11)$.

7. Find the line perpendicular to $y = 3x - 5$ that passes through $(6, 11)$.

8. Find the line parallel to $3x + 4y = 7$ that passes through $(4, 2)$.

9. Find the line perpendicular to $3x + 4y = 7$ that passes through $(3, 10)$.

10. Find the line parallel to $y = 5$ that passes through $(2, 16)$.

11. Find the line perpendicular to $y = 5$ that passes through $(2, 16)$

12. Find the line parallel to $y = -\frac{1}{3}x - 4$ that passes through $(6,8)$.

13. Find the line perpendicular to $y = -\frac{1}{3}x - 4$ that passes through $(6,8)$.

14. Line a passes through the point $(2,4)$ and $(3,6)$. Line b passes through the points $(6,7)$ and $(11,17)$. Are lines a and b parallel, perpendicular, or neither?

15. Line a passes through the point $(1,-1)$ and $(6,14)$. Line b passes through the points $(9,3)$ and $(-6,8)$. Are lines a and b parallel, perpendicular, or neither?

16. A new amusement park is going to be built near two major highways. On a coordinate grid of the area, with scale 1 unit representing 1 km, the park is located at point $P(3,4)$. Highway 2 is represented by the equation $y = 2x+5$, and Highway 10 is represented by the equation $y = -0.5x+2$. Determine the coordinates of the exits that must be built on each highway to result in the shortest road to the park.

17. Determine if the points A(1, -2), B(4, 4), and C(5, 6) are collinear.

18. Find the value of b given that A(-6, 2), B(b, 0), and C(3, -4) are collinear.

19. Determine the value of k for which the lines $kx - 2y - 1 = 0$ and $8x - ky + 3 = 0$ are parallel. How does the value change if the lines are perpendicular? Can there be more than one value for k?

20. Construct the equations of lines that form a square or a rectangle such that no sides are vertical or horizontal.

Review (Answers)

To see the Review answers, open this PDF file and look for section 10.4.

1.8 References

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CHAPTER 2 Reasoning and Proof

Chapter Outline

2.1 Theorems and Proofs

Learning Objectives

Here you will learn what it means to write a proof and different methods for writing proofs.

Most of the geometry concepts and theorems that are learned in high school today were first discovered and proved by mathematicians such as Euclid thousands of years ago. Given that these geometry concepts and theorems have been known to be true for thousands of years, why is it important that you learn how to prove them for yourself?

Theorems and Proofs

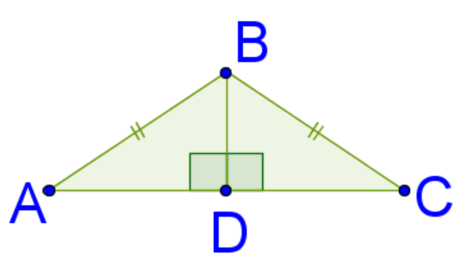
In geometry, a **postulate** is a statement that is *assumed* to be true based on basic geometric principles. An example of a postulate is the statement *"through any two points is exactly one line"*. A long time ago, postulates were the ideas that were thought to be so obviously true they did not require a proof. A **theorem** is a mathematical statement that can and must be proven to be true. You've heard the word theorem before when you learned about the **Pythagorean Theorem**. Much of your future work in geometry will involve learning different theorems and proving they are true.

Theorems and postulates may be more easily understood and applied via **conditional statements**. Conditional statements are a study in themselves, but a quick introduction may prove useful. Here are a few examples of theorems and postulates written as conditional statements (also known as if-then statements):

- **Postulate**: Through any two points exists a straight line.
  - **Conditional statement**: **If** any two points are defined, **then** exactly one line may be drawn through them both.
- **Postulate:** A circle may be drawn with any given radius and center.
  - **Conditional statement**: **If** you have a radius and center, **then** you may draw a circle.
- **Theorem**: Angles that form a linear pair are supplementary.
  - **Conditional statement**: **If** two angles form a linear pair, **then** they are supplementary.

What does it mean to "prove" something? In the past you have often been asked to "*justify your answer*" or "*explain your reasoning*". This is because it is important to be able to show your thinking to others so that ideally they can follow it and agree that you must be right. A **proof** is just a formal way of justifying your answer. In a **proof** your goal is to use given information and facts that everyone agrees are true to show that a new statement must also be true.

Suppose you are given the picture below and asked to prove that $\overline{AD} \cong \overline{DC}$. *This means that you need to give a convincing mathematical* ***argument*** *as to why the line segments MUST be congruent.*



Here is an example of a paragraph-style proof. This is similar to a detailed explanation you might have given in the past.

$\overline{AB} \cong \overline{BC}$ because it is marked in the diagram. *Also,* **$\angle ADB$ and $\angle CDB$ are both right angles because it is marked in the diagram.** *This means that* **$\triangle ADB$ and $\triangle CDB$ are right triangles because right triangles are triangles with right angles. Both triangles contain segment $\overline{BD}$. $\overline{BD} \cong \overline{BD}$ because of the reflexive property that any segment is congruent to itself. $\triangle ADB \cong \triangle CDB$ by $HL \cong$ because they are right triangles with a pair of congruent legs and congruent hypotenuses. $\overline{AD} \cong \overline{DC}$ because they are corresponding segments and corresponding parts of congruent triangles must be congruent.**

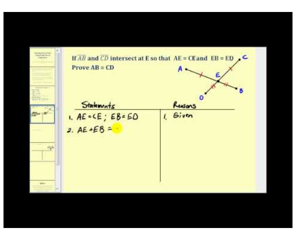
There are two key components of any proof – **statements** and **reasons**.

- The **statements** are the claims that you are making throughout your proof that lead to what you are ultimately trying to prove is true. *Statements are written in red throughout the previous proof.*
- The **reasons** are the reasons you give for why the statements must be true. *Reasons are written in blue throughout the previous proof.* If you don't give reasons, your proof is not convincing and so is not complete.

When writing a proof, your job is to make everything as clear as possible, because you need other people to be able to understand and believe your proof. Skipping steps and using complicated words is not helpful!

There are many different styles for writing proofs. In American high schools, a style of proof called the **two-column proof** has traditionally been the most common (see Example 3). In college and beyond, **paragraph proofs** are common. An example of a style of proof that is more visual is a **flow diagram proof** (see Example 4)**.** No matter what style is used, the key components of statements and reasons must be present. You should be familiar with different styles of proof, but ultimately can use whichever style you prefer.

Learning to write proofs can be difficult. One of the best ways to learn is to study examples to get a sense for what proofs look like.



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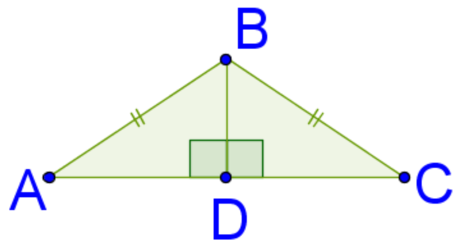
1. Rewrite the proof from the guidance in a two-column format.

Using the picture below, prove that $\overline{AD} \cong \overline{DC}$.



In a two-column proof, the statements and reasons are organized into two columns. All of the same logic that was used in the paragraph proof will be used here. Look at the proof below and compare it to the paragraph proof from the guidance.

TABLE 2.1:

Statements	**Reasons**
$\overline{AB} \cong \overline{BC}$	Given
$\angle ADB$ and $\angle CDB$ are right angles	Given
ΔADB and ΔCDB are right triangles	definition of right triangles
$\overline{BD} \cong \overline{BD}$	reflexive property
$\Delta ADB \cong \Delta CDB$	$HL \cong$
$\overline{AD} \cong \overline{DC}$	CPCTC (corresponding parts of congruent triangles must be congruent)

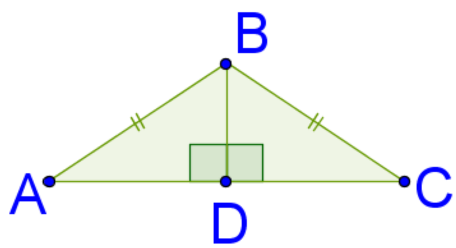
There are a couple of points to note about two-column proofs.

1. For a two-column proof, instead of saying "it is marked in the diagram" as a reason, you just write "given". You can use the reason "given" for anything that was stated up front or marked in a diagram. Typically, the first few rows of your proof will always be the "givens".
2. In a two-column proof you will use less words than in a paragraph proof, because you are not writing in complete sentences.

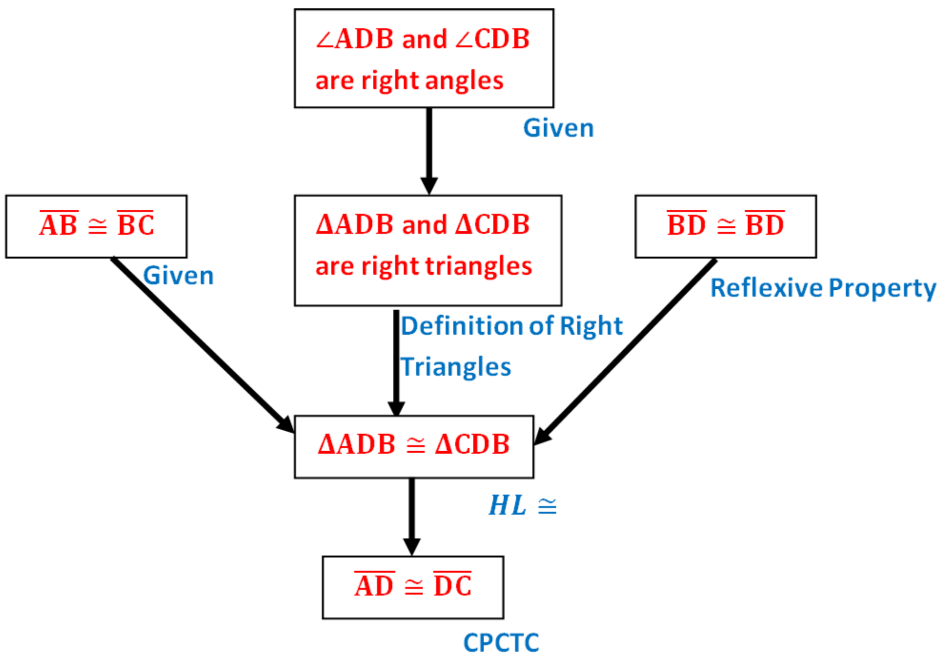
- $HL \cong$ and the other criteria for triangle congruence are always acceptable reasons if you have shown in earlier rows that each part of the criteria has been met. *You do not need to write a sentence explaining why you can use* $HL \cong$.
- Instead of stating right triangles are triangles with right angles as a reason, you can just say "definition of right triangles". Definitions are always acceptable reasons.
- CPCTC is an abbreviation for the statement "corresponding parts of congruent triangles are congruent". The abbreviation was developed because this reason is used often, and it can be cumbersome to write it over and over.

2. Rewrite the proof from the guidance in a flow diagram format.

Using the picture below, prove that $\overline{AD} \cong \overline{DC}$.



In the proof below, statements are written in red and reasons are written in blue. In a flow diagram, the statements and reasons will be organized into boxes that are connected with arrows to show the flow of logic. Look at the proof below and compare it to the two-column and paragraph versions of the same proof.



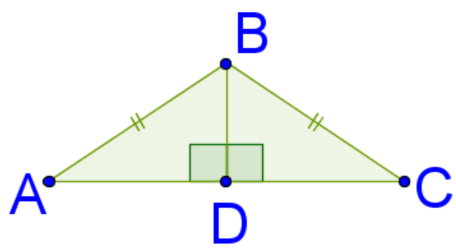
There are a couple of points to note about flow diagram proofs.

1. Statements are written inside the boxes and the reasons the statements must be true are written below the boxes.
2. The arrows show the flow of logic. If two boxes are connected by arrows it means that the statement in the lower box can be made *because* the statement in the upper box is true. Notice that three boxes point towards the statement that $\Delta ADB \cong \Delta CDB$. This is because all three of those statements were necessary for making the conclusion that the two triangles are congruent.
3. Just like in the two-column format, "given" is the reason used for anything that was stated up front or marked in the diagram. The "given" reasons will be towards the top of the flow diagram.
4. Just like in the two-column format, you use abbreviations where possible. $HL \cong$, other triangle congruence criteria, *CPCTC*, and definitions are all acceptable reasons.

Identifying Mistakes

Each proof below has a mistake, can you figure out where the mistake is and why it is a mistake?

Using the picture below, prove that $\overline{AD} \cong \overline{DC}$.



PROOF A:

TABLE 2.2:

Statements	Reasons
$\overline{AB} \cong \overline{BC}$	Given
$\Delta ADB \cong \Delta CDB$	$HL \cong$
$\overline{AD} \cong \overline{DC}$	CPCTC (corresponding parts of congruent triangles must be congruent)

PROOF B:

***$\triangle ABD$ looks to be the same size and shape as $\triangle CBD$*, so the two triangles are congruent. $\overline{AD} \cong \overline{DC}$ because they are corresponding segments and corresponding parts of congruent triangles must be congruent**.

PROOF A is incorrect because it is missing steps. You can't say that the two triangles are congruent by $HL \cong$ without having shown that all the parts of the HL criteria have been met (congruent leg pair, congruent hypotenuse pair, right triangles). Be careful when writing proofs that you don't skip over steps, even if the steps seem obvious.

PROOF B is incorrect because it did not convincingly explain why the two triangles have to be congruent. *Looking congruent* is not a good enough reason. For proving triangles are congruent, there are five triangle congruence criteria to use. If you don't have enough information to use one of those five criteria, you can't prove that the triangles are congruent.

Remember, your goal when writing a proof is to convince everyone else that what you are trying to show is true *actually is true*. If you skip steps or use reasons that aren't convincing, other people won't believe your proof.

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Examples

Example 1

Earlier, you were asked why is it important to learn how to prove geometry concepts and theorems yourself.

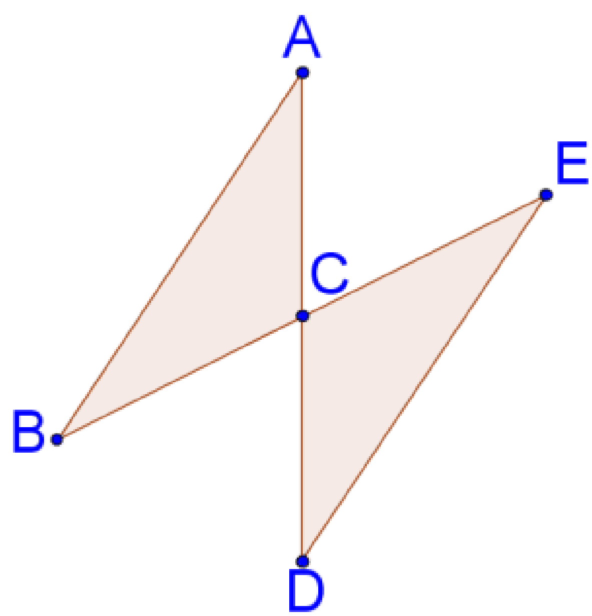
Most of the geometry concepts and theorems that are taught at the high school level today were first discovered and proved by mathematicians such as Euclid thousands of years ago. Given that these geometry concepts and theorems have been known to be true for thousands of years, why is it important that you learn how to prove them for yourself?

There are many reasons why it is valuable to learn to write proofs for yourself. Even though all of the theorems you will learn in geometry have already been proven, mathematicians today are working on trying to prove new ideas that will hopefully help to advance science/technology/medicine. Writing proofs in geometry class allows you to see what proofs are all about and practice writing them. That way, when you someday want to prove something new, you can feel confident in your proof writing abilities.

Writing proofs is all about logic. If you get good at writing proofs, this logical thinking can transfer to other subjects. Writing a persuasive essay about any topic is very similar to writing a paragraph proof. Knowing how to persuade others to believe your way of thinking can be very helpful in many careers and life in general.

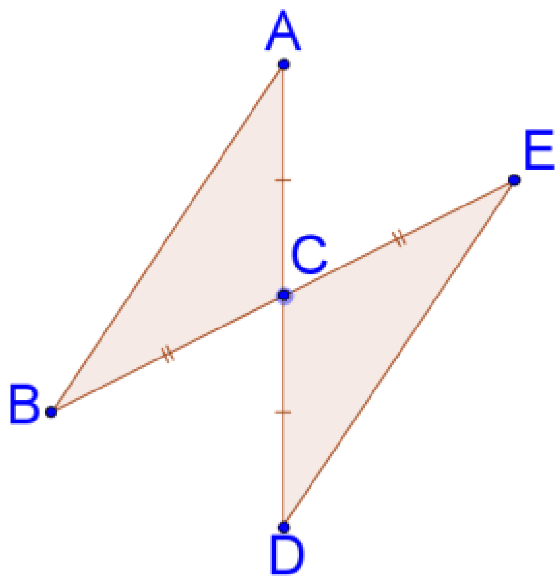
Given: C is the midpoint of $\overline{BE}$ and of $\overline{AD}$. $\angle ACB \cong \angle DCE$.

Prove: $\overline{AB} \cong \overline{DE}$

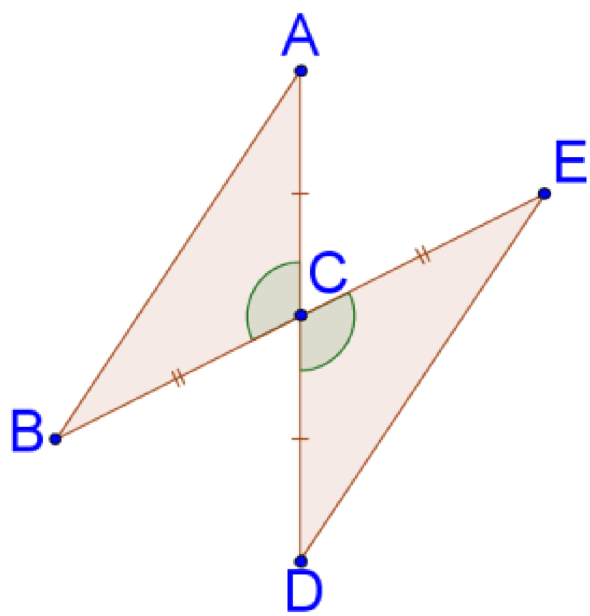


No matter which style of proof you use, before starting to write you should **brainstorm** what you will say in your proof. Start by looking at the given information and thinking about what you know based on each given fact.

- The fact that C is a midpoint means it is right in the middle of the two line segments. This means there are two pairs of segments that must be congruent. *Mark these congruent segments on the diagram as you brainstorm. This will help you to keep track of what you know!*



- You also are given that $\angle ACB \cong \angle DCE$. *This should be marked on the diagram as well.*



Next think about what other conclusions you can make based on what you have now marked on the diagram. You have SAS $\cong$ criteria marked, so you can say that the two triangles are congruent. This will allow you to be able to say that $\overline{AB} \cong \overline{DE}$, because they are corresponding parts of the triangles.

Once you have thought through the proof and your approach, start writing. *In all proofs, the statements have been written in red and the reasons have been written in blue.*

Example 2

Write a paragraph proof that shows that $\overline{AB} \cong \overline{DE}$.

C is the midpoint of $\overline{BE}$ and $\overline{AD}$ **because it is given information.** *This means that* $\overline{AC} \cong \overline{CD}$ **and** $\overline{EC} \cong \overline{CB}$, **because midpoints divide segments into two congruent segments**. Also, $\angle ACB \cong \angle DCE$ **because it is given information.** $\Delta ACB \cong \Delta DCE$ **by** $SAS \cong$ **because they are triangles with two pairs of corresponding sides congruent and included angles congruent.** $\overline{AB} \cong \overline{DE}$ **because they are corresponding segments and corresponding parts of congruent triangles must be congruent.**

Example 3

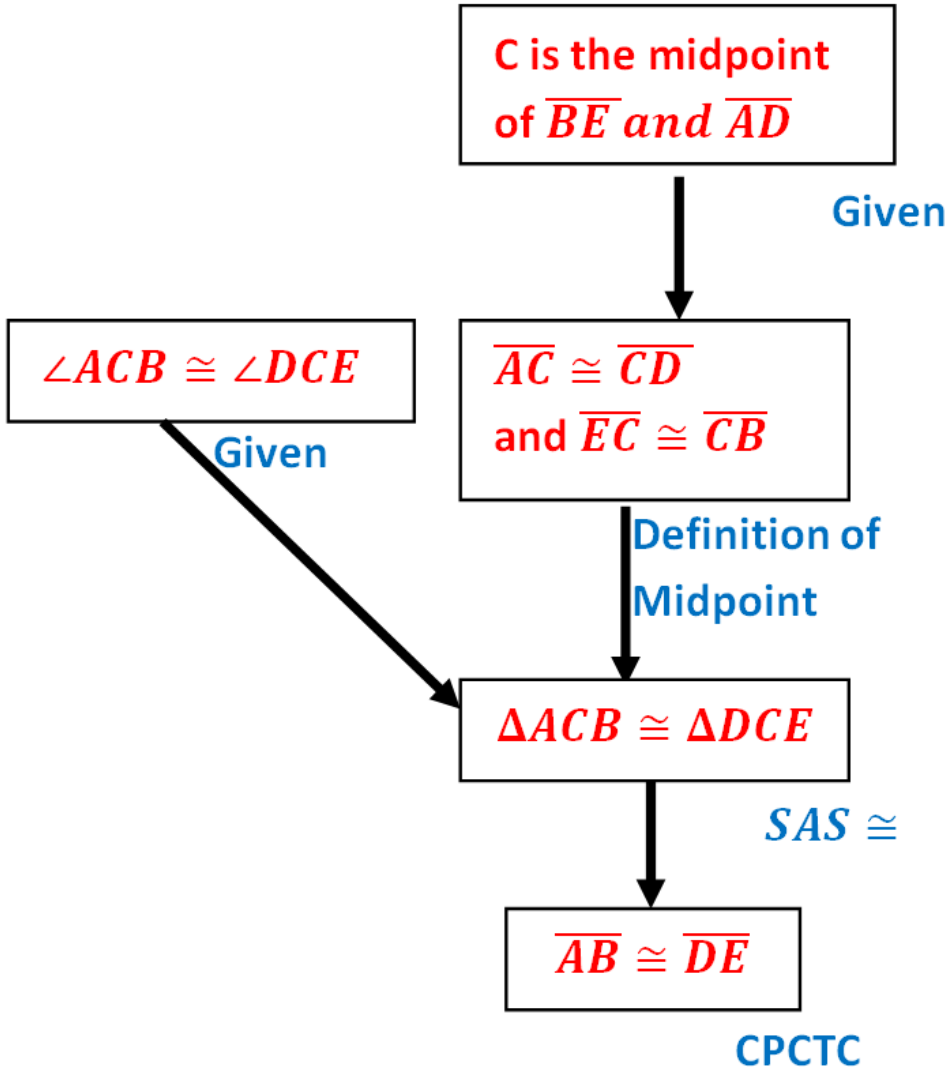
Write a two-column proof that shows that $\overline{AB} \cong \overline{DE}$.

TABLE 2.3:

Statements	Reasons
C is the midpoint of $\overline{BE}$ and $\overline{AD}$	Given
$\overline{AC} \cong \overline{CD}$ and $\overline{EC} \cong \overline{CB}$	definition of midpoint
$\angle ACB \cong \angle DCE$	Given
$\Delta ACB \cong \Delta DCE$	$SAS \cong$
$\overline{AB} \cong \overline{DE}$	CPCTC

Example 4

Write a flow diagram proof that shows that $\overline{AB} \cong \overline{DE}$.



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1. What's the difference between a postulate and a theorem?
2. What are the two main components of any proof?
3. What does it mean when a reason in a proof is "given"?

4. What should the last line/sentence/box for any proof be?
5. What are three styles of proof?

For 6-8, consider the proof below.

Given triangles ΔACB and ΔACD as marked, prove that $\overline{AB} \cong \overline{AD}$.

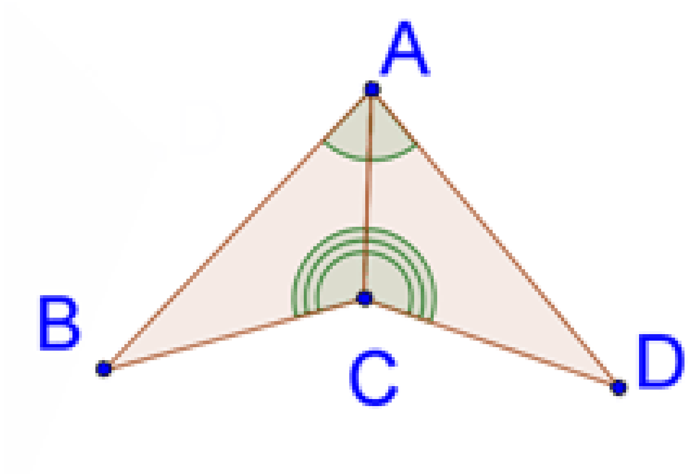


TABLE 2.4:

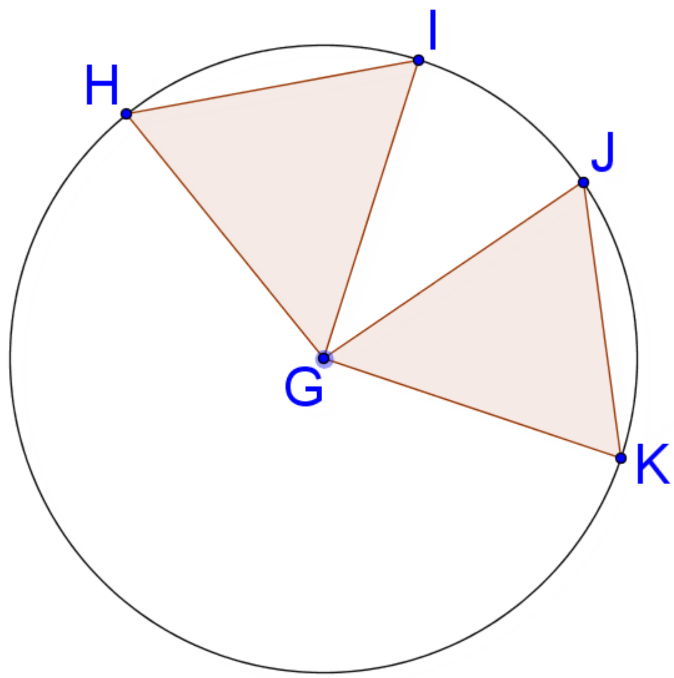
Statements	Reasons
???	Given
$\angle BAC \cong \angle DAC$	Given
$\overline{AC} \cong \overline{AC}$	???
$\Delta ACB \cong \Delta ACD$	???
???	CPCTC (corresponding parts of congruent triangles must be congruent)

6. Fill in the missing statements and reasons.
7. Rewrite this proof as a paragraph proof.
8. Rewrite this proof as a flow diagram proof.

For 9-11, consider the proof below.

Given: Circle G with center G. $\angle HGI \cong \angle JGK$.

Prove: $\Delta HGI \cong \Delta JGK$



_________________________ because it is given information. Point G is the center of the circle because ___________-______________. $\overline{HG}, \overline{GI}, \overline{GJ}, \overline{GK}$ are all radii of the circle, because they are segments that connect the center of the circle with the circle. $\overline{HG} \cong \overline{GK}$ and $\overline{GI} \cong \overline{GJ}$ because all ______ are congruent. $\Delta HGI \cong \Delta JGK$ _______ because they are triangles with two pairs of corresponding sides congruent and included angles congruent.

9. Fill in the blanks.

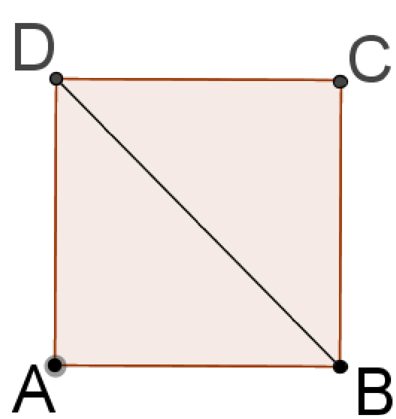
10. Rewrite this proof as a two-column proof.

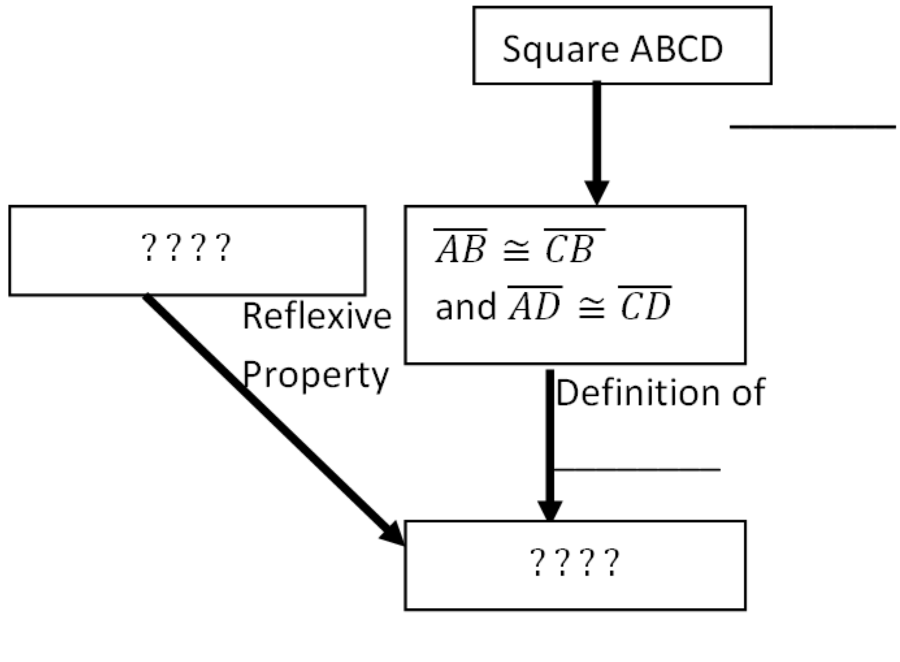
11. Rewrite this proof as a flow diagram proof.

For 12-14, consider the proof below.

Given: Square $ABCD$

Prove: $\Delta ABD \cong \Delta CBD$





12. Fill in the missing boxes/reasons.

13. Rewrite this proof as a paragraph proof.

14. Rewrite this proof as a two-column proof.

15. Give an example of a real life situation where being able to persuade someone else that something is true would be helpful.

16. Before proving something in geometry, one often begins with a conjecture, that is, a prospective idea of something that might be true. For each of the following conjectures, state whether you think the conjecture is true or not. If not, provide a counter-example, that is, an example that shows the statement is false.

a. If two rectangles have the same area, their dimensions are the same.
b. If two triangles have the same angle measures, their areas are the same.
c. Any two lines in the same plane which do not intersect are parallel.
d. Any two lines in space which do not intersect are parallel.
e. If two circles have the same radius, their circumferences are the same.
f. If two circles intersect in two points, they must have radii of different length.
g. Two angles that are complementary must be adjacent to each other.
h. Two angles that are vertical must have the same measure.

17. Geometry includes definitions, postulates, theorems, and properties. Each of these can be written as **conditional statements**, that is, if-then statements. For example, the Pythagorean Theorem states that, **if** a triangle is right with hypotenuse c, **then** $a^2 + b^2 = c^2$. Rewrite each of the following as conditional statements. State whether you think the statement is true or false. If false, write a similar, true statement.

a. Supplementary angles sum to 90° .
b. Vertical angles have the same measure.
c. All rectangles are parallelograms.
d. All right triangles are scalene.

e. In a right triangle, $a^2+b^2=c^2$.
f. Opposite sides are parallel in a rhombus.

18. There is a big difference between a conditional statement and its **converse**. The converse of a conditional statement reverses the order. So if the original conditional is **if *a* then *b***, the converse is **if *b* then *a***. Sometimes the original conditional will be true and the converse will be false:

If a quadrilateral is a square, then it's a parallelogram. **True**.

If a quadrilateral is a parallelogram, then it's a square. **False**.

In other cases, the original can be false and the converse true, or they can both be true, or they can both be false. Write each of the following as a conditional statement. Decide if the statement is true or not. Write the converse. Decide if the converse is true or not.

a. The area of a rectangle is given by the formula Area $=$ base $\times$ height .
b. Lines that intersect at a 90° angle are perpendicular.
c. The sum of the interior angles of a triangle is 180°.
d. All quadrilaterals are trapezoids.
e. If two triangles are congruent, then corresponding angles are congruent.
f. If a figure has reflection symmetry, it also has rotation symmetry.
g. If a figure undergoes a rotation, that is equivalent to a sequence of two reflections.

19. Given $h \parallel i$, and the information in the diagram as shown, find x. Add labels to angles as needed, then use a flow chart, paragraph, or two column proof to explain and justify each step in your solution.

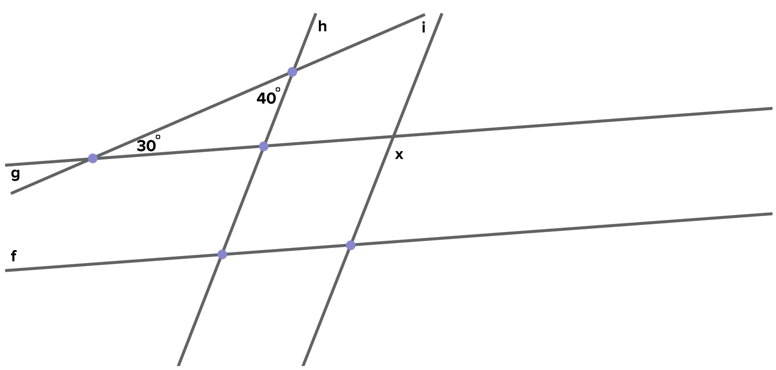


FIGURE 2.1

20. Given: $\overline{AC} \cong \overline{AF}$; $\overline{CB} \cong \overline{FB}$

Prove: $\Delta ABC \cong \Delta ACF$

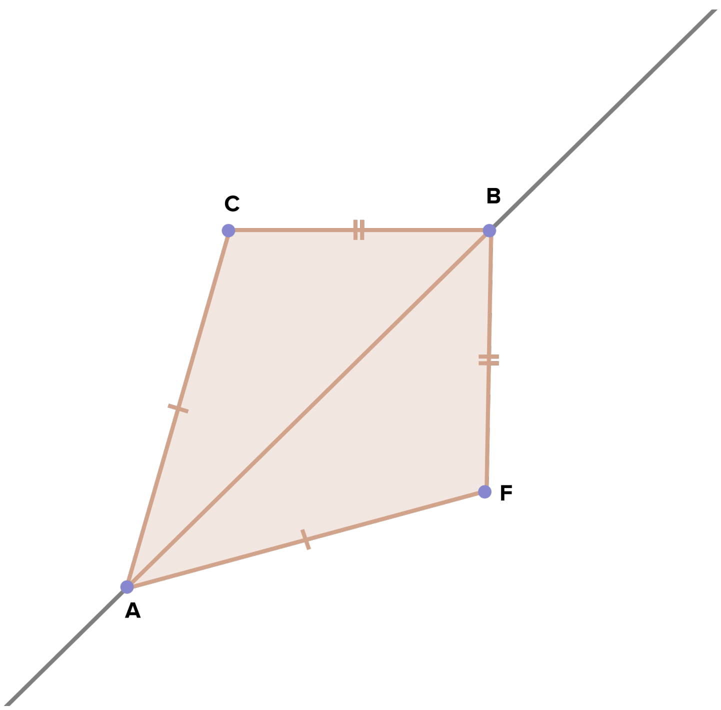


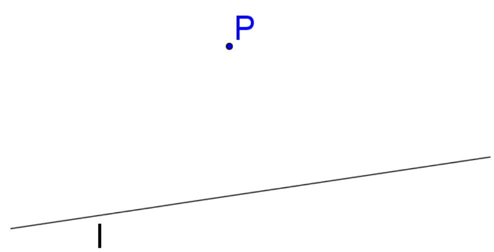
FIGURE 2.2

2.2 Theorems about Lines and Angles

Learning Objectives

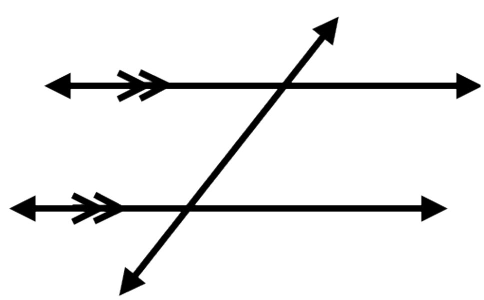
Here you will learn theorems about lines and angles and how to prove them.

Consider line l and a point P that is not on line l. How many lines exist that are parallel to l and pass through point P?

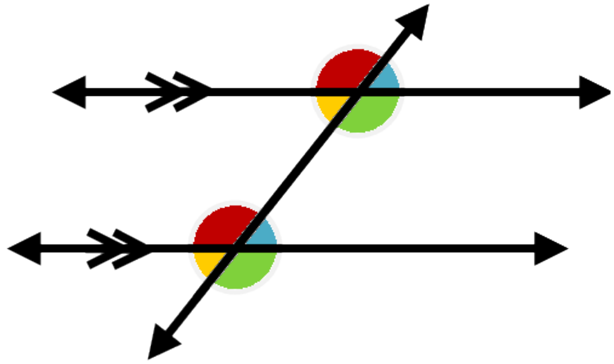


Line and Angle Theorems

Consider two parallel lines that are intersected by a third line. (*Remember that tick marks* ($\gg$) *can be used to indicate that two lines are parallel.)*



This third line is called a **transversal.** Note that four angles are created where the transversal intersects each line. Each angle created by the transversal and the top line has a **corresponding angle** with an angle create by the transversal and the bottom line. These corresponding angle pairs are shown color-coded below. How do you think these corresponding angles are related?



Your intuition and knowledge of translations might suggest that these angles are congruent. Imagine translating one of the angles along the transversal until it meets the second parallel line. It will match its corresponding angle exactly. This is known as the corresponding angle postulate:

If two parallel lines are cut by a transversal, then the corresponding angles are congruent.

Remember that a postulate is a statement that is accepted as true without proof. Your knowledge of translations should convince you that this postulate is true.

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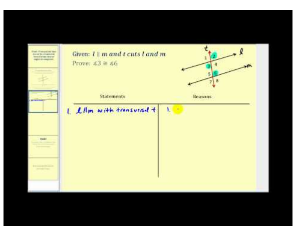

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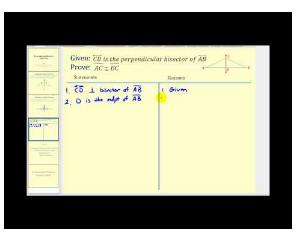

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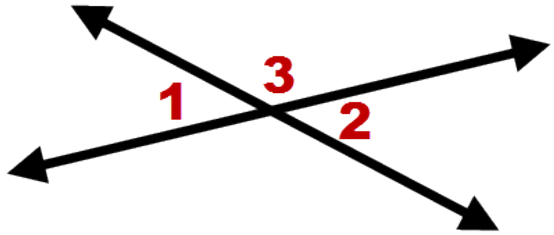
Let's take a look at some example problems.

1. Recall that vertical angles are a pair of opposite angles created by intersecting lines. Prove that **vertical angles are congruent.**

For this proof, you are not given a specific picture. When not given a picture, it helps to create a generic picture to reference in your proof. It's important that the picture does not include any information that you cannot assume. Below is a generic picture of intersecting lines with angles numbered for reference.



In this picture, $\angle 1$ and $\angle 2$ are vertical angles. Your job is to prove that $\angle 1 \cong \angle 2$. You can use any style of proof you prefer. Here is a two-column proof.

TABLE 2.5:

Statements	Reasons
$m\angle 1 + m\angle 3 = 180^\circ$ and $m\angle 2 + m\angle 3 = 180^\circ$	Two angles that form a line are supplementary
$m\angle 1 + m\angle 3 = m\angle 2 + m\angle 3$	Algebraic substitution
$m\angle 1 = m\angle 2$	Subtraction property of equality
$\angle 1 \cong \angle 2$	If two angles have the same measure, they are congruent.

Vertical angles are congruent is a **theorem**. Now that it has been proven, you can use it in future proofs without proving it again.

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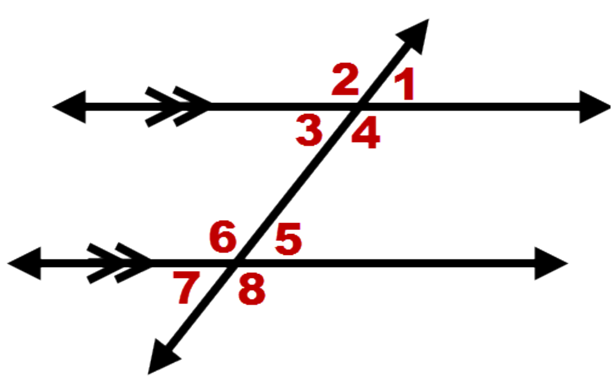
2. When two parallel lines are cut by a transversal, two pairs of **alternate interior angles** are formed. In the diagram below, $\angle 3$ and $\angle 5$ are alternate interior angles. Similarly, $\angle 4$ and $\angle 6$ are alternate interior angles.



Prove that **if two parallel lines are cut by a transversal, alternate interior angles are congruent.**

Use the diagram above, and prove that $\angle 3 \cong \angle 5$. (The same exact proof would show that $\angle 4 \cong \angle 6$). Again, in general you can use any style of proof that you prefer. Here is a paragraph proof.

$\angle 1 \cong \angle 3$ because they are vertical angles and vertical angles are always congruent. $\angle 1 \cong \angle 5$ because they are corresponding angles created by parallel lines and corresponding angles are congruent when lines are parallel. $\angle 3 \cong \angle 5$ because if two angles are congruent to the same angle, they are congruent to each other by the transitive property.

*Note: The transitive property states that if two objects are equal/congruent to the same third object, then they are equal/congruent to each other. The transitive property is a form of substitution. You can use it in any proof.

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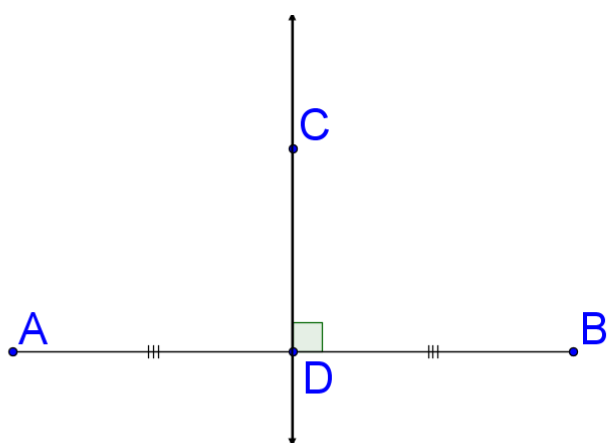
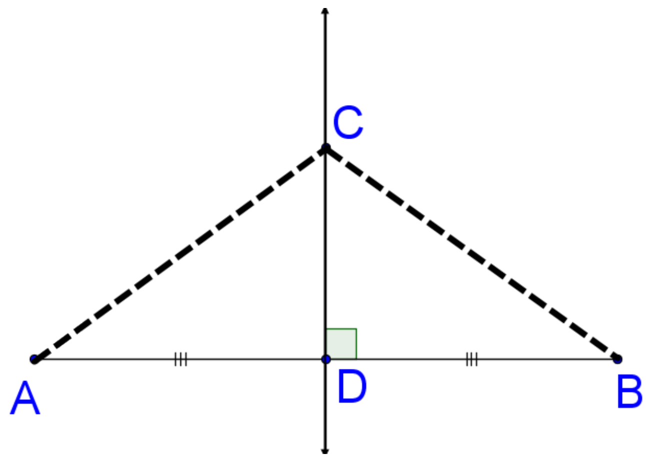
3. The statement *"if two parallel lines are cut by a transversal, then alternate interior angles are congruent*" is a **theorem.** Now that it has been proven, you can use it in future proofs without proving it again.

Prove that **points on the perpendicular bisector of a line segment are equidistant from the endpoints of the line segment**.

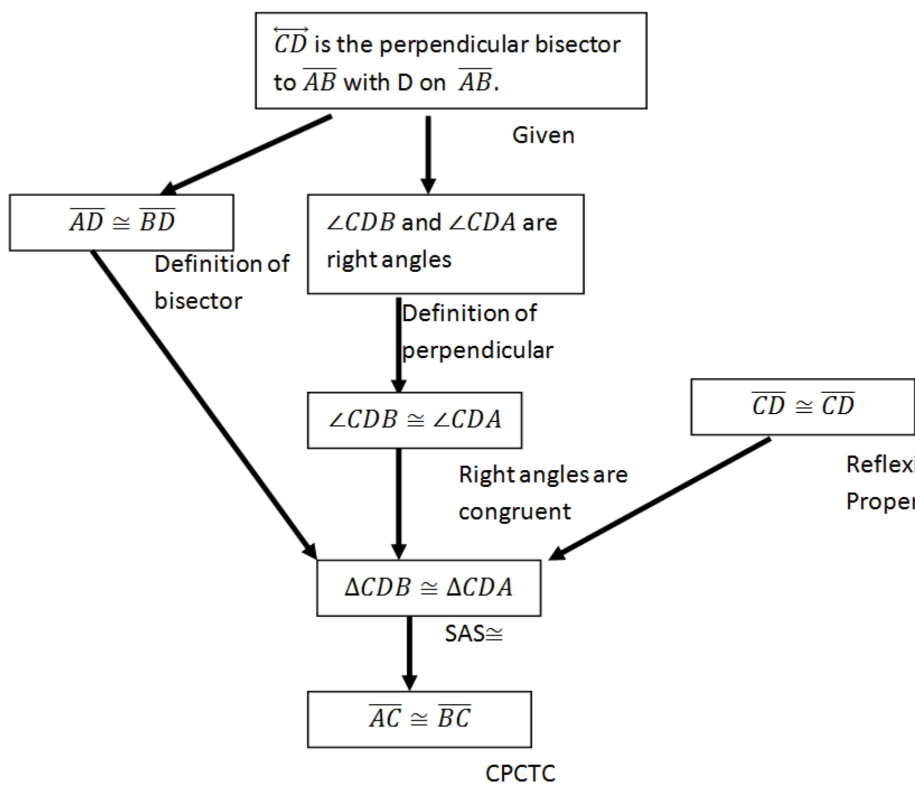
Start by exploring this claim. Draw a picture of a line segment and a perpendicular bisector of this segment. Remember that a perpendicular bisector is **perpendicular** to the line segment (meets it at a right angle) and **bisects** the line segment (cuts it in half).



The claim is that any point on the perpendicular bisector (such as point C), is the same distance away from each endpoint. In other words, the claim is that for some generic point C on the perpendicular bisector, $\overline{AC} \cong \overline{BC}$.



To prove the original statement, it will suffice to prove that if $\overleftrightarrow{CD}$ is the perpendicular bisector to $\overline{AB}$ with D on $\overline{AB}$, then $\overline{AC} \cong \overline{BC}$ using the above diagram as a reference. Remember, in general you can use any style of proof that you prefer. Here is a flow diagram. *To prove this statement, you will show that the two triangles are congruent and then that the $\overline{AC}$ and $\overline{BC}$ are corresponding parts so must be congruent.*



The statement "*points on a perpendicular bisector of a line segment are equidistant from the segment's endpoints*" is a **theorem**. Now that it has been proven, you can use it in future proofs without proving it again.

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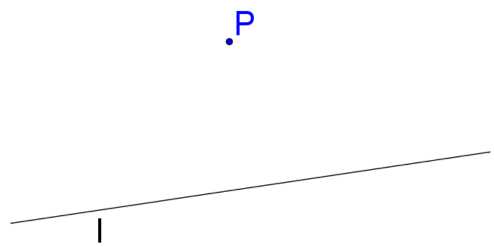
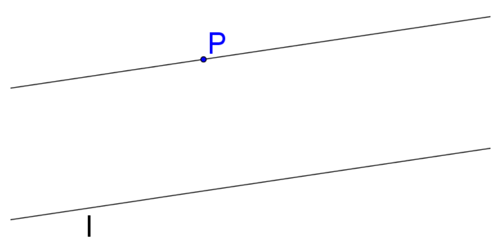
Examples

Example 1

Earlier, you were asked how many lines exist that are parallel to l and pass through point P.



Common sense should tell you that there is only **one** line through P that is parallel to l.



Interestingly, for hundreds of years people have attempted to **prove** this statement from simpler statements with no luck. Eventually, it was accepted that it simply must be a **postulate**, a statement that is assumed to be true without proof. This is known as the **parallel postulate.**

Example 2

If corresponding angles are congruent, then are lines parallel?

This is known as the **converse** of the corresponding angles postulate. The original postulate said:

> ***Original:*** If **lines are parallel**, then **corresponding angles are congruent.**

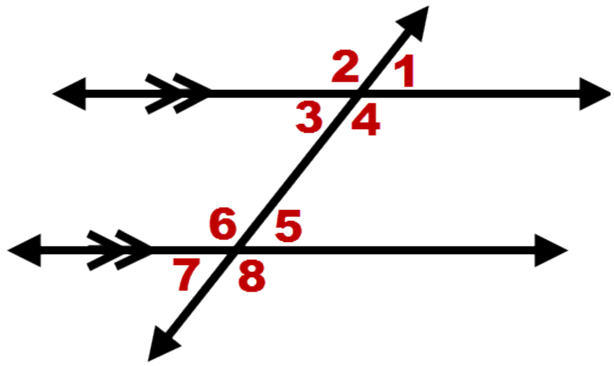
Here, the "if" part of the statement (known as the hypothesis) is switched with the "then" part of the statement (known as the conclusion).

> ***Converse:*** If **corresponding angles are congruent**, then **lines are parallel.**

In general, just because a statement is true doesn't necessarily mean its converse is true. In this case, the converse does happen to be true. The only way for corresponding angles to be congruent is for the lines to be parallel. The **corresponding angles converse** is also a **postulate**, which means it is accepted as true without proof.

Example 3

Prove that if lines are parallel, then same side interior angles (such as $\angle 3$ and $\angle 6$) are supplementary.



In general you can use any style of proof that you prefer. Here, use a paragraph proof.

$\angle 3 \cong \angle 5$ because they are alternate interior angles created by parallel lines and alternate interior angles are congruent when lines are parallel. $m\angle 3 = m\angle 5$ because congruent angles have the same measure. $m\angle 5 + m\angle 6 = 180^\circ$ because two angles that form a line are supplementary. By substitution, $m\angle 3 + m\angle 6 = 180^\circ$. $\angle 3$ and $\angle 6$ are supplementary because two angles with measures that add to 180° are supplementary.

The statement *"if two parallel lines are cut by a transversal, then same side interior angles are supplementary"* is a **theorem.** Now that it has been proven, you can use it in future proofs without proving it again.

Example 4

Prove that if alternate interior angles are congruent, then lines are parallel.

This is the converse of the alternate interior angles theorem.

Original: If **lines are parallel**, then **alternate interior angles are congruent.**

Here, the "if" part of the statement (known as the hypothesis) is switched with the "then" part of the statement (known as the conclusion).

Converse: If **alternate interior angles are congruent**, then **lines are parallel.**

To prove this statement, start with a picture of alternate interior angles that are assumed to be congruent, but don't assume the lines are parallel. In the picture below, assume $\angle 1 \cong \angle 2$. Prove that $m \parallel n$ (two parallel bars indicate parallel lines).

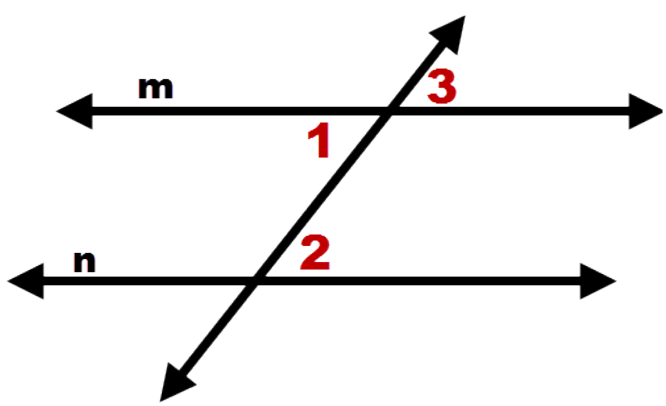


TABLE 2.6:

Statements	Reasons
$\angle 1 \cong \angle 2$	Given
$\angle 1 \cong \angle 3$	Vertical angles are congruent
$\angle 2 \cong \angle 3$	Transitive property of congruence
$m \parallel n$	If corresponding angles are congruent then lines are parallel.

Review

1. In problem 1 from the Line and Angle Theorems section above, the theorem "vertical angles are congruent" was proved with a two-column proof. Rewrite this proof in a paragraph format.

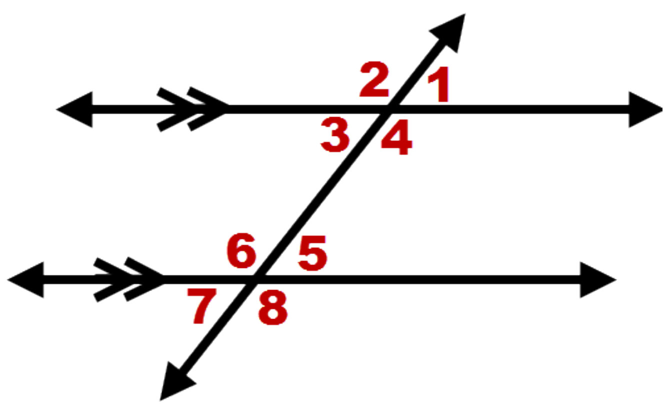
2. In problem 2 from the Line and Angle Theorems section above, the theorem "if lines are parallel then alternate interior angles are congruent" was proved with a paragraph proof. Rewrite this proof with a flow diagram.

3. In problem 3 from the Line and Angle Theorems section above, the theorem "points on a perpendicular bisector of a line segment are equidistant from the segment's endpoints" was proved with a flow diagram. Rewrite this proof in a two-column format.

4. In Example 3, the theorem "if lines are parallel then same side interior angles are supplementary" was proved with a paragraph proof. Rewrite this proof in a two-column format.

5. In Example 4, the theorem "if alternate interior angles are congruent then lines are parallel" was proved with a two-column proof. Rewrite this proof with a flow diagram.

6. Alternate exterior angles are outside a pair of lines and on opposite sides of a transversal. $\angle 2$ and $\angle 8$ are an example of alternate exterior angles. $\angle 1$ and $\angle 7$ are another example of alternate exterior angles.



The theorem "**if lines are parallel then alternate exterior angles are congruent**" is partially proved below. Fill in the blanks to complete the proof. *Note, the angles referenced are from the above picture.*

TABLE 2.7:

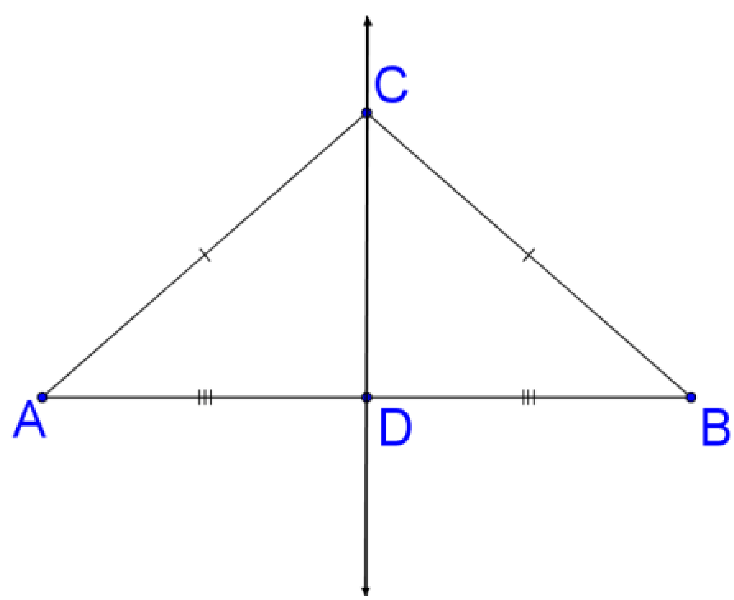
Statements	Reasons
Two parallel lines are cut by a transversal	__________
$\angle 2 \cong \angle 6$	__________
$\angle 6 \cong$__________	Vertical angles are congruent.
__________	Transitive property.

7. What is the **converse** of the theorem "if lines are parallel, then same side interior angles are supplementary"?

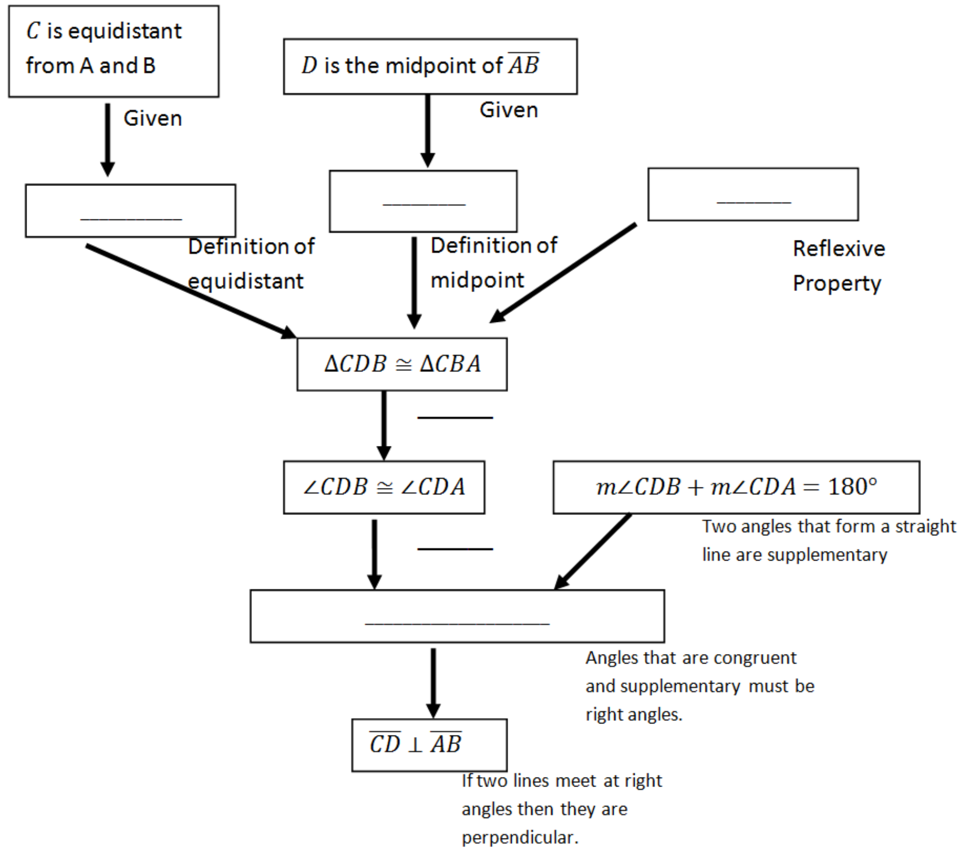
8. Prove the converse that you wrote in #7. Use any style of proof you prefer.

9. In problem 3 from the Line and Angle Theorems section above, the theorem "**points on a perpendicular bisector of a line segment are equidistant from the segment's endpoints**" was proved. This theorem could be rewritten as "**if a point is on the perpendicular bisector of a line segment, then the point is equidistant from the endpoints of the line segment**". What is the **converse** of this theorem? *To check your answer, look at #10.*

10. The converse of the theorem in #9 is "**if a point is equidistant from the endpoints of a line segment, then the point is on the perpendicular bisector of the line segment**". To prove this new theorem you can use the picture below.



Assume point C is a random point that is equidistant from endpoints A and B. Point D is the midpoint of line segment $\overline{AB}$. Your goal is to show that $\overline{CD}$ must be ***perpendicular to*** $\overline{AB}$ $(\overline{CD} \perp \overline{AB})$. This new theorem is partially proved below. Fill in the blanks to complete the proof.



In 11-13, you will prove that **if two angles are complementary to the same angle, then the two angles are congruent.**

11. Draw a generic picture of this situation and label the three angles.

12. What are the "givens" from your picture? What are you trying to prove?

13. Write a proof of the statement using whatever proof style you prefer.

14. Using your work from 11-13 to help, prove that **if two angles are supplementary to the same angle, then the two angles are congruent.**

15. Give at least 3 methods for proving that lines are parallel.

16. Given: $\angle B$ supplementary $\angle D$; $\angle F$ supplementary $\angle D$.

Prove: $\angle B \cong \angle F$

17. Prove that the exterior angle of a triangle is equal to the sum of the two remote interior angles. Use the diagram below to assist you.

18. Given: A and B are on the perpendicular bisector of $\overline{CD}$

Prove: ACBD is a kite

19. Use the diagram above to prove that $\angle 1 \cong \angle 2$. Anything proven in the last proof can also be used here without re-writing.

20. Given: $\angle 1 \cong \angle 2$; $\overline{EC} \cong \overline{CB}$; point C is the midpoint of $\overline{DA}$

Prove: $\angle D \cong \angle A$.

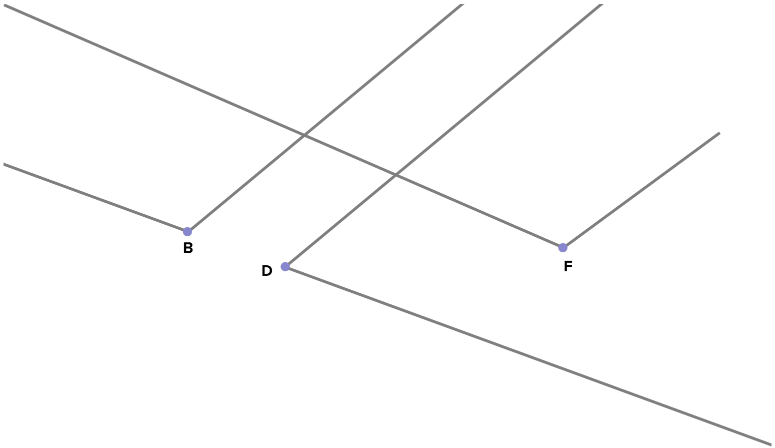


FIGURE 2.3

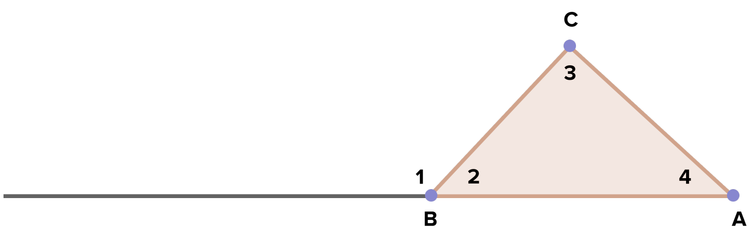


FIGURE 2.4

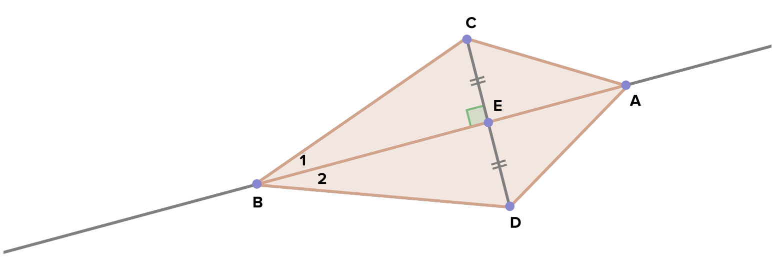


FIGURE 2.5

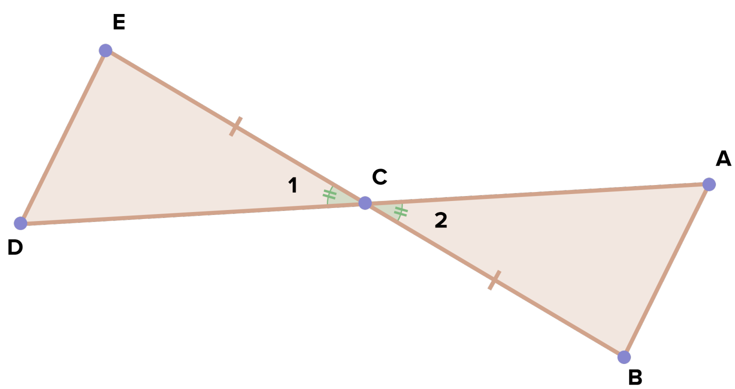


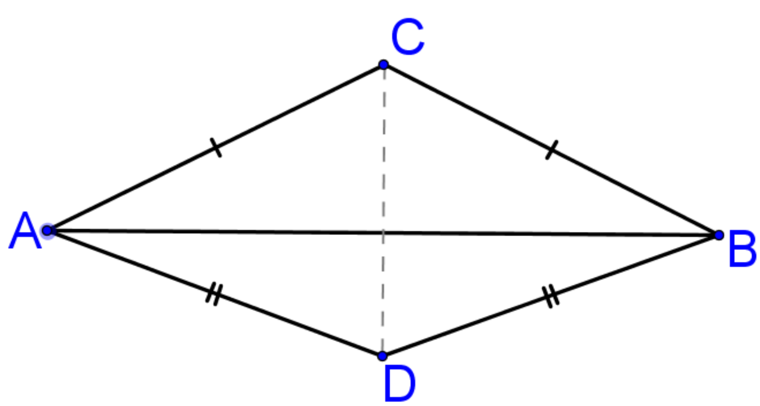
FIGURE 2.6

2.3 Applications of Line and Angle Theorems

Learning Objectives

Here you will use theorems about lines and angles to solve problems.

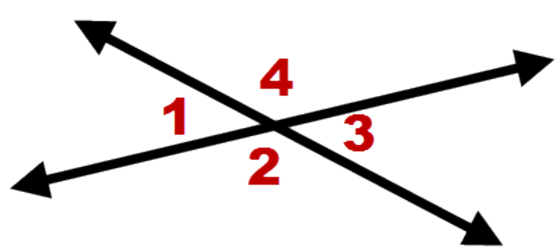
What can you say about the relationship between $\overline{AB}$ and $\overline{CD}$? What does this have to do with kites?



Applying Line and Angle Theorems

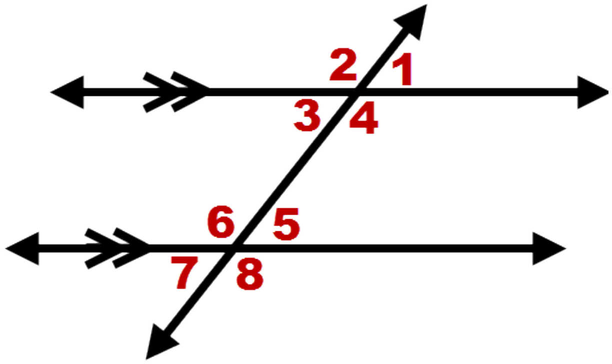
There are four categories of theorems to remember that have to do with lines and angles.

1) When two lines intersect, two pairs of vertical angles are formed.



In the diagram above, $\angle 1$ and $\angle 3$ are vertical angles. $\angle 2$ and $\angle 4$ are also vertical angles. **Vertical angles are always congruent**.

2) When two lines are cut by a transversal, many different angle pairs are formed. **If the two lines are parallel**, these angle pairs have special properties.



- $\angle 1$ and $\angle 5$ are ***corresponding angles*** because their locations are corresponding. **If lines are parallel, then corresponding angles are congruent**. Other examples of corresponding angles are $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, and $\angle 4$ and $\angle 8$.
- $\angle 4$ and $\angle 5$ are ***same side interior angles*** because they are inside the lines and on the same side of the transversal. **If lines are parallel, then same side interior angles are supplementary**. Another example of same side interior angles is $\angle 3$ and $\angle 6$.
- $\angle 3$ and $\angle 5$ are ***alternate interior angles*** because they are inside the lines and on opposite sides of the transversal. **If lines are parallel, then alternate interior angles are congruent**. Another example of alternate interior angles is $\angle 4$ and $\angle 6$.
- $\angle 1$ and $\angle 7$ are ***alternate exterior angles***. **If lines are parallel, then alternate exterior angles are congruent**. Another example of alternate exterior angles is $\angle 2$ and $\angle 8$.
- $\angle 2$ and $\angle 7$ are ***same side exterior angles***. **If lines are parallel, then same side exterior angles are supplementary**. Another example of same side exterior angles is $\angle 1$ and $\angle 8$.

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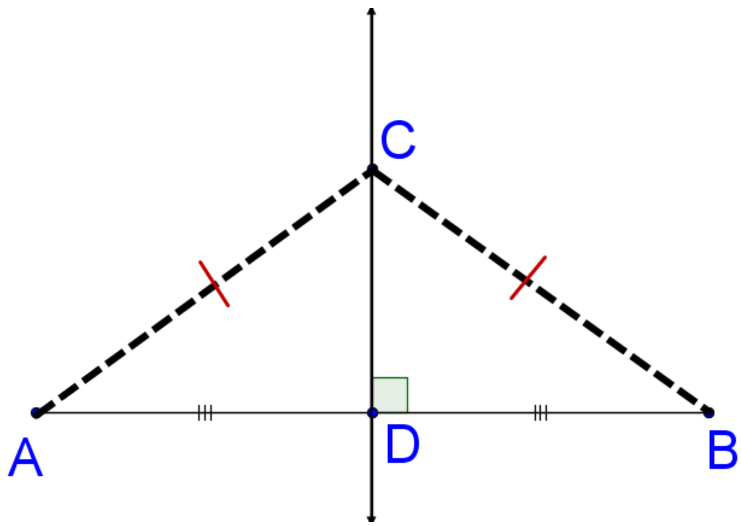
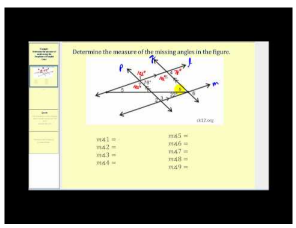
3) The **converses** of all of the above theorems and postulates are also true and are ways to show that lines are parallel. For example, if corresponding angles are congruent then lines must be parallel. Similarly, if same side interior angles are supplementary then lines must be parallel.

4) When a line segment is bisected by a perpendicular line, the points on the perpendicular bisector are exactly those equidistant from the segment's endpoints.



For the figure above, as C moves along the perpendicular bisector, it will always be true that $\overline{AC} \cong \overline{CB}$.

If you remember all of the above postulates and theorems, you can use them to help solve problems.



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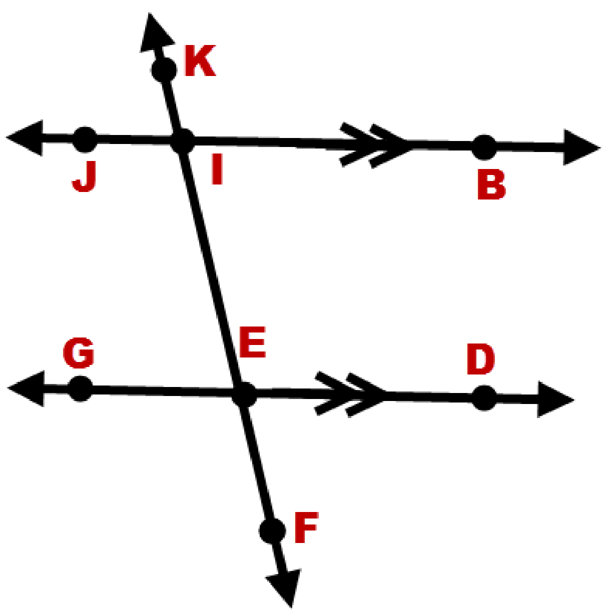
Measuring Angles



If $m\angle KIB = 105^\circ$, what is:

a. $m\angle JIE$

$m\angle JIE = 105^\circ$ because it is a vertical angle with $\angle KIB$ and vertical angles are congruent.

b. $m\angle DEI$

$m\angle DEI = 105^\circ$ because it is a corresponding angle with $\angle KIB$ and corresponding angles are congruent when lines are parallel (note that the $\gg$ markings indicate that the lines are parallel).

c. $m\angle GEI$

$m\angle GEI = 75^\circ$ because it forms a straight line with $\angle DEI$ and so those angles are supplementary.

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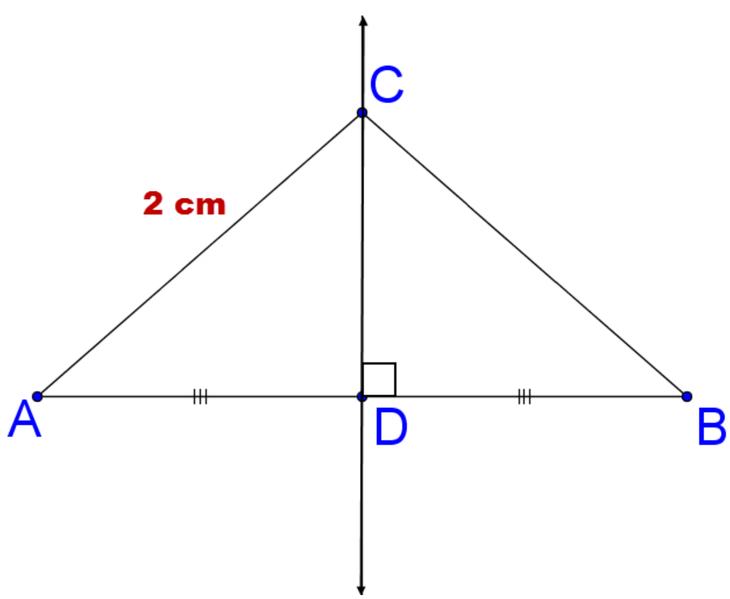
Measuring Line Segments

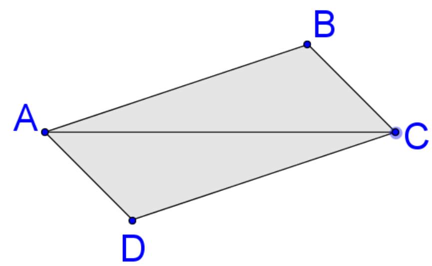
Find the length of $\overline{CB}$.



The markings in the picture indicate that D is the midpoint of $\overline{AB}$ and $\angle CDB$ is a right angle. This means that $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$. Therefore, C must be equidistant from A and B, and $CB = 2\ cm$.

Writing a Proof

Parallelogram $ABCD$ is shown below. Prove that $\Delta ABC \cong \Delta CDA$.



Recall that the definition of a parallelogram is a quadrilateral with two pairs of parallel sides. Since this is a parallelogram, you know that $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$ (Remember that $\parallel$ means parallel). With parallel lines comes lots of congruent angles. These angles will help you to show that the triangles are congruent.

TABLE 2.8:

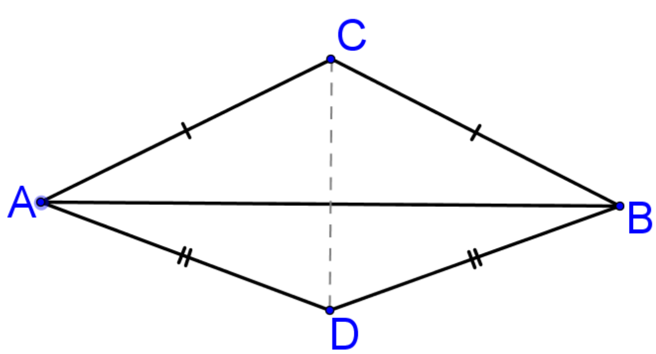
Statements	**Reasons**
$\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$	Definition of a parallelogram
$\angle DAC \cong \angle BCA$, $\angle ACD \cong \angle CAB$	Alternate interior angles are congruent if lines are parallel
$\overline{AC} \cong \overline{AC}$	Reflexive Property
$\Delta ABC \cong \Delta CDA$	$ASA \cong$

If you have trouble seeing the alternate interior angles, try extending the lines that form the parallelogram and focusing on one pair of parallel lines at a time.

Examples

Example 1

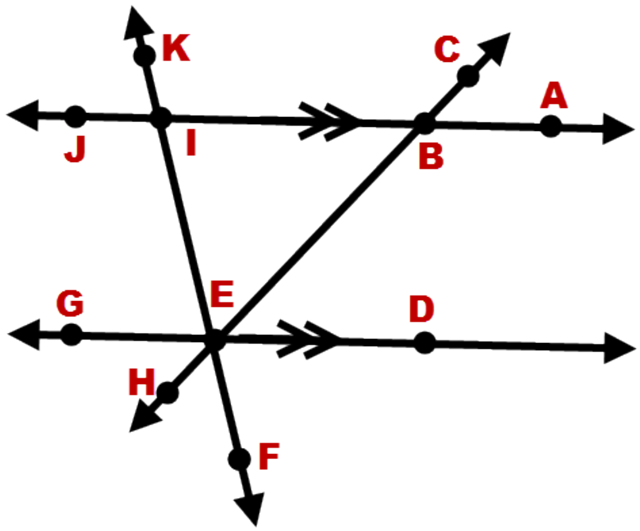
Earlier, you were asked what does the relationship between $\overline{AB}$ and $\overline{CD}$ have to do with kites.



From the markings in the picture, you can see that $\overline{AC} \cong \overline{CB}$ and $\overline{AD} \cong \overline{DB}$. This means that both C and D are equidistant from A and B. Therefore, both C and D are on the perpendicular bisector of $\overline{AB}$. Therefore, $\overline{CD}$ must BE the perpendicular bisector of $\overline{AB}$.

Quadrilateral $ACBD$ is a kite because it has two pairs of adjacent congruent sides. This shows that one of the diagonals of a kite is the perpendicular bisector of the other diagonal.

In the diagram below, $m\angle ABC = 50^\circ$ and $m\angle KIJ = 80^\circ$.



Example 2

Find $m\angle EBI$.

$m\angle EBI = 50^\circ$ because it is a vertical angle with $\angle ABC$ and vertical angles are congruent.

Example 3

Find $m\angle BIE$.

$m\angle BIE = 80^\circ$ because it is a vertical angle with $\angle KIJ$ and vertical angles are congruent.

Example 4

Find $m\angle BEI$.

$m\angle BEI = 50^\circ$ because $\angle EBI$, $\angle BIE$, and $\angle BEI$ form a triangle, and the sum of the measures of the interior angles of a triangle is 180°.

Example 5

Find $m\angle GEI$.

$m\angle GEI = 80^\circ$ because it is a corresponding angle with $\angle KIJ$ and corresponding angles are congruent when lines are parallel.

CK-12 PLIX Interactive

PLIX

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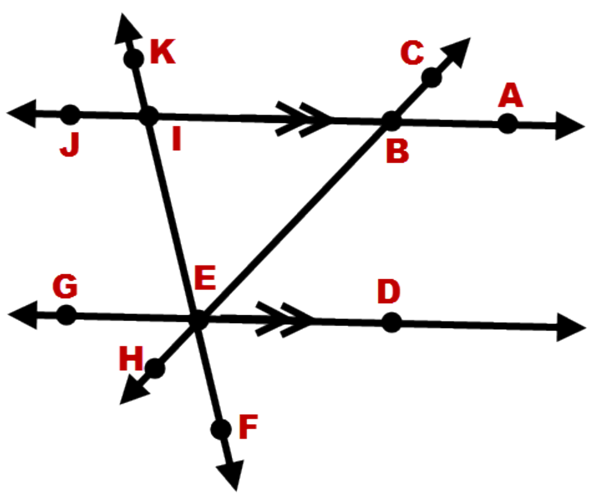
Review

1. Draw an example of vertical angles.

Use the diagram below for questions 2-4.



2. Give an example of same side interior angles. Name each angle with three letters.

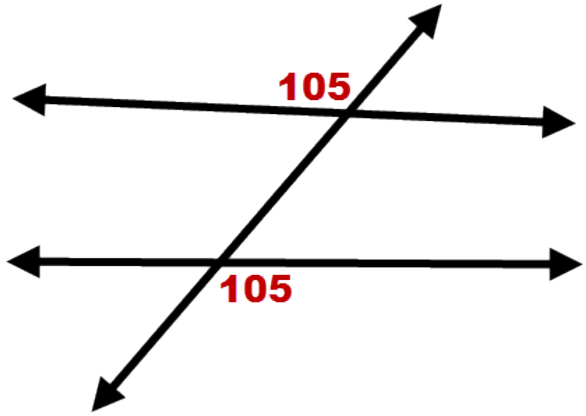
3. Give an example of alternate interior angles. Name each angle with three letters.

4. Give an example of corresponding angles. Name each angle with three letters.

5. If lines are not parallel, are corresponding angles still congruent?

For 6-9, determine whether or not the lines are parallel based on the given angle measures. Explain your answer in each case.

6.



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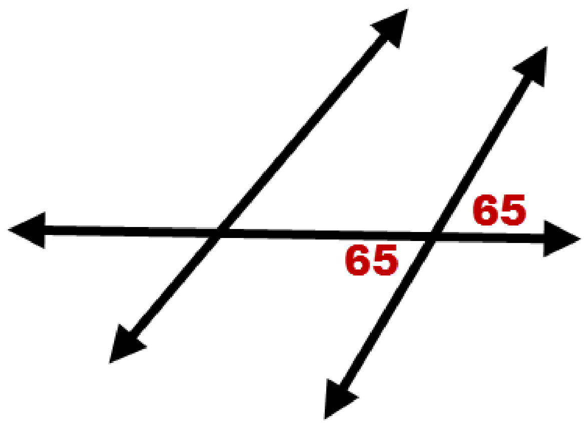

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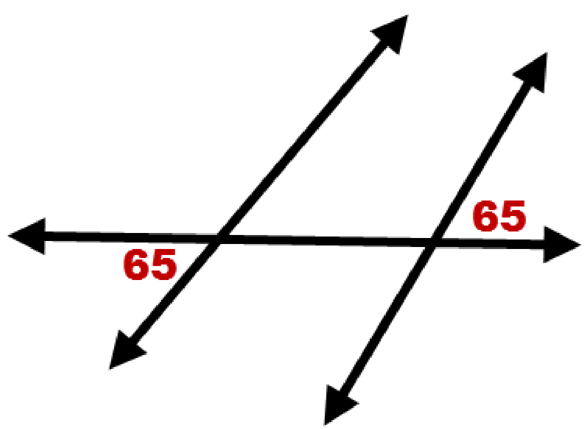

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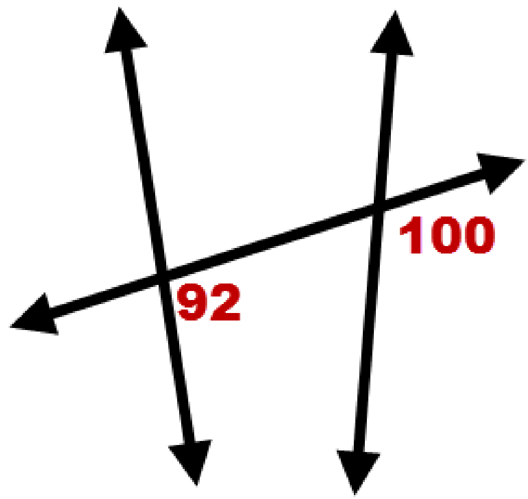
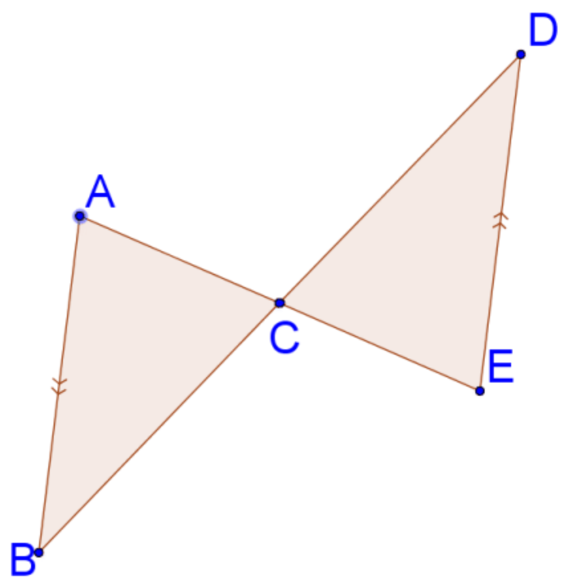
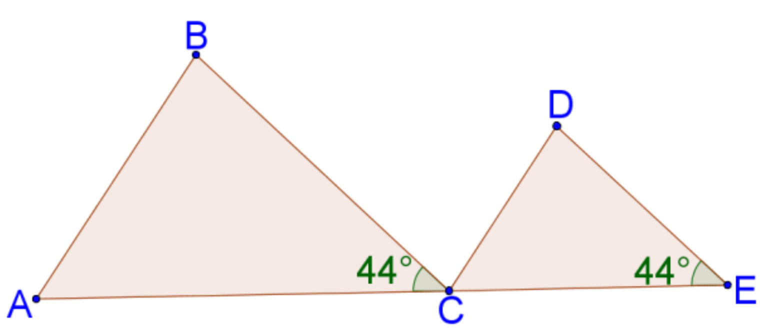


10. In the diagram below, C is the midpoint of $\overline{BD}$. Prove that $\Delta ABC \cong \Delta EDC$.

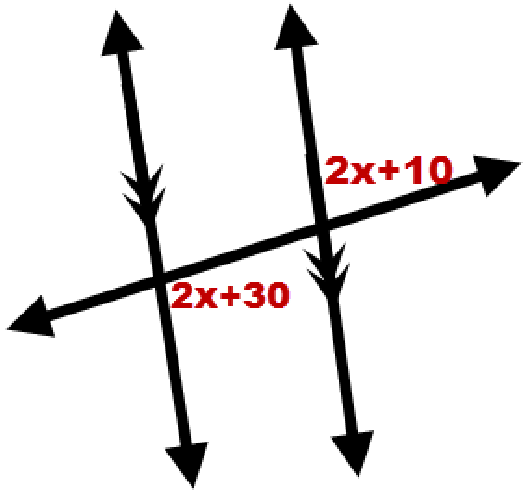


11. Extend your proof from #10 to prove that $\overline{AC} \cong \overline{CE}$.

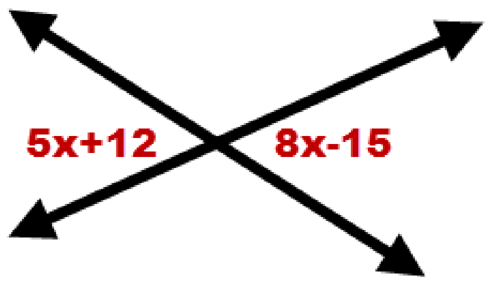
12. Which two line segments must be parallel in the picture below?



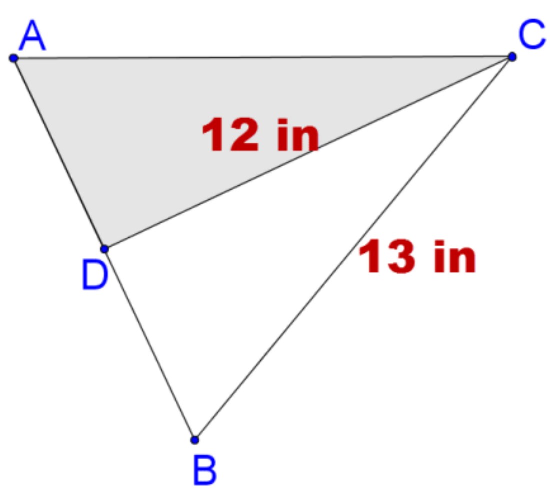
13. The measures of two angles are given below. Solve for x.



14. The measures of two angles are given below. Solve for x.



15. D is the midpoint of $\overline{AB}$ and $\overline{AC} \cong \overline{CB}$. Find the length of $\overline{AB}$.



16. Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$; $\overline{AB} \cong \overline{DE}$; $\overline{CB} \cong \overline{EF}$

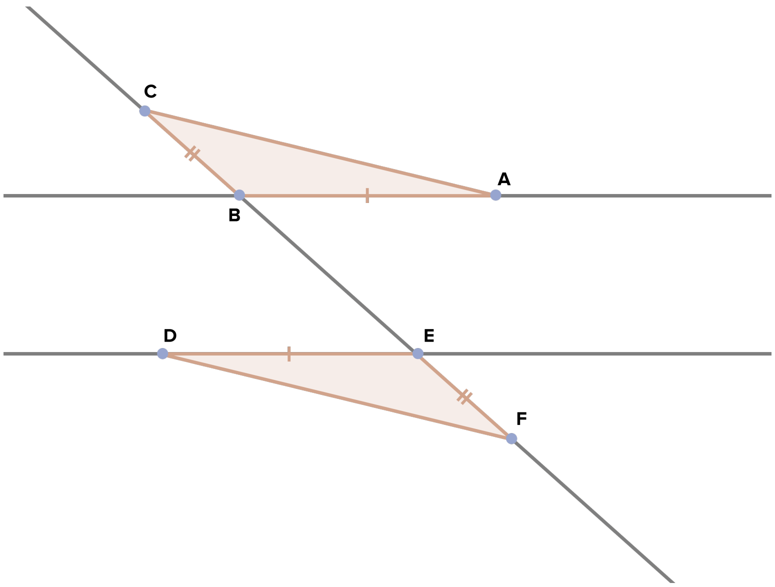


FIGURE 2.7

Prove: $\angle BCA \cong \angle EFD$

17. Given: $\angle 1 \cong \angle 2$; congruent segments as marked in diagram

Prove: $\angle BCA \cong \angle EFD$

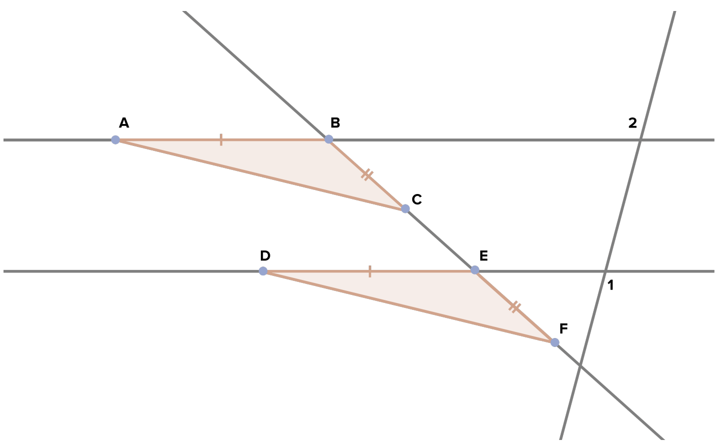


FIGURE 2.8

18. Given $h \parallel i$, find x in the diagram shown.

19. Given: $f \parallel g$

Prove: $\angle 1$ and $\angle 3$ are supplementary

20. Given: $f \parallel g$; $h \parallel i$

Prove: $\angle 1$ and $\angle 3$ are congruent

21. Given $f \parallel g$, find x and y in the diagram below.

22. Given: $\angle 1 \cong \angle 2$

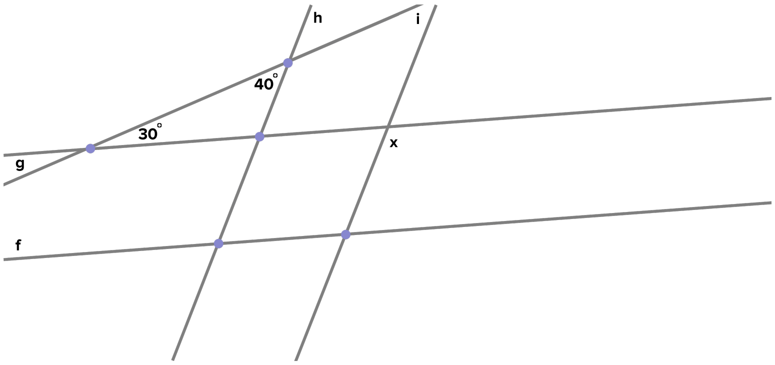


FIGURE 2.9

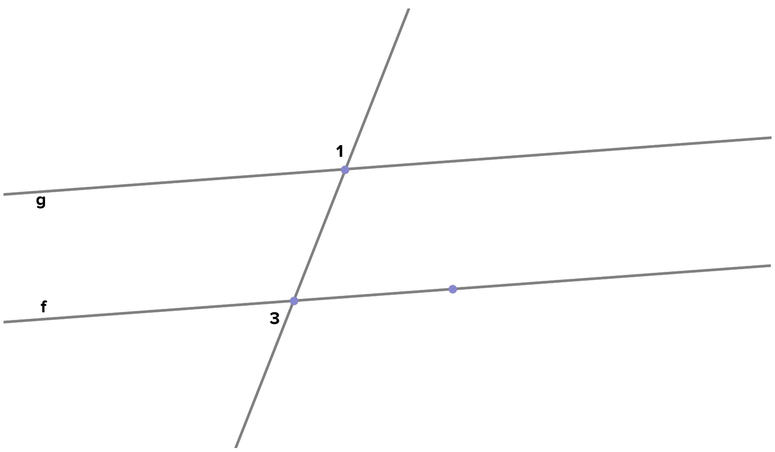


FIGURE 2.10

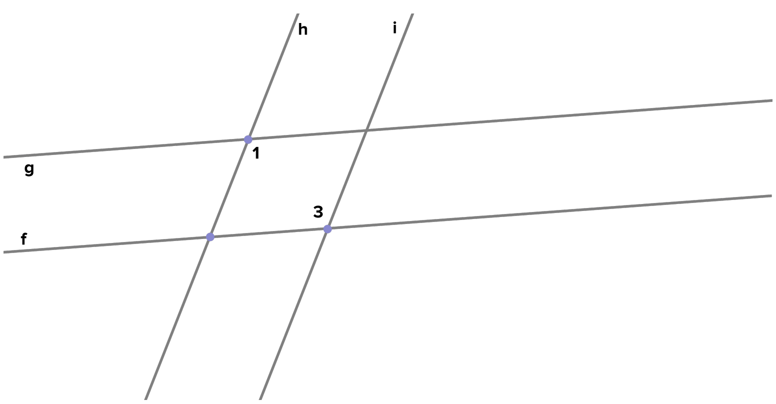


FIGURE 2.11

Prove: $\angle 3 \cong \angle 4$

23. Given: $g \parallel f$ and M is the midpoint of $\overline{AC}$. Describe a rigid motion transformation that maps $\angle BAM$ to $\angle DCM$.

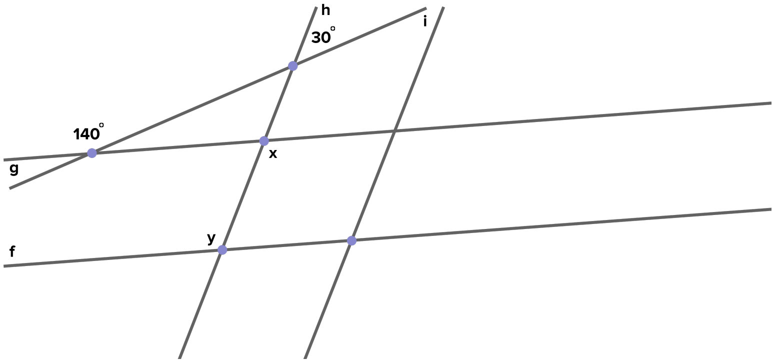


FIGURE 2.12

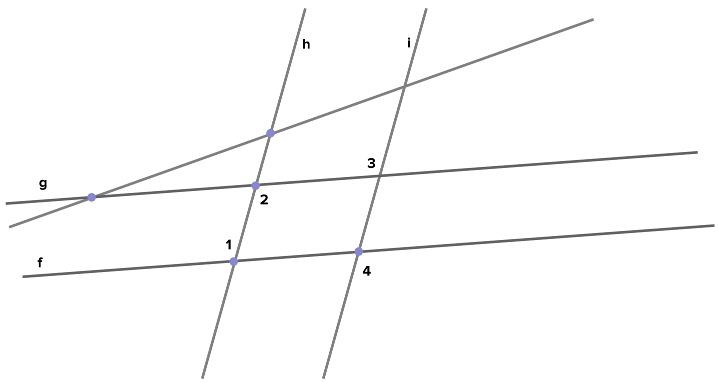
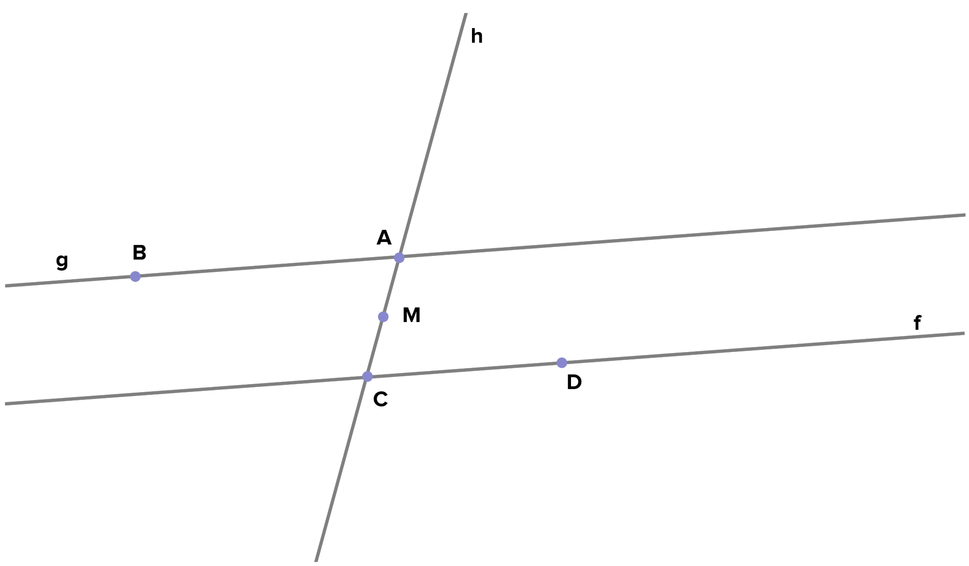


FIGURE 2.13



Review (Answers)

To see the Review answers, open this PDF file and look for section 4.3.

2.4 References

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CHAPTER 3 Transformations

Chapter Outline

3.1 Transformations in the Plane

Here you will learn about transformations in the plane. You will learn what makes a transformation a *rigid* transformation.

Transformation is a process that changes the shape, size or position of a figure to create a new image. It is a function that takes points in the plane as inputs and gives other points as outputs. You can think of a transformation as a rule that tells you how to create new points.

Suppose you have a transformation F that applies a horizontal stretch factor of two to each point. Below, this transformation is applied to triangle S to create triangle S'.

- S' is considered the image of S by F.
- It is also correct to say that F maps S to S'.
- Each of the points in the image are labeled with a $'$ symbol, which is read as "prime."

This helps to show how points on S correspond to points on S'. For example, you could say that "point A maps to point A-prime."

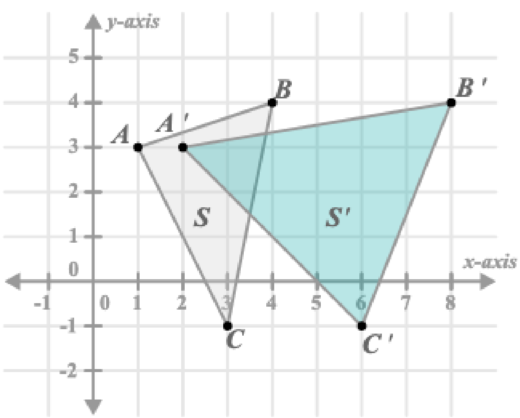
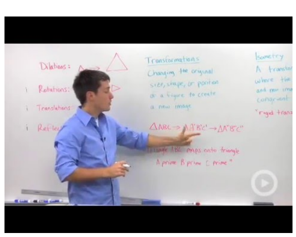


FIGURE 3.1

Some transformations preserve length and angles. Preserving length means that if a line segment is 3 units, its image will also be 3 units. Similarly, preserving angles means if an angle is 60^{o}, its image will also be 60^{o}.

- A transformation that preserves length and angles is called a rigid transformation.



MEDIA

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URL: http://www.ck12.org/flx/render/embeddedobject/71077

Recognizing Rigid Transformations

1. Is a horizontal stretch an example of a rigid transformation?

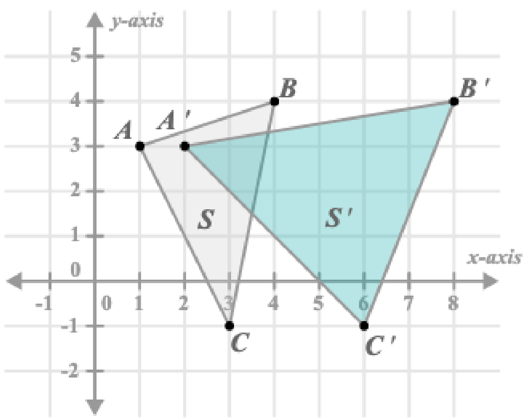

FIGURE 3.2

No. You can prove this using the picture above by showing that length is not preserved.

$$\begin{aligned}
&\textit{The length of } \overline{AC} \\
&= \sqrt{(3-1)^2 + (-1-3)^2} \\
&= \sqrt{(2)^2 + (-4)^2} \\
&= \sqrt{4+16} \\
&= \sqrt{20} \\
&= 2\sqrt{5} \quad \textit{units}
\end{aligned}$$

FIGURE 3.3

$$\begin{aligned}
&\textit{The length of } \overline{A'C'} \\
&= \sqrt{(6-2)^2 + (-1-3)^2} \\
&= \sqrt{(4)^2 + (-4)^2} \\
&= \sqrt{16 + 16} \\
&= \sqrt{32} \\
&= 4\sqrt{2} \;\; \textit{units}
\end{aligned}$$

FIGURE 3.4

2. A transformation reflects points in shape K across $\overleftrightarrow{AB}$ to create shape K'. Is this reflection a rigid transformation?

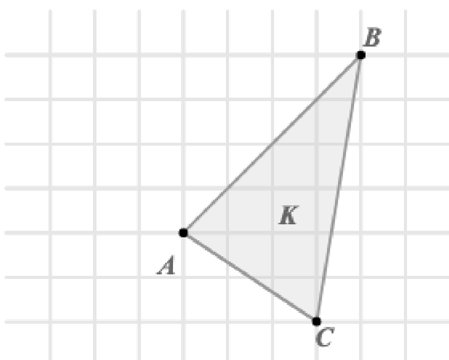

FIGURE 3.5

Yes, reflections are rigid transformations. You can verify that the distances between the points are preserved.

3. A transformation translates points in shape K along vector $\vec{v}$ to create shape K'. Is this translation a rigid transformation?

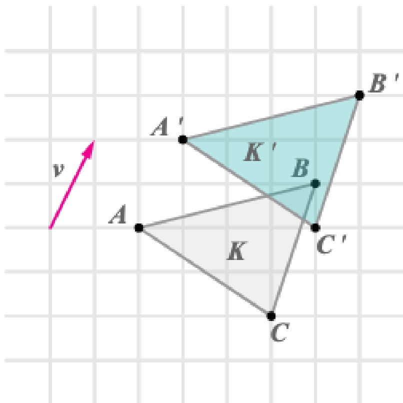

FIGURE 3.6

Yes, translations are rigid transformations. You can verify that the distances between the points are preserved.

MEDIA

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URL: http://www.ck12.org/flx/render/embeddedobject/218128

Examples

Example 1

You slide a book across your desk. You pour soda from a can into a big glass. Describe these actions as transformations.

Sliding a book across your desk is a rigid transformation because a book is a rigid object that does not change shape. The distances and angles that make up the book don't change once the book is in a new location. Pouring soda, on the other hand, is not a rigid transformation. Liquid is not a rigid object and it can change shape depending on its surroundings. The overall shape of the soda in the can will be different from the overall shape of the soda in the glass.

Example 2

A transformation rotates points in shape K around point D to create shape K'. Does this rotation look like a rigid transformation? Use algebra to prove your answer.

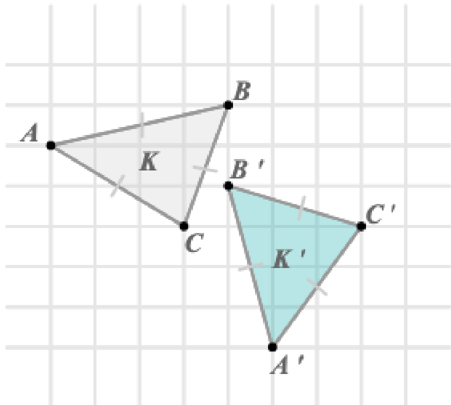

FIGURE 3.7

This appears to be a rigid transformation.

With the Pythagorean Theorem, you can show that the corresponding sides are the same length. For example:

The length of $\overline{AC}$
$= \sqrt{3^2+2^2}$
$= \sqrt{9+4}$
$= \sqrt{13}$ units

The length of $\overline{A'C'}$
$= \sqrt{3^2+2^2}$
$= \sqrt{9+4}$
$= \sqrt{13}$ units

Thus it is indeed a rigid transformation.

Example 3

What makes a transformation a rigid transformation?

Rigid transformations preserve distance and angles. All corresponding sides will be the same length and all corresponding angles will be the same measure.

Review

For 1-7, define each statement as true or false and justify your answer.

1. Translations are rigid transformations.
2. Rotations are rigid transformations.
3. Horizontal stretches are rigid transformations.
4. Rigid transformations preserve location in the plane.
5. Corresponding sides in rigid transformations are the same length.
6. If it's not a rigid transformation, it's not a real transformation.

7. Reflections are rigid transformations.

Use the following image for 8-9.

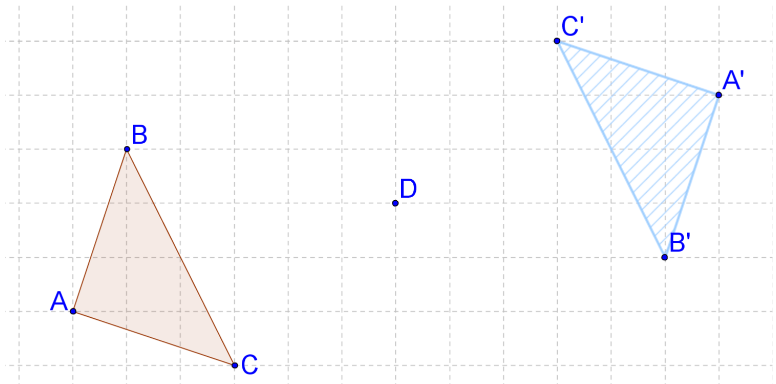

FIGURE 3.8

8. Describe the transformation in your own words. Does it look like a rigid transformation?

9. Prove your answer to #8 by comparing the lengths of two sides.

Use the following image for 10-11.

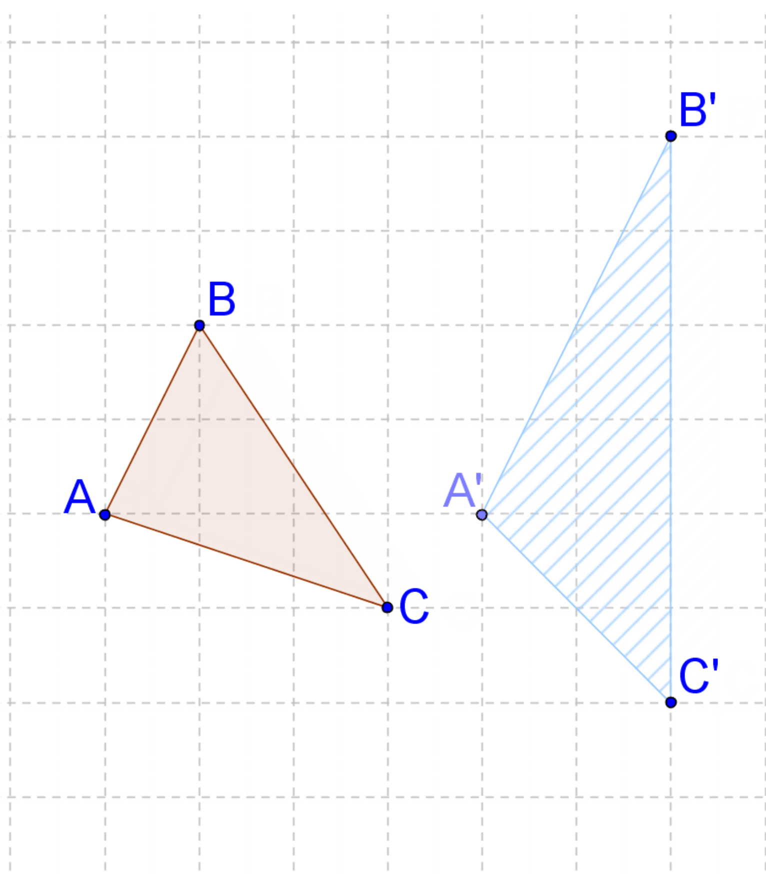

FIGURE 3.9

10. Describe the transformation in your own words. Does it look like a rigid transformation?

11. Prove your answer to #10 by comparing the lengths of two sides.

Use the following image for 12-13.

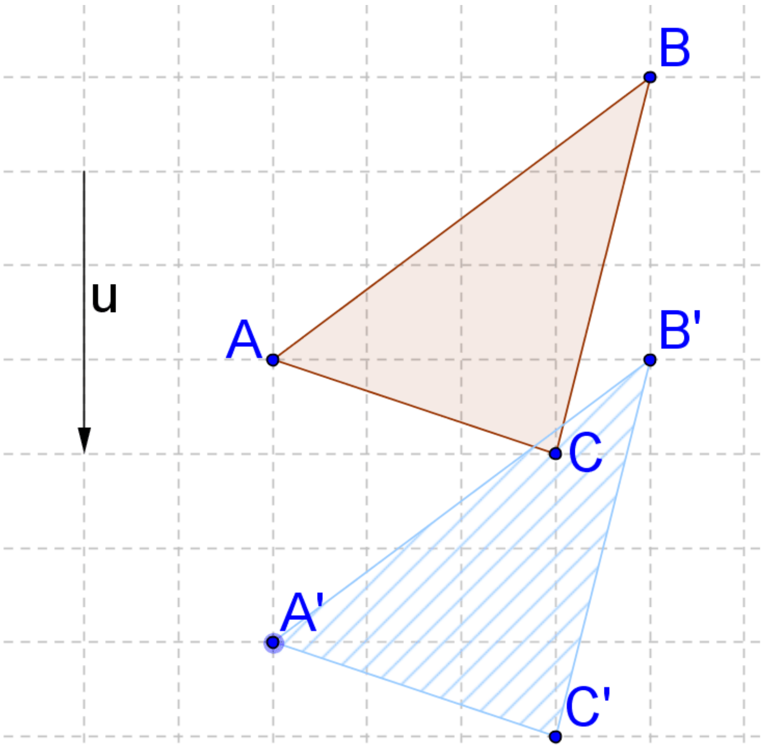

FIGURE 3.10

12. Describe the transformation in your own words. Does it look like a rigid transformation?

13. Prove your answer to #12 by comparing the lengths of two sides.

Use the following image for 14-15.

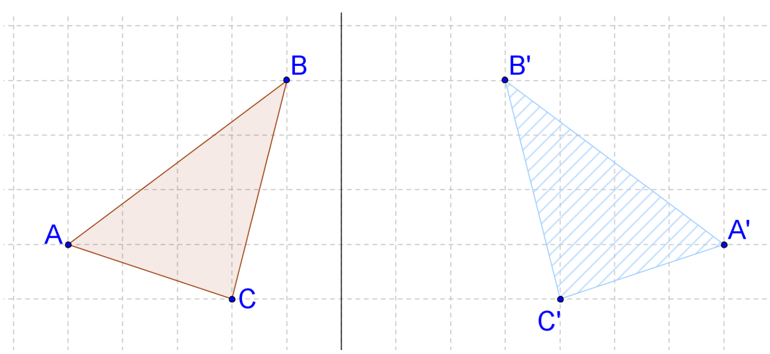

FIGURE 3.11

14. Describe the transformation in your own words. Does it look like a rigid transformation?

15. Prove your answer to #14 by comparing the lengths of two sides.

16. A transformation can be thought of as a movement of an object in a plane such that angle measures and segment lengths are preserved. Imagine a triangle that you've cut out, lying flat on a piece of poster paper. How can the triangle be moved so that the measures of angles and the lengths of segments are preserved? How can the different types of movements be categorized and defined? Be creative. Draw, write, and discuss how to more specifically describe and define the types and methods of rigid motion transformations.

17. There is one type of rigid motion transformation which seems to require the triangle described above to be lifted from the poster board and flipped. (Mathematically, that's not what happens, but it looks that way from our 3-dimensional perspective.) How can this transformation be defined and described?

18. Sometimes we can describe transformations in the coordinate plane. For example, one can specifically alter the coordinates of the vertices of a polygon according to a rule. Think about how you can modify the coordinates of a point and what the consequences of each type of modification would be. Experiment. Draw a polygon on a coordinate plane, and alter the coordinates according to different rules, then draw the resulting images. Write about and discuss your conclusions.

19. Define and describe the transformations that map A to A' to A" to A"' below.

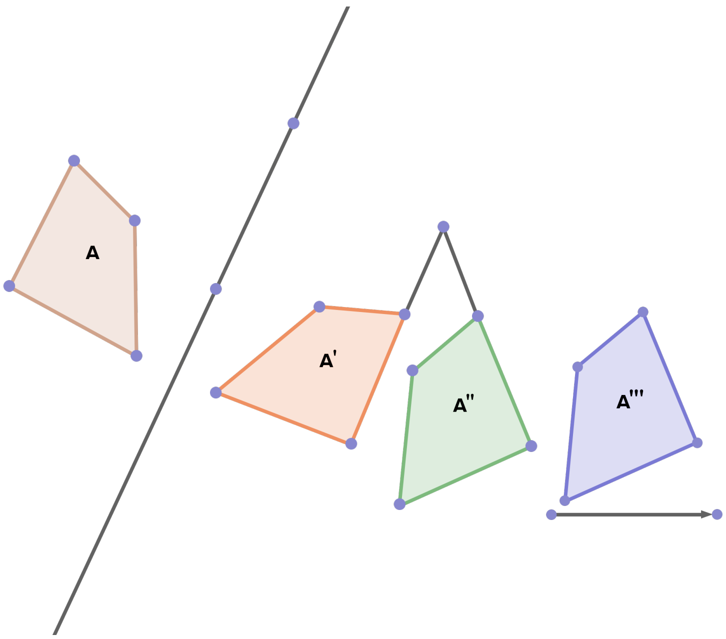

FIGURE 3.12

20. Some of the images below represent rigid motion transformations of polygon B and some do not. Decide which do and which don't, and explain your choices. Define and describe the ones that do.

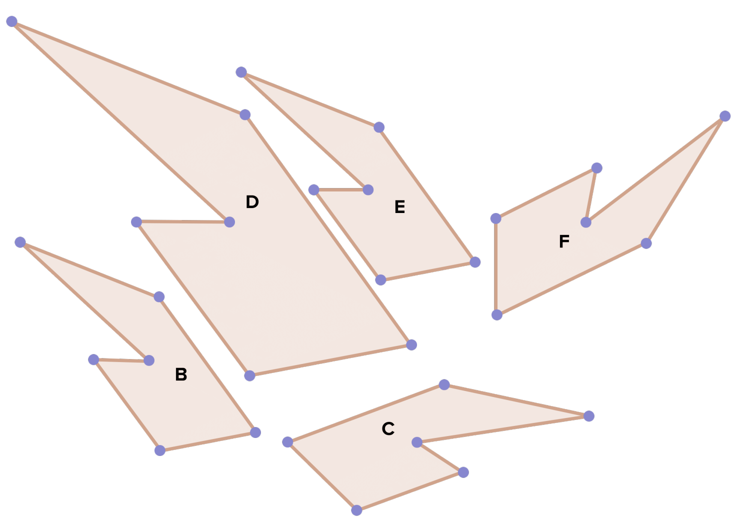

FIGURE 3.13

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.1.

3.2 Translations

Learning Objectives

Here you will learn about translations.

Translations

In Latin, the word "translate" means "carried across". A translation moves ("slides") an object a fixed distance in a given direction, as defined by a vector.

- Translations do not change the size or shape of an object, and do not rotate or flip it.
- The original object is called the pre-image, and the translation is called the image.
- An object and its translation have the same shape and size, and face in the same direction.
- A translation is one example of a rigid transformation.

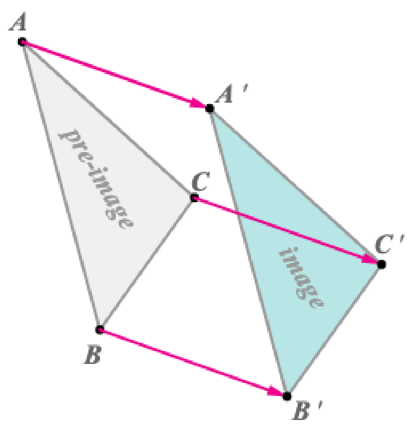

FIGURE 3.14

Below, the parallelogram $ABCD$ has been translated along vector $\vec{v}$ to create a new parallelogram $A'B'C'D'$ (the image).

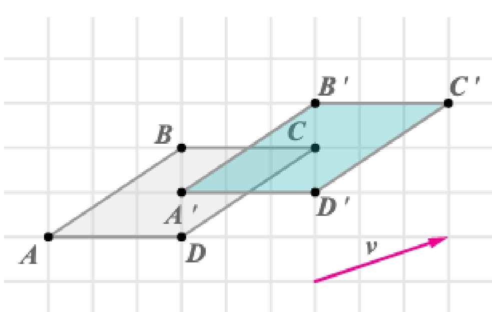

FIGURE 3.15

- Note that the location of vector $\vec{v}$ does not matter, but the slope (the angle of the vector compared to the grid) does matter.
- Vectors have a direction and a magnitude (a length). They can be thought of as directions for moving points from preimage to image.
- Vector $\vec{v}$ essentially tells you that all points move three units to the right and one unit up.

The lines that connect corresponding points will all be parallel to vector $\vec{v}$.

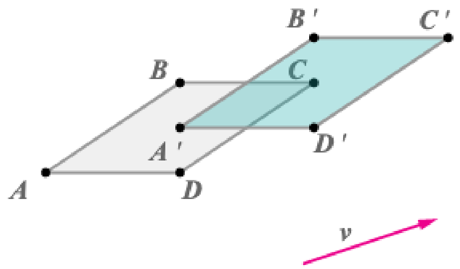

FIGURE 3.16

The slope of vector $\vec{v}$ and each line is $\frac{1}{3}$. This should make sense because each point in the original parallelogram was moved 3 units to the right and 1 unit up to create its corresponding point in the image.

One way to think about translations is that they move points a specified distance along lines parallel to a given line. In this case, all points were moved a distance of $\sqrt{10} = \sqrt{3^2 + 1^2}$ (found using the Pythagorean Theorem) along lines parallel to vector $\vec{v}$.

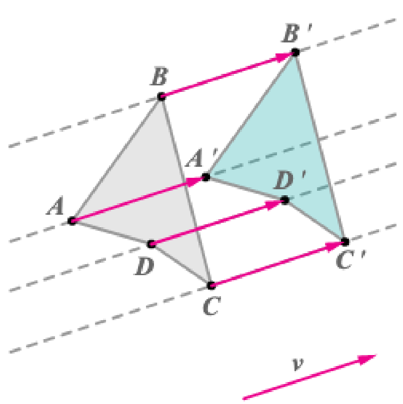

FIGURE 3.17

Notice that lines parallel to vector $\vec{v}$ have been drawn through each of the original points. Vector $\vec{v}$ has been copied onto each of those lines at the points that define the original quadrilateral. The ends of the vectors define the points on the image.

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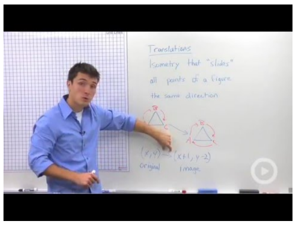
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/223125



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/71062

Recognizing Translations

Is the following transformation a translation?

One way to check if a transformation is a translation is to look at how each point moves to create its image. If all points are not moving in the same way, then it is not a translation.

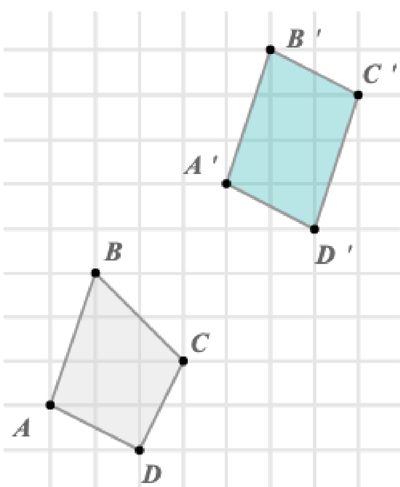


FIGURE 3.18

TABLE 3.1:

Point to Image Point	**Description of Motion**
A to A'	4 to the right and 5 up
B to B'	4 to the right and 5 up
C to C'	**4 to the right and 6 up**
D to D'	4 to the right and 5 up

Because C to C' is different, this is not a translation.

Is the following transformation a translation?

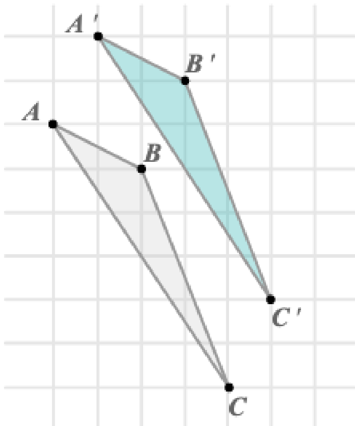


FIGURE 3.19

The above transformation is _____.

CK-12 PLIX: Translation Notation

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/translation-notation/plix/Moving-a-Truck-56a117dcda2cfe6cbc2ee919

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/257125

Describing a Translation Vector

Describe the vector that defined the translation below.

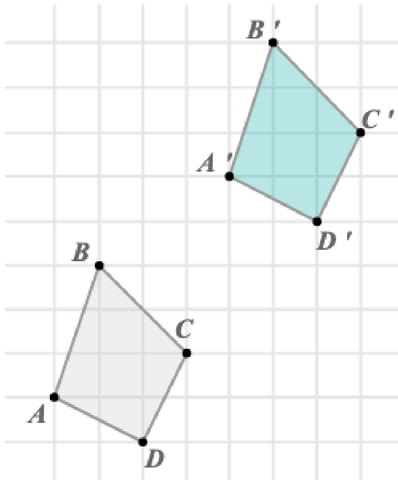


FIGURE 3.20

The vector moved each point four units to the right and five units up.

Describe the vector that defined the translation below.

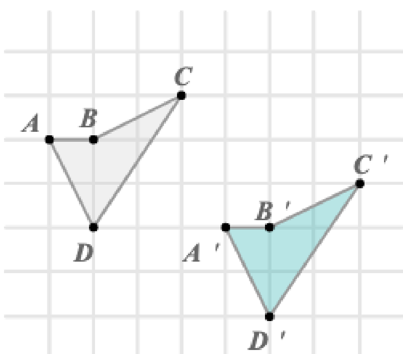


FIGURE 3.21

The vector moved each point _____.

Performing Translations

Observe how the translation defined by vector $\vec{u}$ is performed on the quadrilateral below.

With the grid in the background, you can see that vector $\vec{u}$ tells you to move each point 2 units up and 1 unit to the left.

Here is the translation:

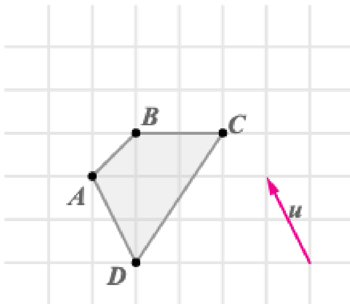


FIGURE 3.22

Translation

- Use the dropdown selector to select a shape to translate to a new location.
- Explore different distances and directions of translation by clicking or tapping the blue arrow and then dragging any of the orange vertices of the shape that appear.

Shape: _____

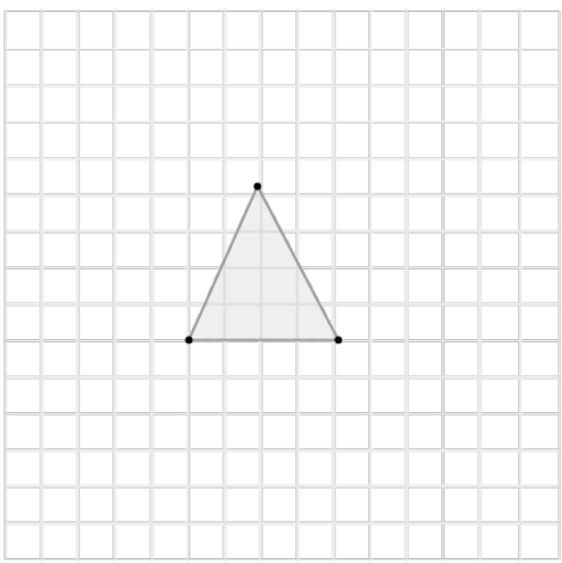


FIGURE 3.23

Examples

Example 1

Translations are often informally called "slides". Why is this?

A translation is informally called a slide because it essentially slides a shape to a new position. The orientation and relative position of the points does not change.

Example 2

Perform the translation defined by vector t ($\vec{t}$) on the triangle below.

Since $\vec{t}$ indicates 1 unit to the right and 3 units up, each of the points A, B, and C, should be translated by that amount.

Hover or tap translation to see the translation in action.

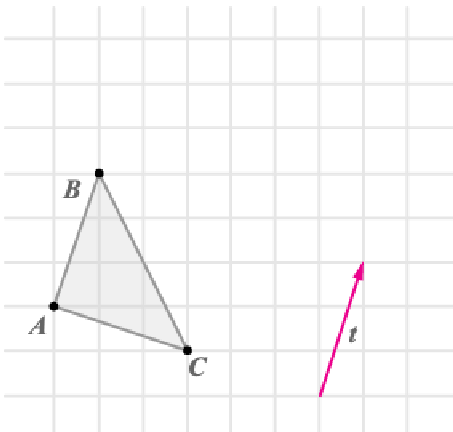


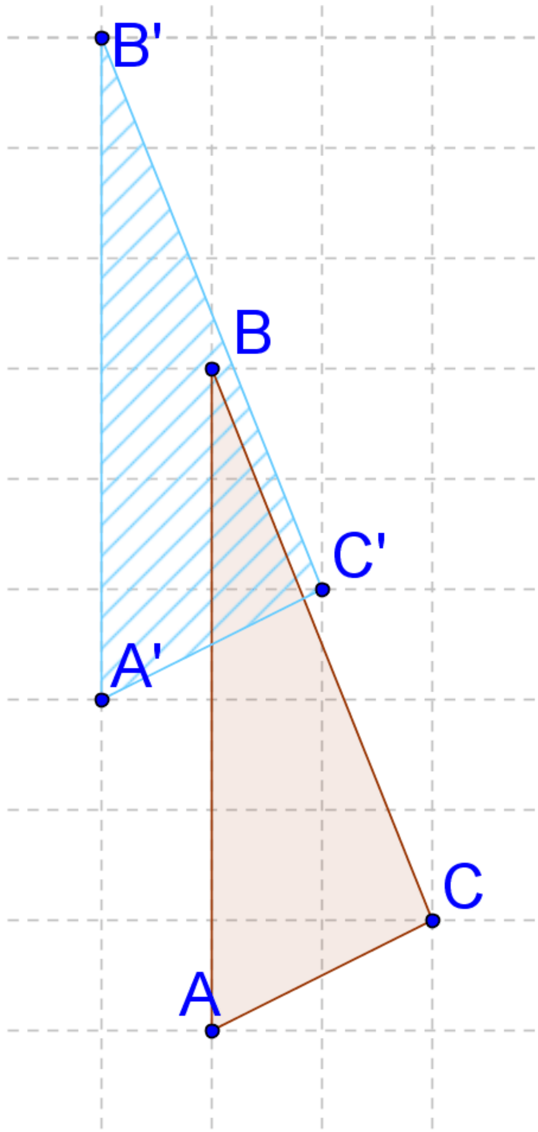
FIGURE 3.24

Review

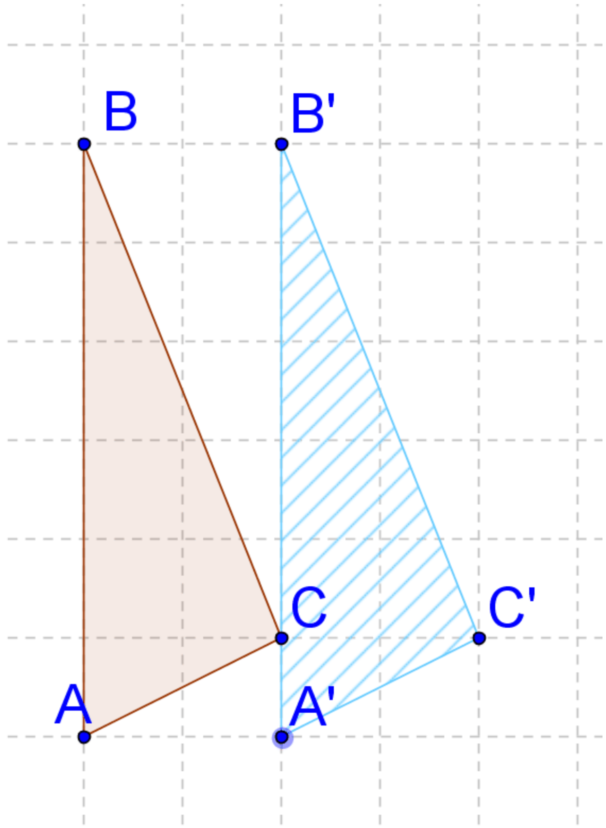
1. Is a translation a rigid transformation? Explain.
2. What role does a vector play in a translation?
3. How are parallel lines relevant to translations?
4. How can you tell if a transformation is a translation?

Describe the vector that defined each of the following translations.

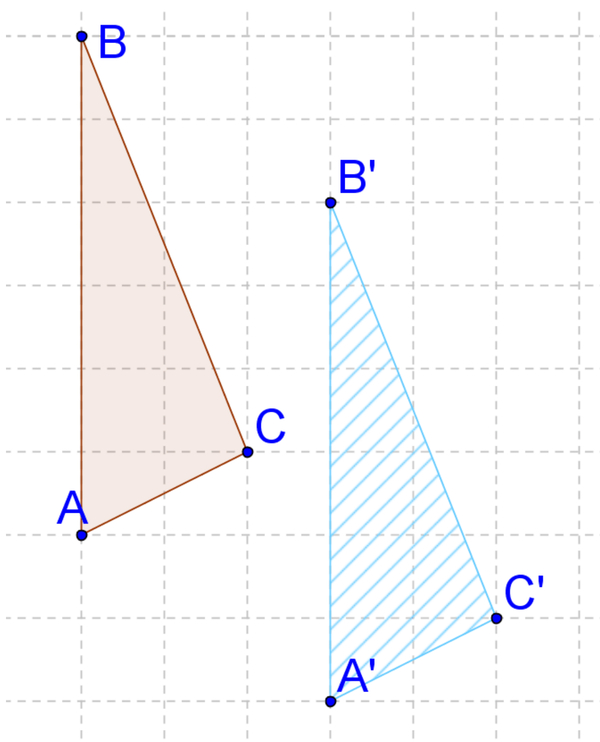
5.



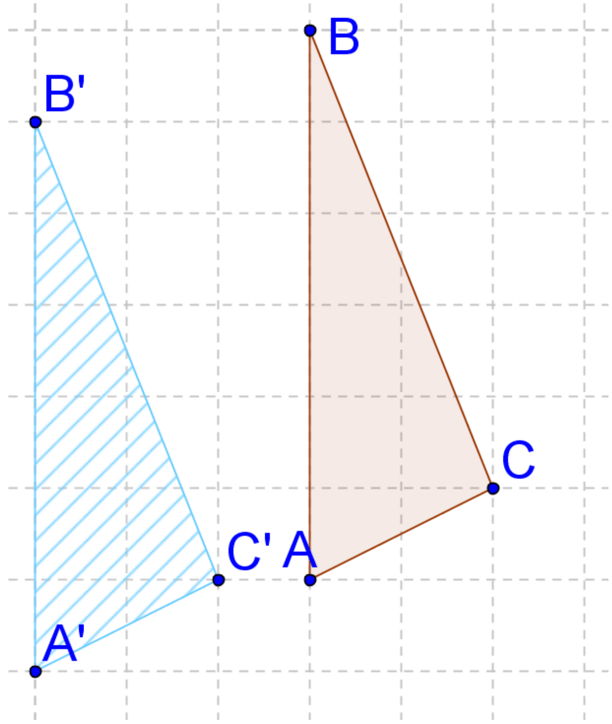
6.



7.

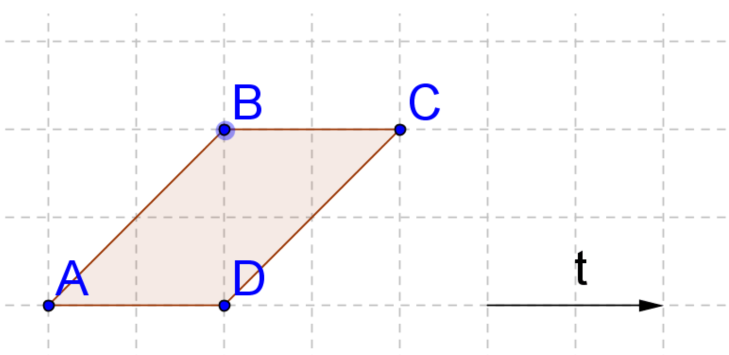


8.



Perform the translation defined by vector $\vec{t}$ on the polygons below.

9.



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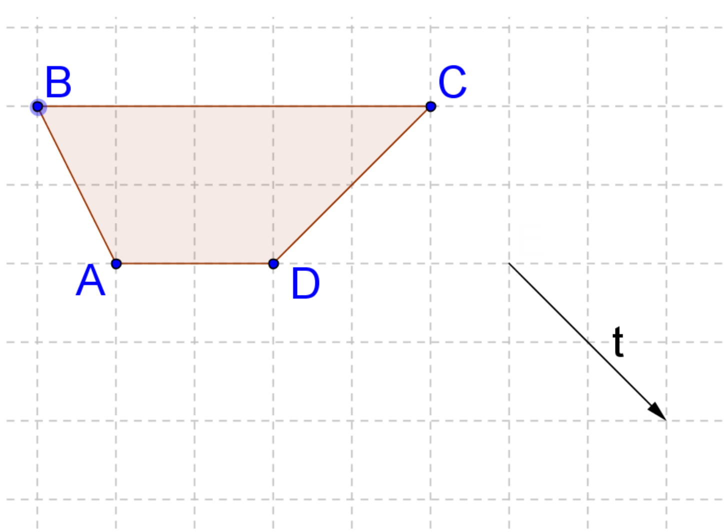
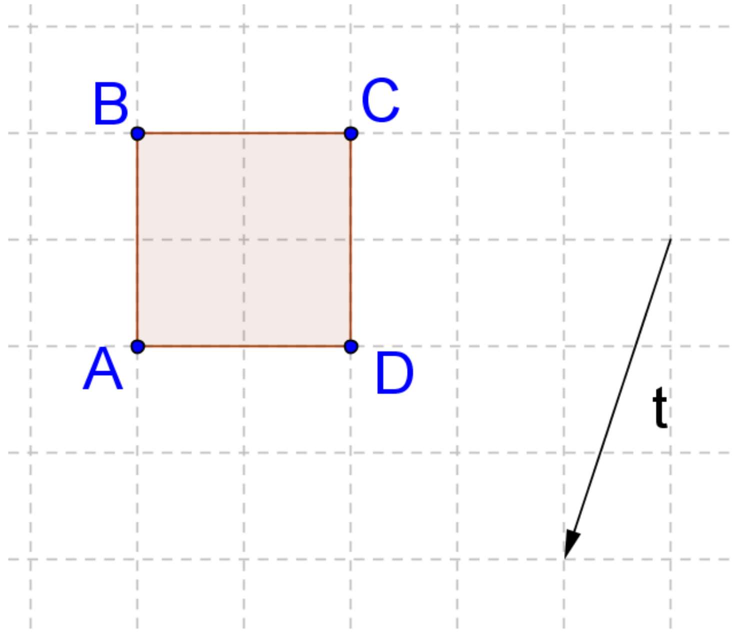
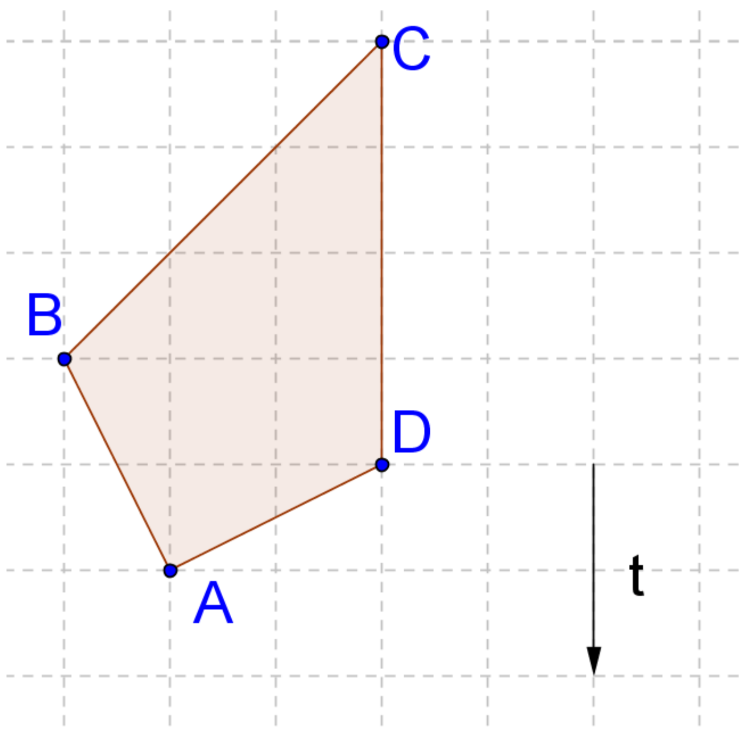


11.

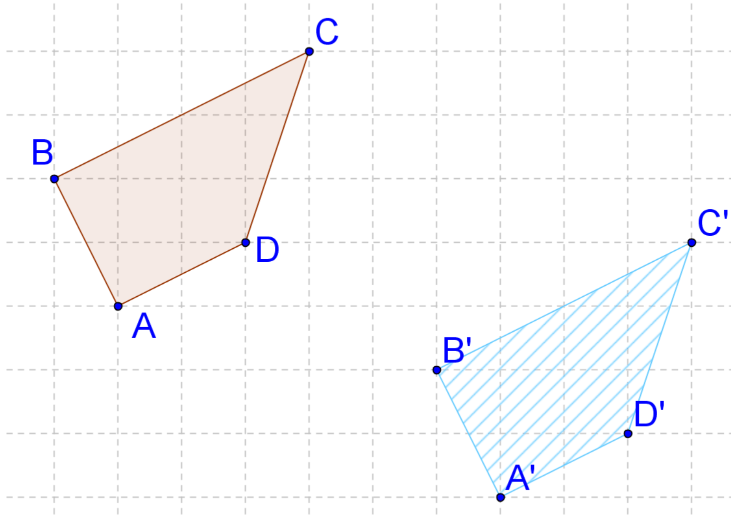


12.

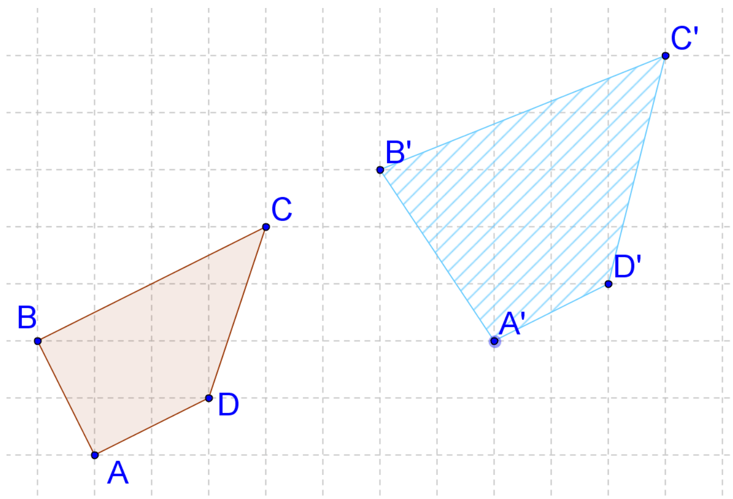


Are the following transformations translations? Explain.

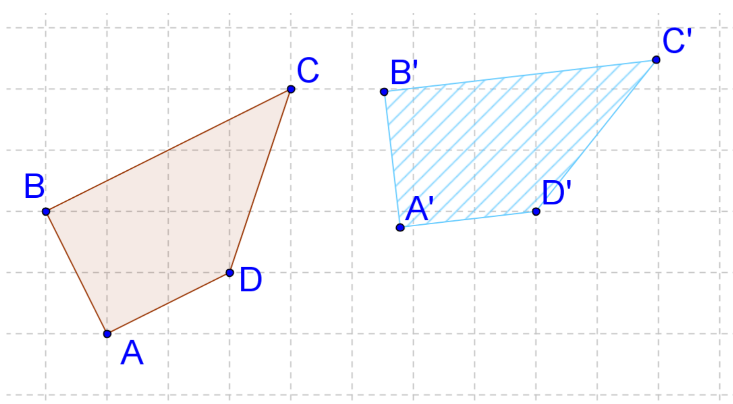
13.



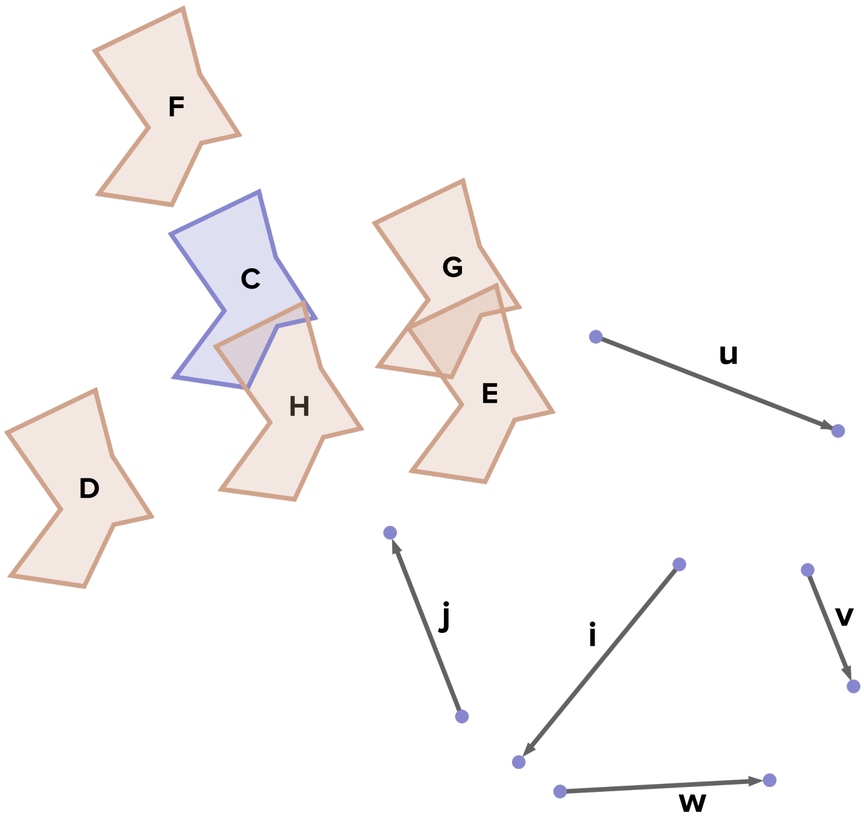
14.



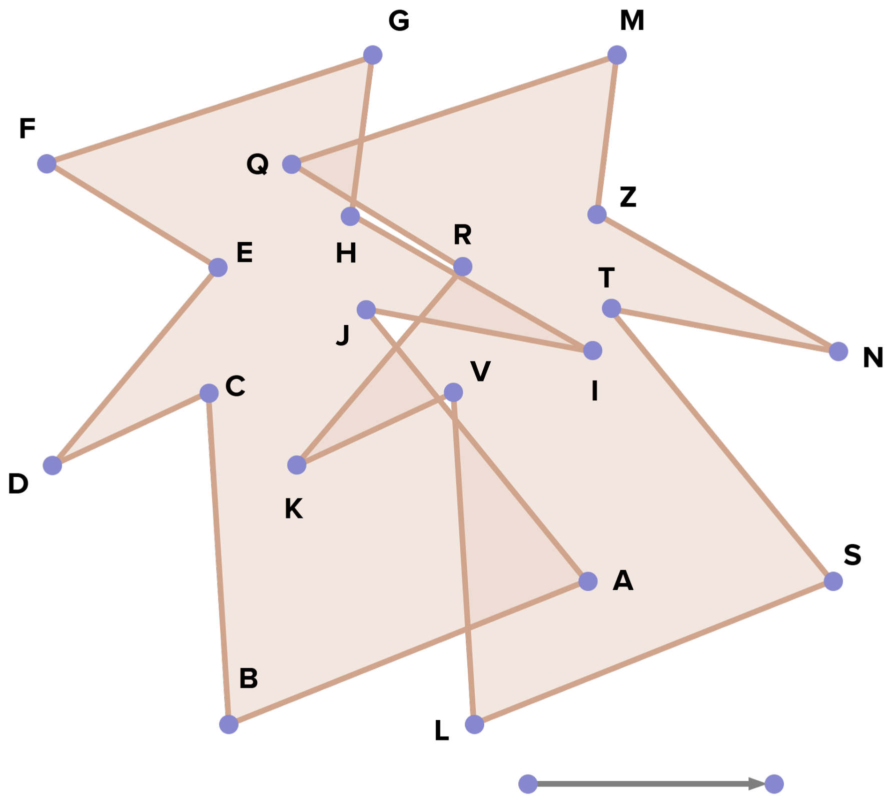
15.



16. A translation moves an object along a vector. A vector is a directed line segment that tells us how far to move the object, and in what direction. It's the signpost that tells the object where to go. In the diagram below, there are several vectors which are independently used to translate polygon C. Match each vector with the correct image. Explain your results.



17. For visualizing translations of polygons, it can be useful to connect vertices of the original polygon to the corresponding vertices of the image by drawing a copy of the vector used for translation. It is not necessary to connect all of the corresponding vertices, just one pair is usually sufficient. Which two points would you connect to visualize the translation below? Why did you choose those two points? What would be your second choice of corresponding points? Why?



18. Sketch a line. Sketch a vector such that the image of the line after translation by the vector is the original line. Does a point on the original line map to itself? Does the line map to itself? Explain.

19. Sketch a line. Sketch a vector such that the image of the line after translation is not the same as the original line. What relationship do these lines have? Why?

20. Sketch a circle. Translate the circle by a vector that maps only one point on the original circle to a point on the new circle.

21. The point $(2,3)$ is translated by a vector such that its image is $(5,4)$. This image is then translated so that *its* image is $(7,9)$. Is there one single vector that would accomplish these two translations in one step? Describe it in terms of its horizontal and vertical components.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.2.

3.3 Geometry Software for Translations

Learning Objectives

Here you will learn how to perform translations using geometry software.

Translation with Grid

Recall that a translation is one example of a rigid transformation. A translation moves each point in a shape a specified distance in a specified direction as defined by a vector. Below, the parallelogram $ABCD$ has been translated along vector $\vec{v}$ to create new parallelogram $A'B'C'D'$ (the image).

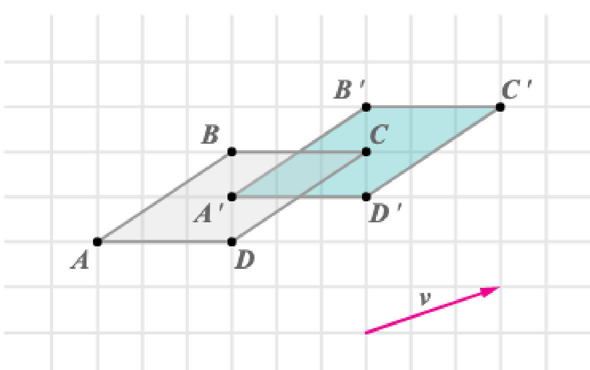


FIGURE 3.25

Translation Without a Grid

When you are working on a grid (or graph paper), performing translations is relatively easy. In the parallelogram translation, you can see that the vector moves each point one unit up and three units to the right. To perform the translation, just move each point that defines the parallelogram one unit up and three units to the right. What if the grid is not there? Then, it is not as easy to describe the translation because there are no grid lines as a guide.

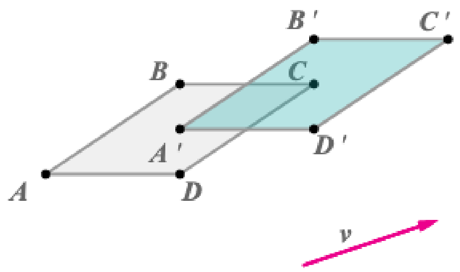


FIGURE 3.26

To perform a translation without a grid, you need to:

1. Find lines parallel to the vector through each of the points that define the shape.
2. Move the length of the vector along each of those lines.

Doing this by hand requires careful construction of parallel lines and copying of line segments. Geometry software simplifies this process, because it commonly has a "translate" tool.

Translation using Geometry Software

The interactive below is an example of geometry software that is free to download. The process is detailed step-by-step with pictures just below the interactive.

In summary, you will:

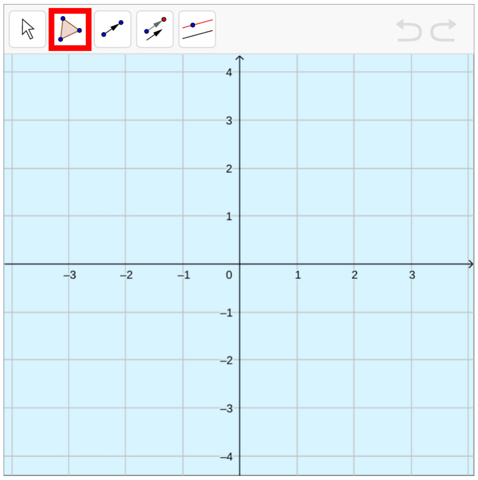
1. Create a polygon with the shape tool (button with a picture of a triangle)
2. Create a vector with the vector tool (button with an arrow between two points)
3. Translate the polygon with the translate tool (button with two parallel arrows)

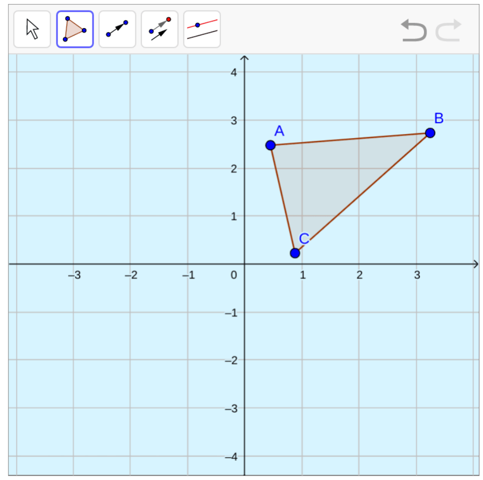
MEDIA

Click image to the left or use the URL below.

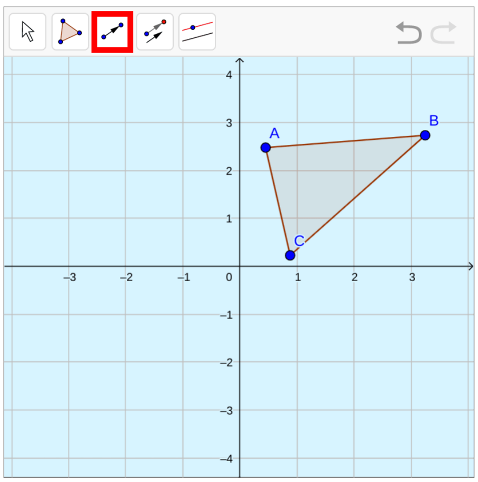
URL: http://www.ck12.org/flx/render/embeddedobject/223142

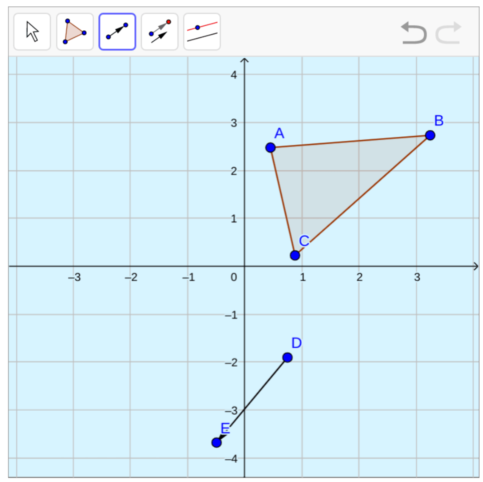
First, select the "polygon" tool. Create a polygon to translate by clicking on the canvas where you want each vertex to be, click the first point again to complete the shape.



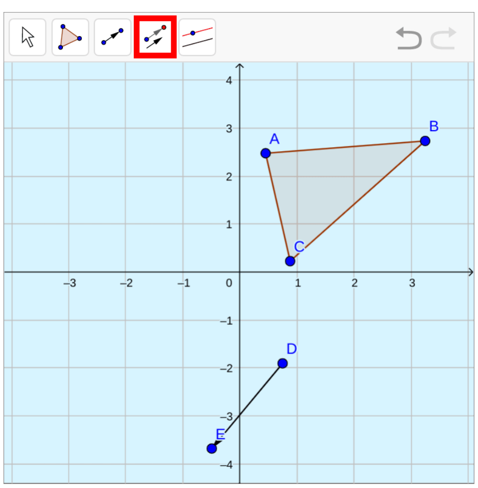


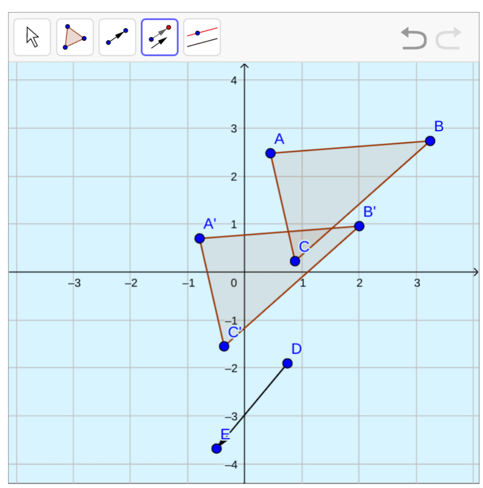
Next, create the vector that will define your translation. Choose the "vector" tool from the menu and click two places on the canvas to create the vector.



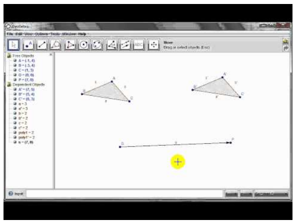


To perform the translation, select the "translate" tool in the menu. Click the polygon first and then click the vector to complete the translation.





Note that the points defining the image are labeled with prime notation. Remember that the specific location of the vector on the graph does not matter. At this point you can drag the vector to another location and your translation will not change. You can also move points D or E to redefine the vector (or change your original polygon) and the location of your image will move accordingly.



MEDIA

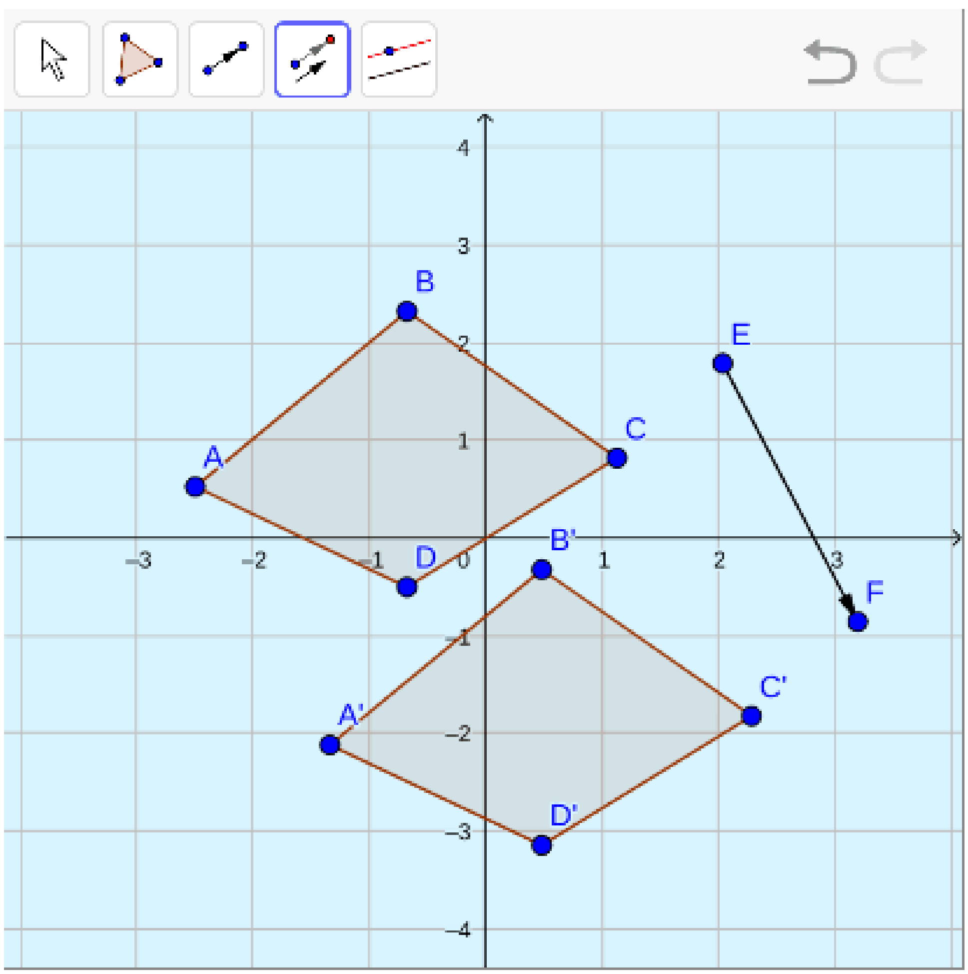
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/71064

Examples

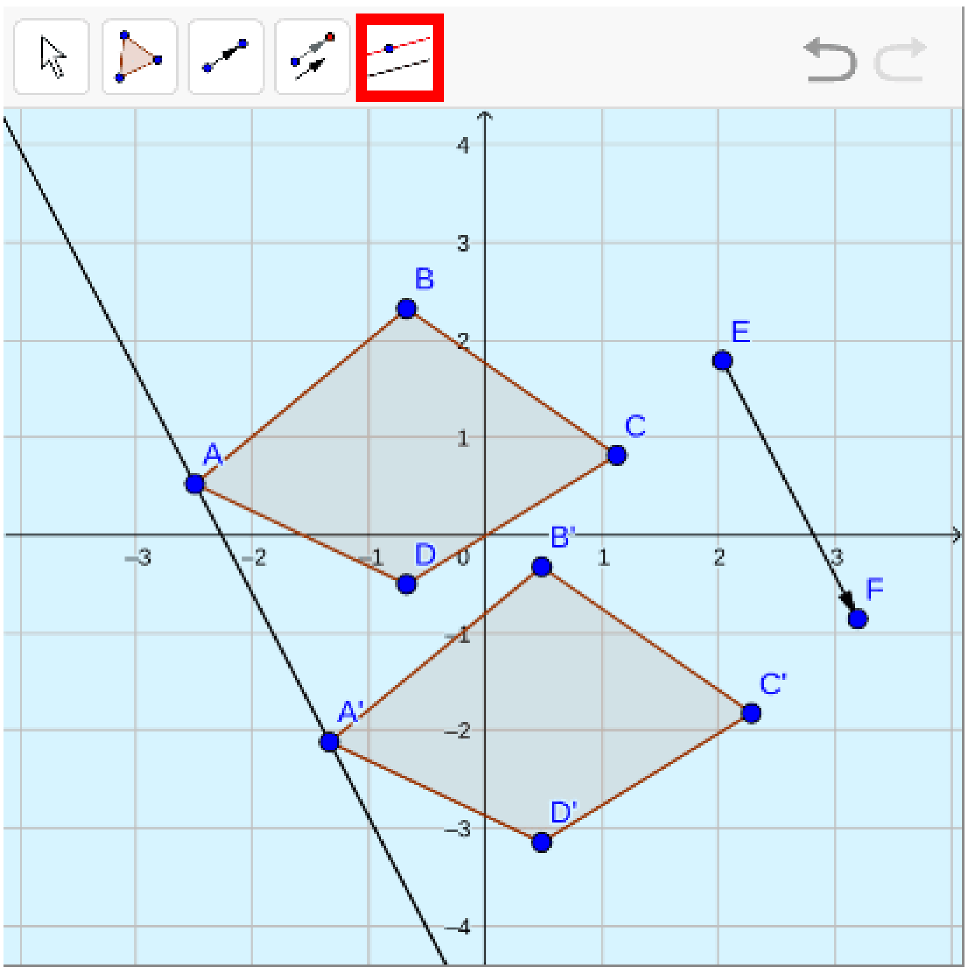
Example 1:

Quadrilateral $ABCD$ is translated along vector $\overrightarrow{EF}$ to create quadrilateral $A'B'C'D'$. Show that the segment connecting A with A' is parallel to the vector $\overrightarrow{EF}$.



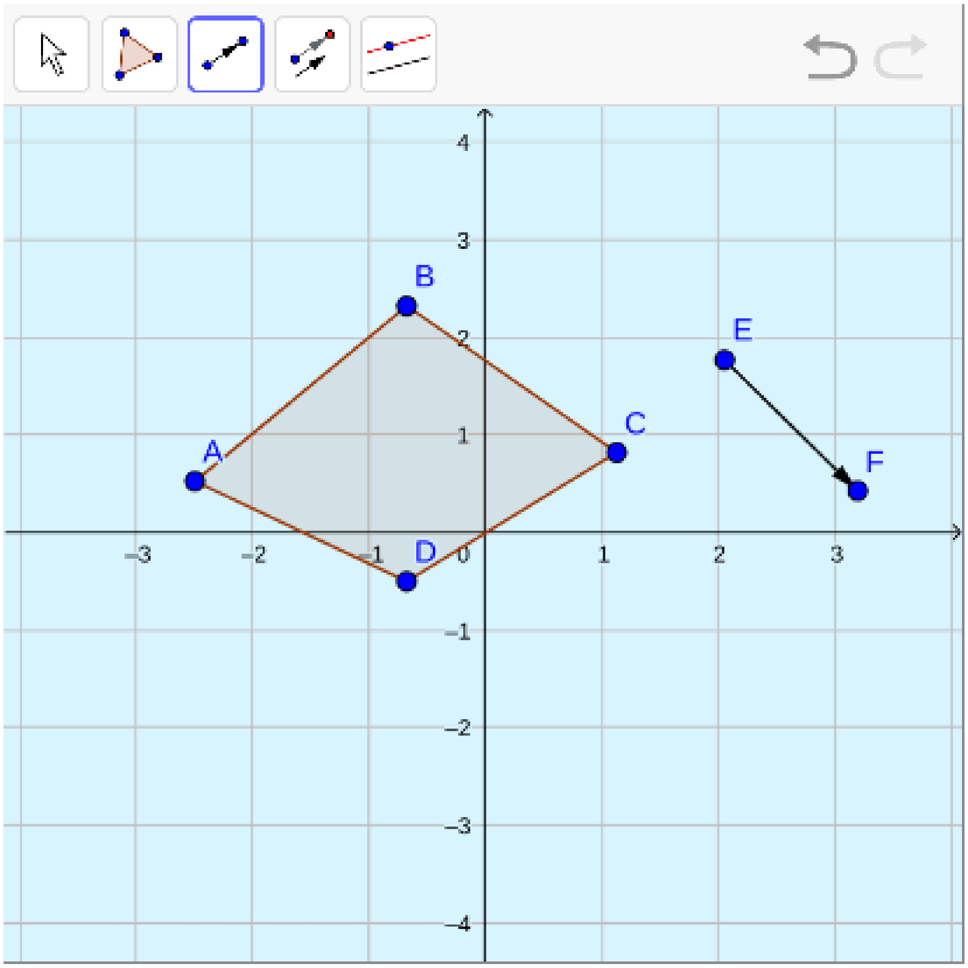
- Follow the steps in the interactive activity above to create and translate a quadrilateral.

- In the interactive geometry software, select the "parallel line" tool in the menu. Click on the point A, then click on the vector $\overrightarrow{EF}$ to construct a line parallel to the vector $\overrightarrow{EF}$ through point A. You can see that it also passes through A'. This means the line segment connecting A with A' is parallel to the vector $\overrightarrow{EF}$.



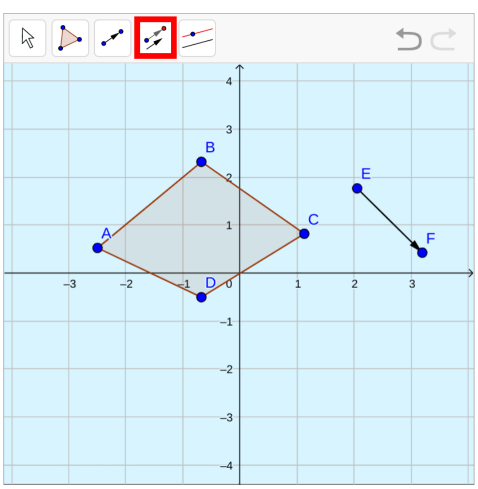
Example 2:

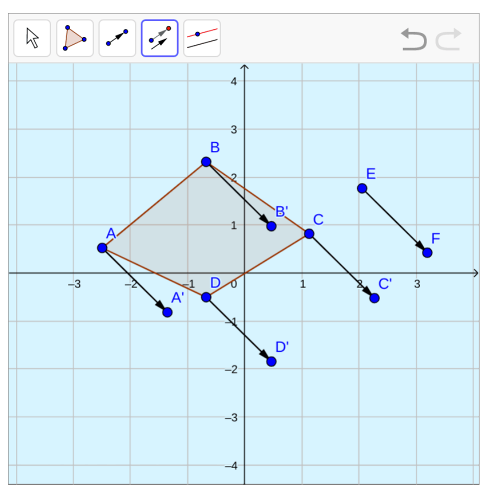
Use the "translate" tool to independently translate *each vertex* of the quadrilateral $ABCD$ along vector $\overrightarrow{EF}$.



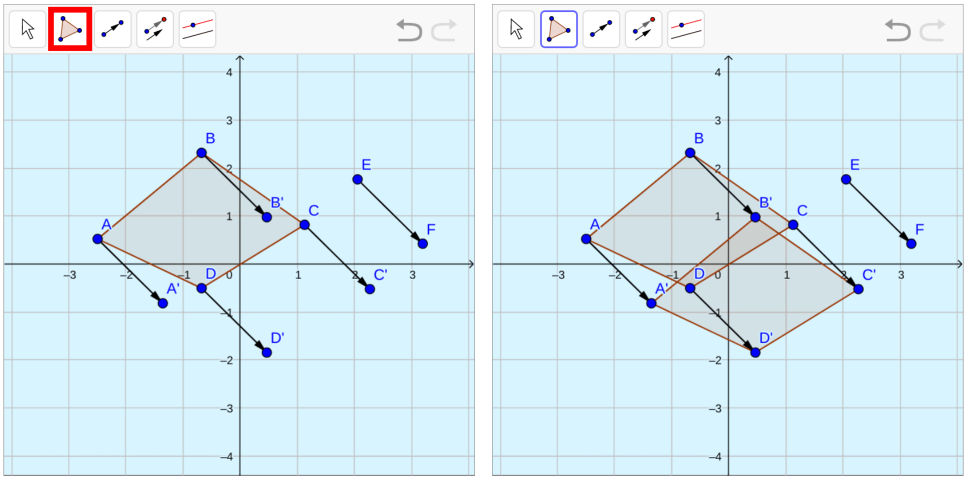
(Note - This example illustrates how to use the interactive geometry software to perform the same steps that the "translate" tool uses to translate an object.)

- Select the "translate" tool in the menu. Click on vertex A of quadrilateral $ABCD$, then click on vector $\overrightarrow{EF}$, to create an identical vector from point A.
- Repeat for vertices B, C and D.

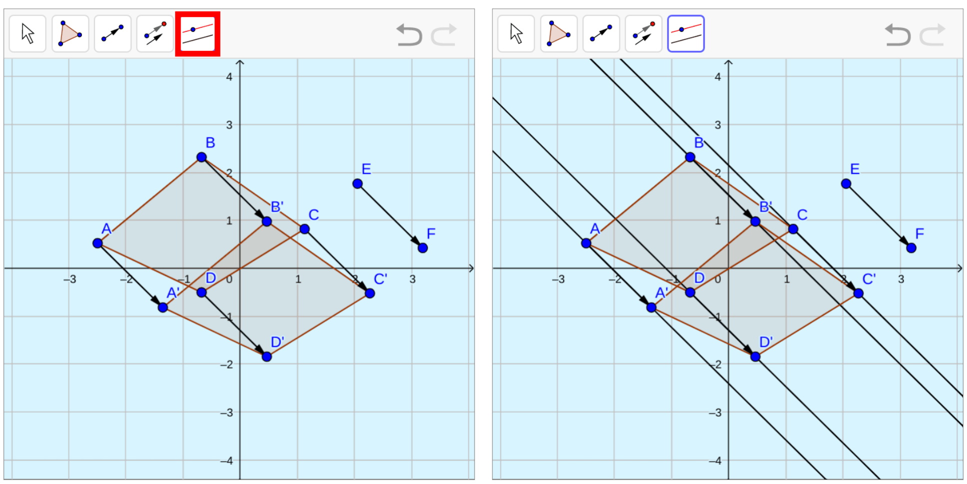




Next, select the "polygon" tool and start making your polygon at the point A'. Continue to the points B', C' and D' to complete the translation of the original polygon.



Use the "parallel line" tool to create a line through vertex A, which is parallel to the vector $\overrightarrow{EF}$. Repeat for vertices B, C and D. Using these lines, you can check that the vectors are parallel and contain both the original and translated point.



Now compare and contrast the two methods for translating a polygon using the interactive geometry software.

- Using the first method is faster, but performing the steps of constructing the parallel vectors also works and helps to show the connection between parallel lines, vectors, and translations.
- In both cases you can move the vector or the original shape after the translation has taken place and the image will change and move accordingly.

Review

For questions 1-8, you can use the interactive from the 'Translation Using Geometry Software'. The remaining questions require access to the full suite of tools commonly available in free online or downloaded software.

1. Create a polygon using geometry software.

2. Create a vector that will move the polygon to the right.

3. Translate the polygon along the vector from #2 using the translate button.

4. Create another vector that will move the polygon up and to the left.

5. Translate the polygon along the vector from #4 using the translate button.

6. Create a third vector that will move the polygon straight down.

7. Translate the polygon along the vector from #6 by constructing parallel vectors and creating the polygon at their endpoints.

8. Check your work to #7 by using the translate button to translate the polygon along the vector from #6. How can you tell if you performed the translation correctly?

9. Translations are rigid transformations which means that distance is preserved. Verify that distance has been preserved by using interactive geometry software to measure sides of your original polygon and their images. *Select "distance or length" from the menu. Then, click on each line segment that you want to measure to see its length.*

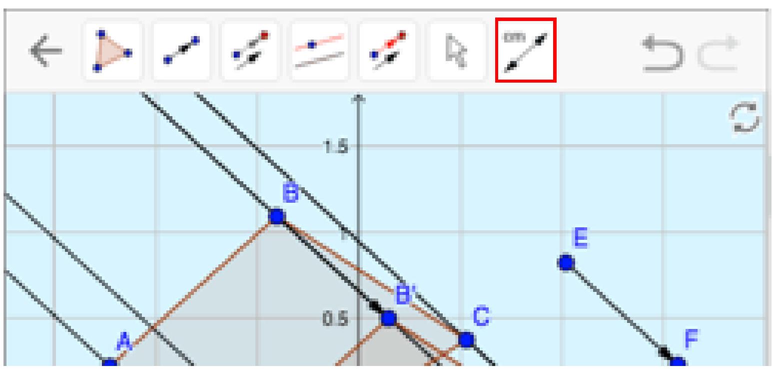


FIGURE 3.27

10. Translations are rigid transformations which means that angles are preserved. Verify that angles have been preserved by using interactive geometry software to measure two corresponding angles. *Select "angle" from the same menu as in #9. Then, tell the interactive geometry software what angle you want to measure by clicking on the three points you would use to name the angle. You must click on the points in clockwise order for it to measure the correct angle.*

11. Construct a circle using the interactive geometry software.

12. Create a vector that will move the circle to the right.

13. Translate the circle along the vector from #12 using the translate button.

14. Create another vector that will move the circle up and to the left.

15. Translate the circle along the vector from #14 using the translate button.

16. Explore how you might be able to translate the circle without using the translate button by creating parallel lines and copying and pasting the vector.

17. Create a polygon using the interactive geometry software. Create several vectors. Transform the original polygon by one of the vectors. Translate the image by another vector and so on until you have a final image resulting from a sequence of translations. Find a vector which generates the same final image from the original in one translation. Find a vector which translates the final image back to the original. (Note that you can create vectors which join corresponding points on a polygon.) Explain the results.

18. Create a triangle using the interactive geometry software. Translate the triangle along 3 vectors equivalent in length and direction to the sides of the triangle. (Note that you can create vectors which are coincident with the sides of the triangle.) Explain the results.

19. Create a regular polygon with the interactive geometry software. (It's in the dropdown menu for creating a polygon.) A popup window will allow you to enter the number of sides. Enter the correct number to create a hexagon. Translate the hexagon such that the image intersects with the original in exactly one complete side of the original. Do the same to create the 5 images which coincide exactly with the remaining 5 sides. Describe the result. Notice that none of the polygons overlap and there are no gaps. Why is this possible?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.3.

3.4 Reflections

Learning Objectives

Here you will learn about reflections.

A reflection is one example of a rigid transformation. A reflection over line is a transformation in which each point of the original figure (the pre-image) has an image that is the same distance from the reflection line as the original point, but is on the opposite side of the line. In a reflection, the image is the same size and shape as the pre-image. A reflection across line l moves each point P to P' such that line l is the perpendicular bisector of the segment connecting P and P'.

In other words, the line segment connecting a point in the pre-image to the matching point in the image is split in half by the mirror line.

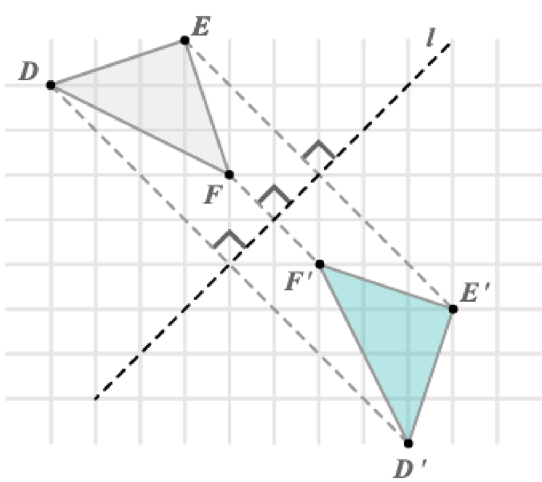


FIGURE 3.28

Reflecting Shapes

Below, the quadrilateral $ABCD$ (the pre-image) has been reflected across line l to create a new quadrilateral, $A'B'C'D'$ (the image).

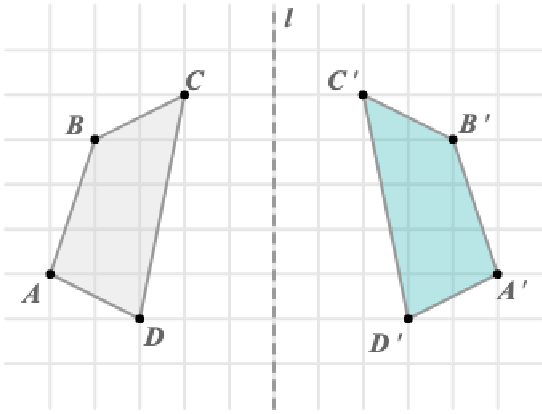


FIGURE 3.29

Reflections move all points of a shape across a line called the line of reflection (sometimes informally called the mirror line). A point and its corresponding point in the image are each the same distance from the line.

Notice that the segments connecting each of the corresponding points are all perpendicular to line l. Because each point and its corresponding point are the same distance from the line, line l bisects each of these segments. This is why line l is called the perpendicular bisector for each of these segments.

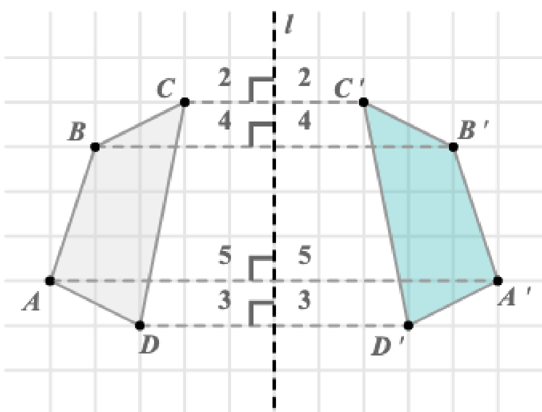
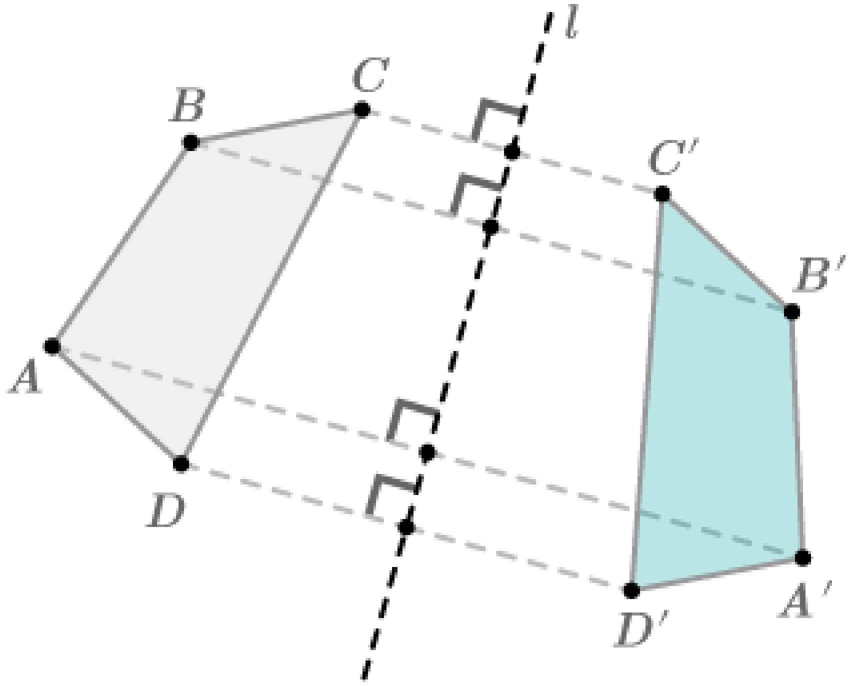


FIGURE 3.30

Keep in mind that you can perform reflections even when the line of reflection is "slanted" or the grid is not visible; however, it is much harder to do by hand.



Reflection

Select the direction in which you want to reflect the shape. Play with different shapes from triangle up by adding or removing points. Explore the reflection of various shapes by dragging any of the points.

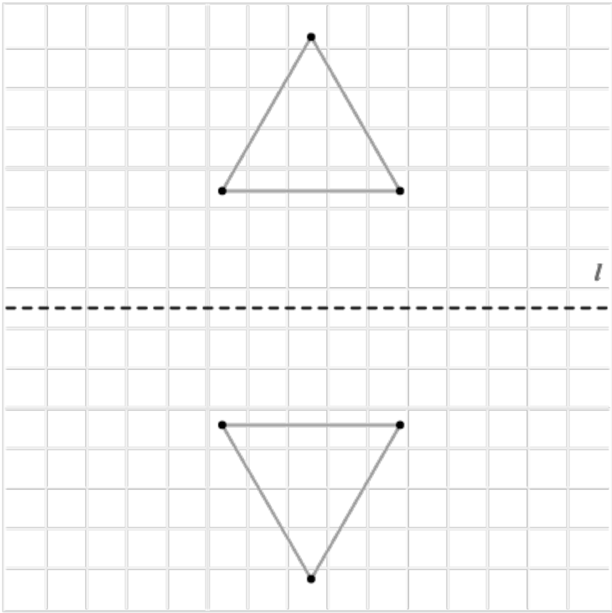


FIGURE 3.31

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/rigid-transformations/plix/The-Transformation-56c36bba8e0e0812a54cfb60

Determining the Line of Reflection

Observe the line of reflection that created the reflected image below.

The fact that point D is in the same location as point D' tells you that the line of reflection passes through point D. Imagine folding the graph so that each point on the original parallelogram matched its point on the image. Where would the fold be?

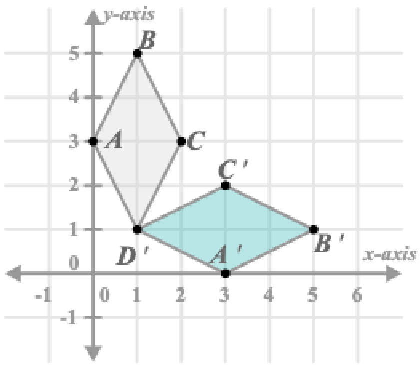


FIGURE 3.32

The line of reflection is the line $y = x$. When reflections are performed on graph paper with axes, you can define the lines of reflections with their equations.

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/223146

Determining Reflection

1. Is the following transformation a reflection?

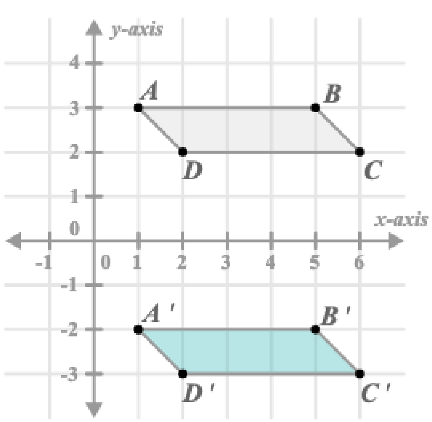


FIGURE 3.33

Even though overall both the parallelogram and its image are 2 units from the x-axis, each individual point is not the same distance from the x-axis as its corresponding image.

- Point A is 3 units from the x-axis and point A' is 2 units from the x-axis.
- The line of reflection for points A and A' would be the line $y = \frac{1}{2}$, which is not the same line of reflection for points D and D'.
- This is not a reflection (it is a translation).

2. Perform the reflection across line l.

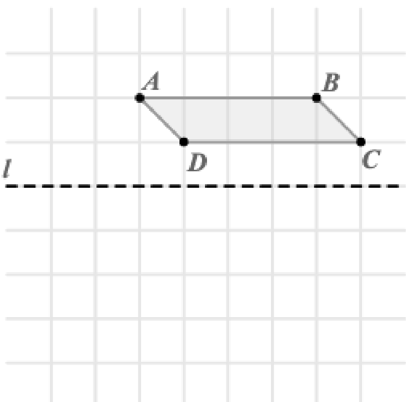


FIGURE 3.34

- Draw a perpendicular line from each point that defines the parallelogram to line l.
- Count the units between each point and line l along the perpendicular lines.
- Count the same number of units on the other side of line l along the perpendicular lines to create the image.

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/223147

Examples

Example 1

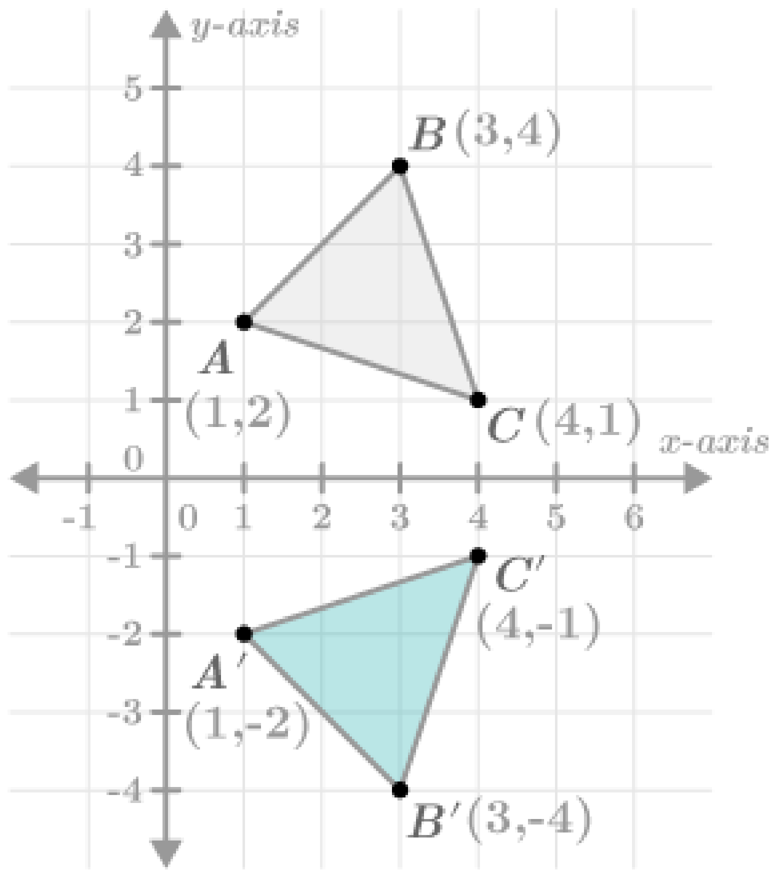
Reflections are often informally called "flips". Why is this?

A reflection is informally called a “flip” because it’s as if you are flipping the shape over the line of reflection.

Example 2

Consider the reflection of a triangle across the x-axis.

What do you notice about the coordinates of each point and its image when a reflection is performed across the x-axis?



The x-coordinate of each point and its image are the same, but the y-coordinate has changed sign. You could describe this as $(x,y) \rightarrow (x,\text{-}y)$.

Example 3

Describe the line of reflection that created the reflected image below.

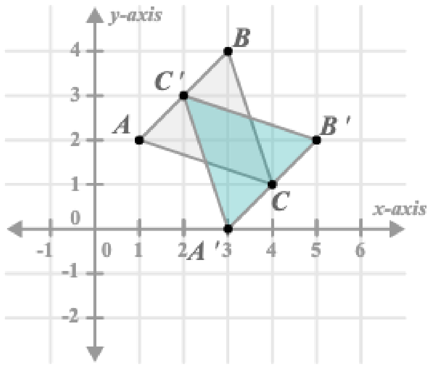


FIGURE 3.35

Connect A with A'. This line segment has a slope of -1 and a midpoint at (2, 1).

The line of reflection is the perpendicular bisector of this segment. This means it passes through its midpoint and has an opposite reciprocal slope $\left(-\frac{1}{1} \rightarrow +\frac{1}{1}\right)$.

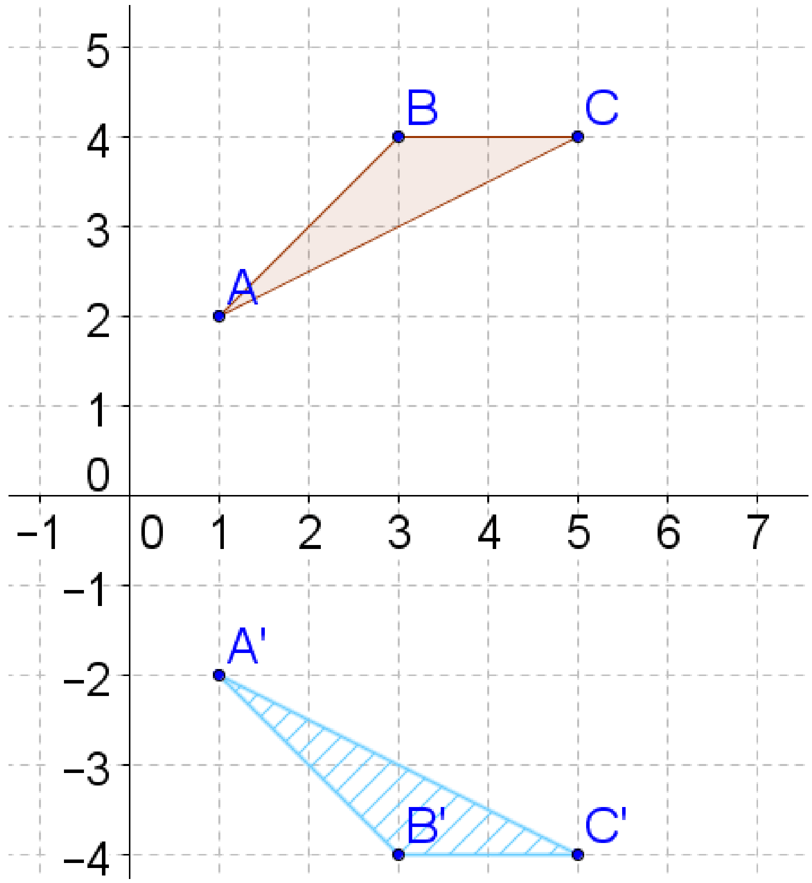
The line of reflection is the line

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 1(x - 2) \\ y - 1 &= x - 2 \\ y &= x - 1 \end{aligned}$$

Review

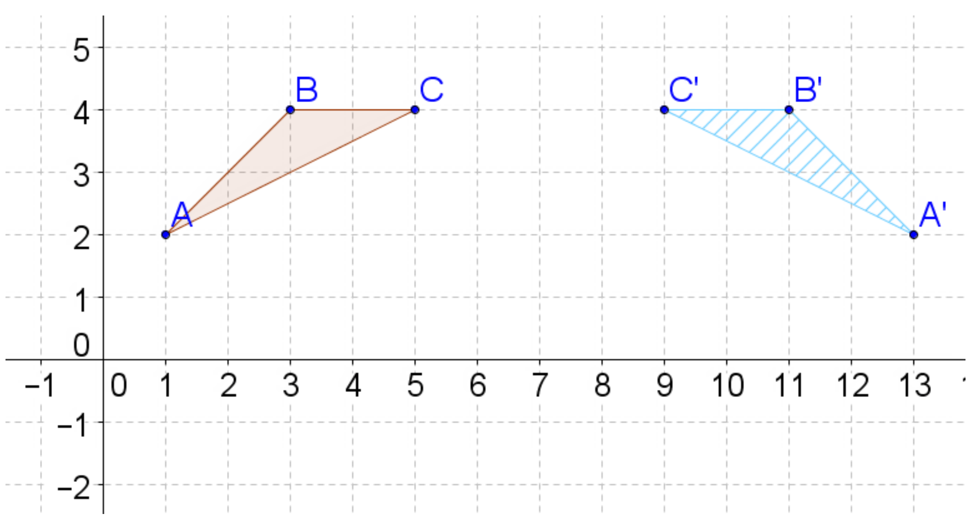
Describe the line of reflection that created each of the reflected images below.

1.



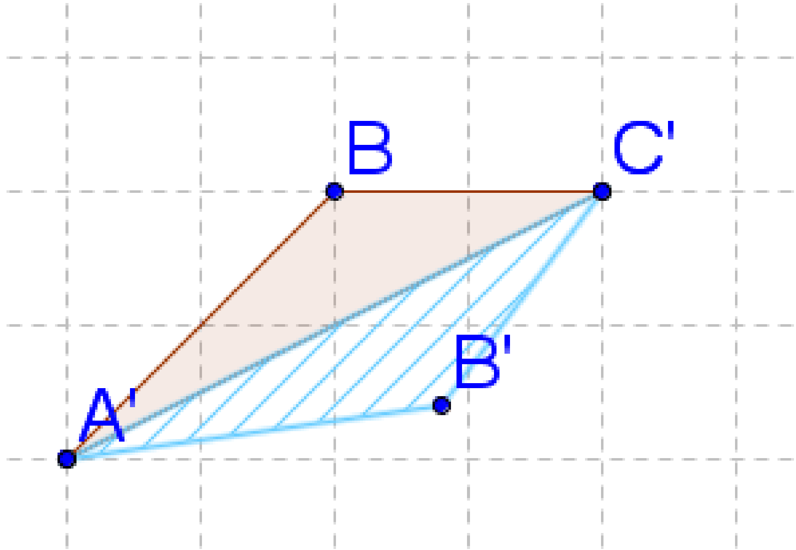
5
4
3
2
1
0
−1
−2
−3
−4
−1
0
1
2
3
4
5
6
7
B
C
A
A'
B'
C'

2.

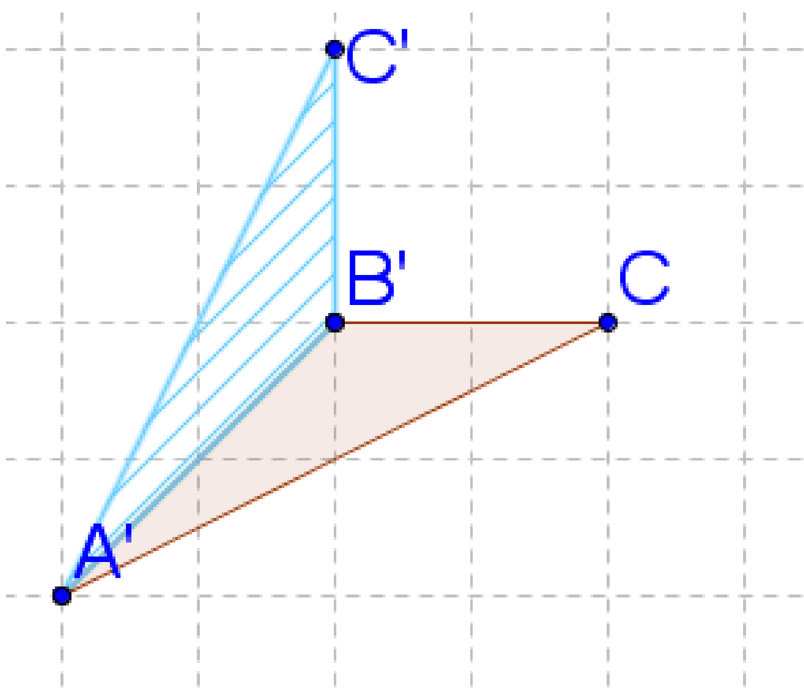


5
4
3
2
1
0
−1
−2
−1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
B
C
A
C'
B'
A'

3.

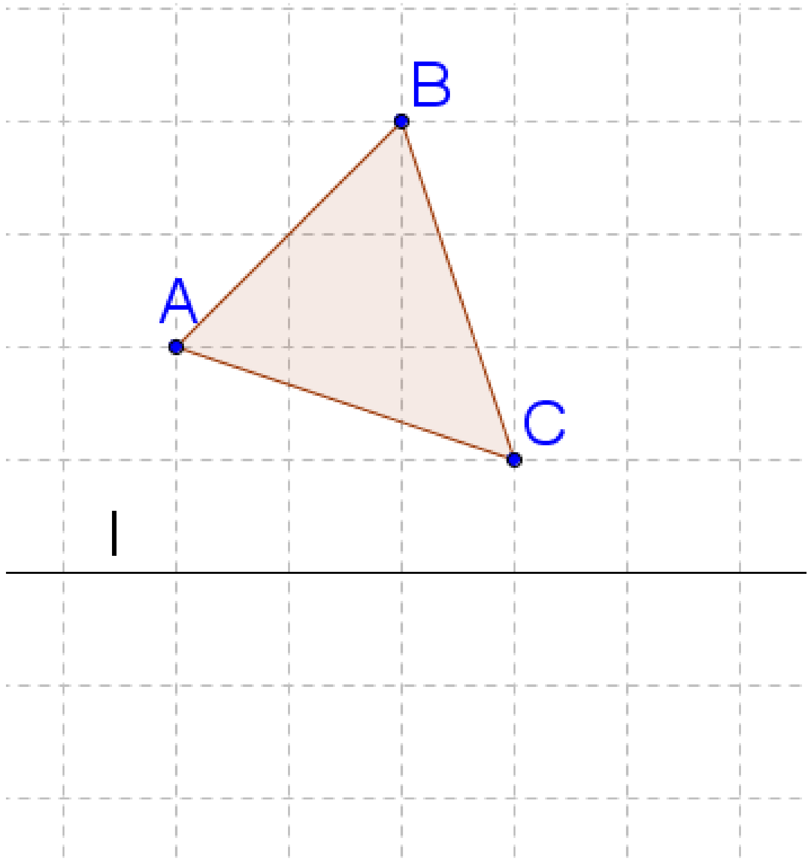


4.

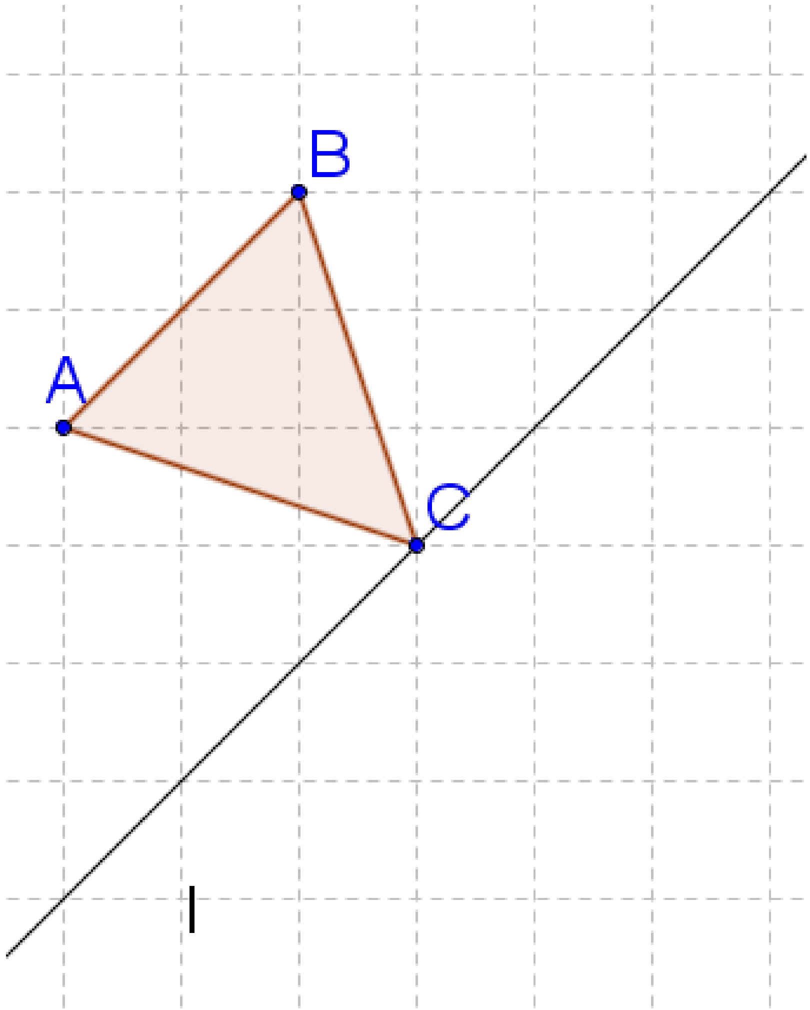


For questions 5 - 8, reflect each shape across line l.

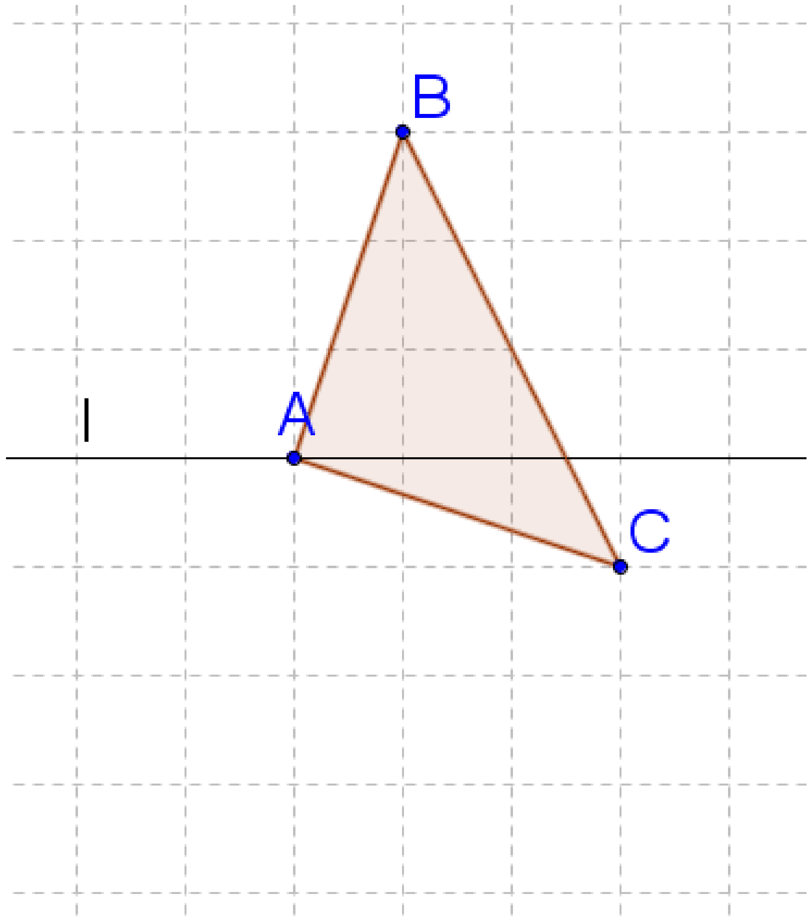
5.



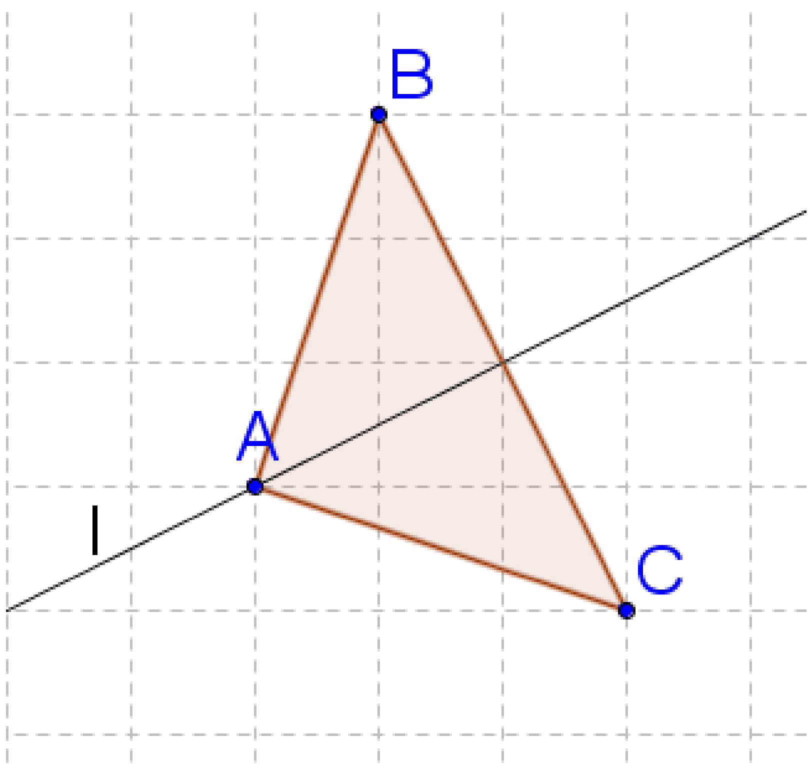
6.



7.

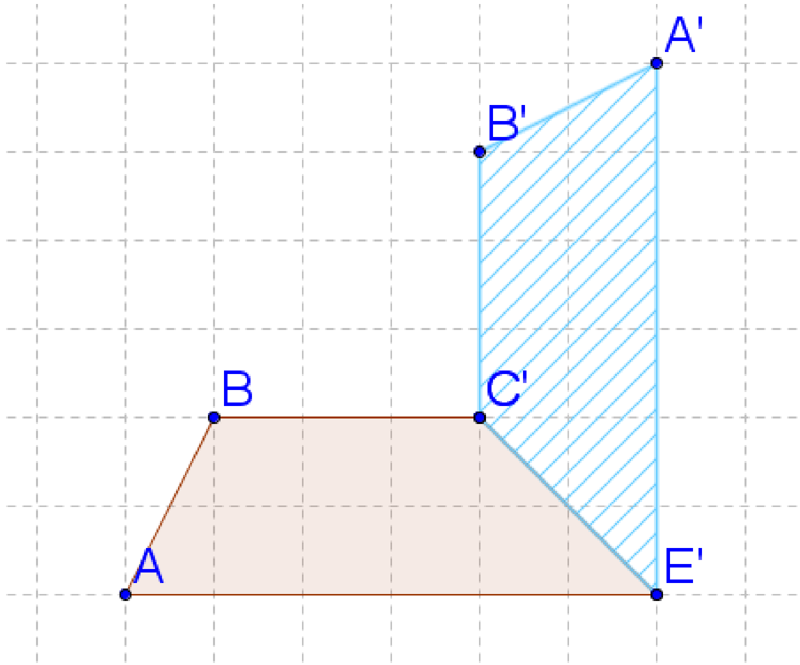


8.

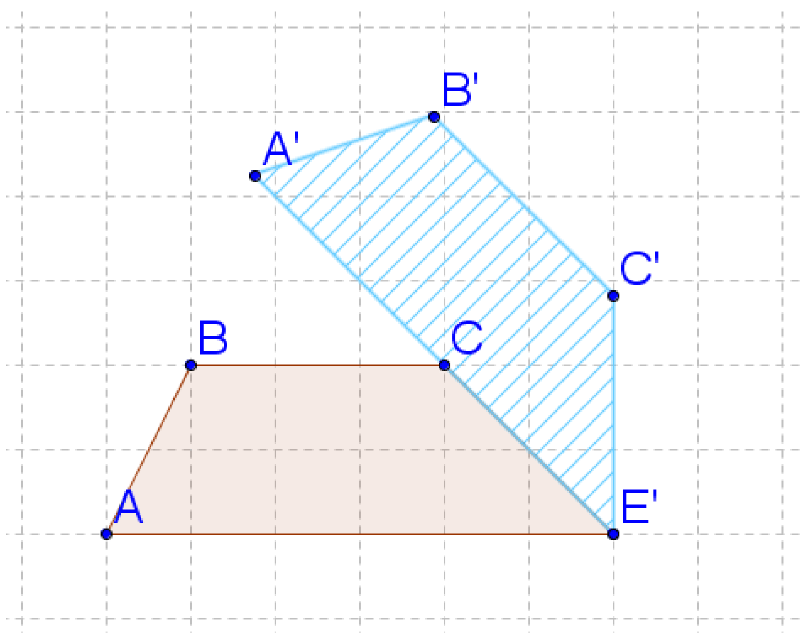


Questions 9 and 10: is the transformation a reflection? Explain.

9.



10.



11. Reflect a shape across the y-axis. How are the points of the original shape related to the points of the image?

12. The point (7, 2) is reflected across the y-axis. Can you find the coordinates of the image point using the relationship you found in the last question?

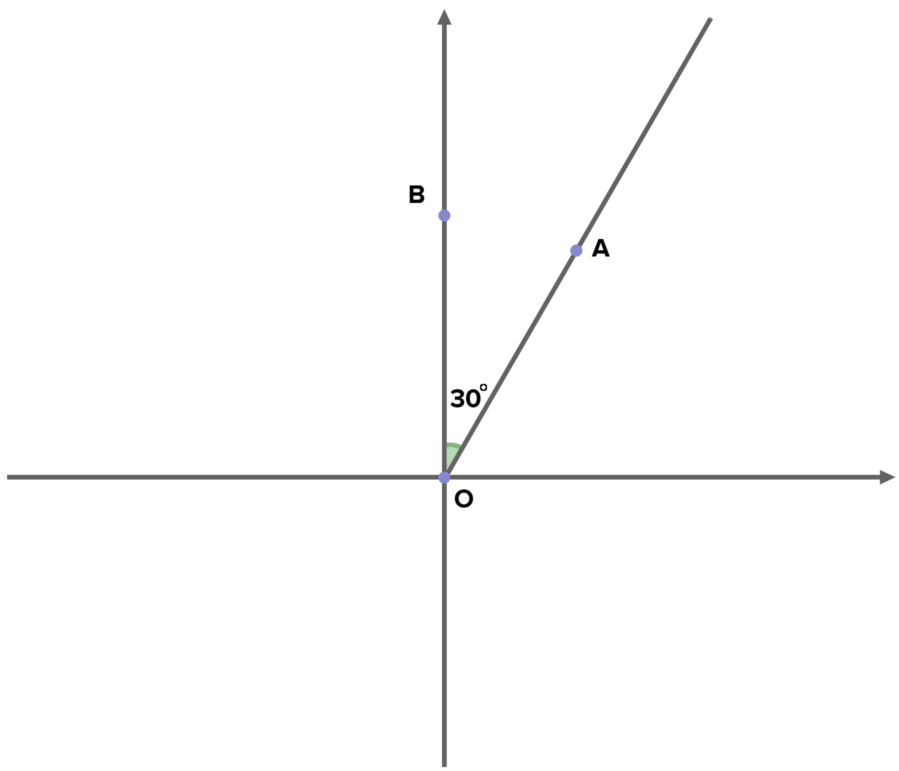
13. Reflect a shape across the line $y = x$. How are the points of the original shape related to the points of the image?

14. The point (7, 2) is reflected across the line $y = x$. Can you find the coordinates of the image point using the relationship you found in the last question?

15. Reflect a shape across the line $y = -x$. How are the points of the original shape related to the points of the image?

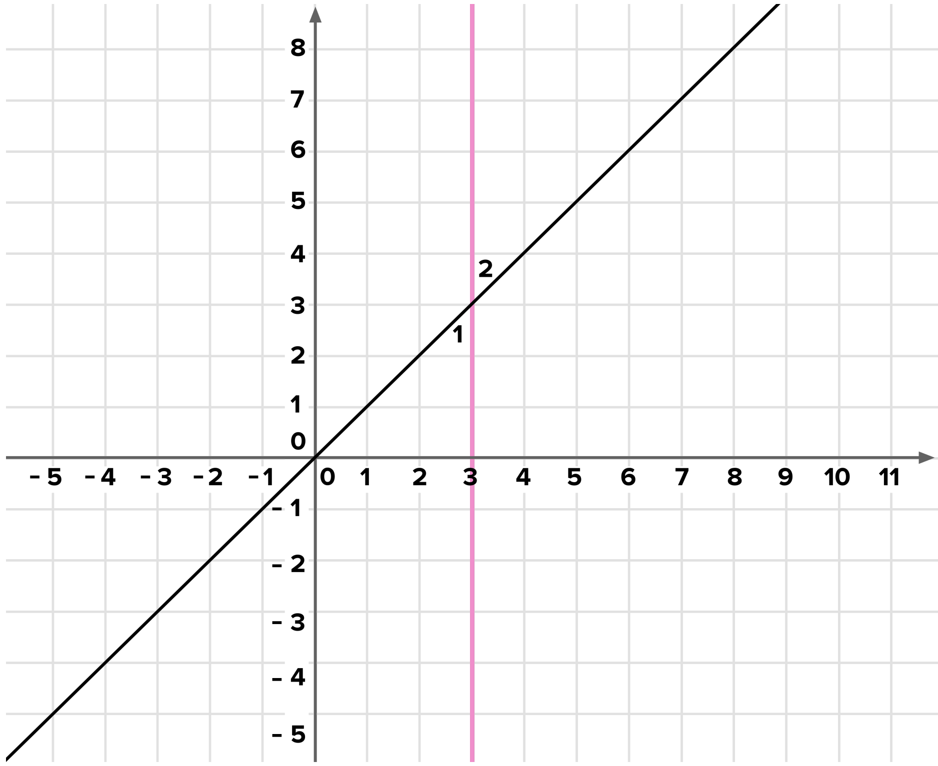
16. The point (7, 2) is reflected across the line $y = -x$. Can you find the coordinates of the image point using the relationship you found in the last question?

17. Given the diagram below, reflect $\angle AOB$ across the y-axis and call the resulting image $\angle COB$. What is the measure of $\angle COB$? What is the measure of $\angle COA$? Draw sketches to experiment with reflecting angles of different measures and describe your results.



18. If it hasn't been completed already, perform the reflection described above for $m\angle AOB = 45°$. What is the relationship between $\overrightarrow{OC}$ and $\overrightarrow{AC}$? Explain.

19. The graph of the line $y = x$ is shown below. A vertical line is also shown. What is the measure of $\angle 2$? How do you know? Reflect the vertical line across the line $y = x$. Describe the image. What is the relationship between the image and the original? Explain.



20. Look at the image below. The line $y = x$ is graphed. What are the coordinates of point A? How do those coordinates relate to the triangle shown? Reflect the triangle shown across the line $y = x$. What are the coordinates of the image of A? How do those coordinates relate to the image triangle? How does this explain the rule for reflecting a point across the line $y = x$?

21. If a line is reflected across a line, is the image always parallel to the original? Why or why not? Is the image ever parallel to the original? If so, under what conditions? Under what conditions is the image of a line reflected across a line coincident with the original?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.4.

3.5 Geometry Software for Reflections

Learning Objectives

Here you will learn how to perform reflections using geometry software.

Reflection with a Grid

Recall that a reflection is one example of a rigid transformation. A reflection across line l moves each point P to P' such that line l is the perpendicular bisector of the segment connecting P and P'.

Below, quadrilateral $ABCD$ (the pre-image) has been reflected across line l to create the new quadrilateral $A'B'C'D'$ (the image).

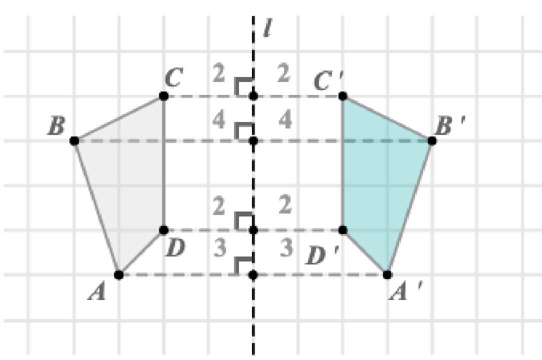


FIGURE 3.36

When you are working on a grid (or graph paper) with lines of reflection that are vertical, horizontal, or have slopes of ± 1, performing reflections is relatively easy. In the above reflection, you can see that each image point is the same distance from line l as its corresponding original point. To perform the reflection, simply count the correct number of units on the right side of line l.

If the grid is not there it is not as easy to do the reflection because there are no lines as a guide.

Reflection without a Grid

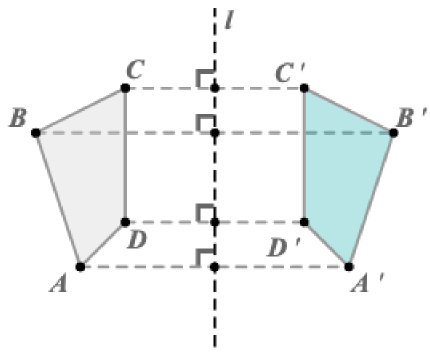


FIGURE 3.37

To perform a reflection without a grid, you need to:

1. Find lines perpendicular to the line of reflection through each of the points that define the shape.
2. Construct line segments along the perpendicular lines that connect each point of the shape to the line of reflection.
3. Copy the line segments to the other side of the line of reflection.
4. Connect the endpoints on the other side of the line of reflection to create the image.

Doing this by hand requires careful construction of perpendicular lines and copying of line segments. Interactive geometry software often simplifies this process with "reflect" tool.

Reflection using Geometry Software

In the interactive below, follow the steps to perform the reflection of a polygon.

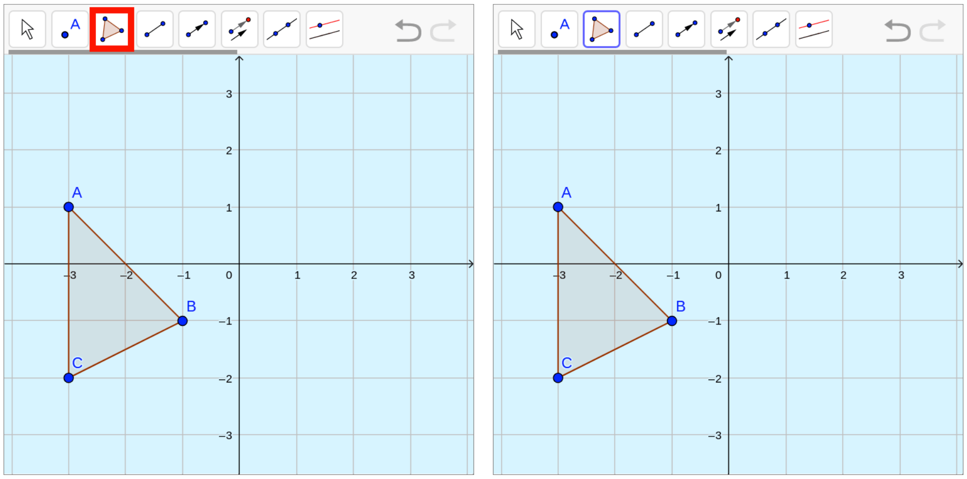
- The steps are stated in the interactive description, and demonstrated in detail with pictures below the interactive.

MEDIA

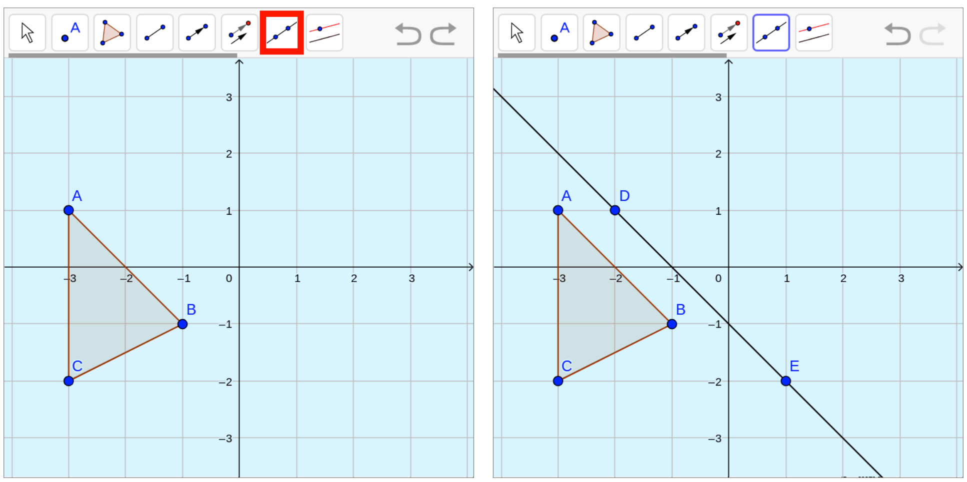
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/223239

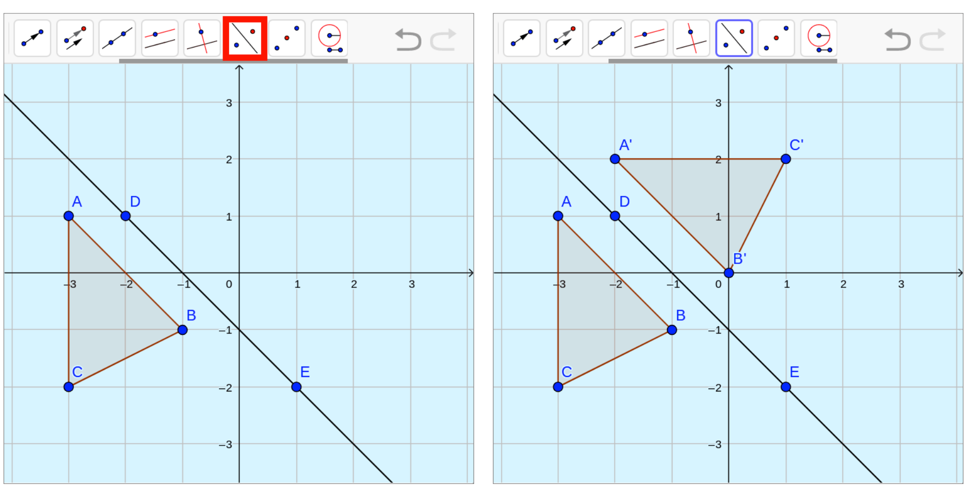
First, select the "polygon" tool and create a polygon.



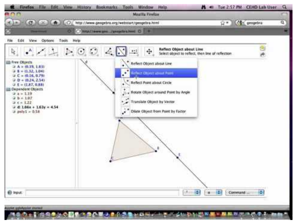
Next, create your line of reflection. Choose the "line" tool from the menu and click two places on the canvas to create the line.



Then, reflect the polygon across the line. First, select the “reflect” tool, click on the polygon, then on the line.



Note that the points defining the image are labeled with prime notation. At this point you can move points *D* or *E* to redefine the line of reflection or change your original polygon and your image will move/change accordingly. *Make sure to select the cursor button before attempting to move points.*



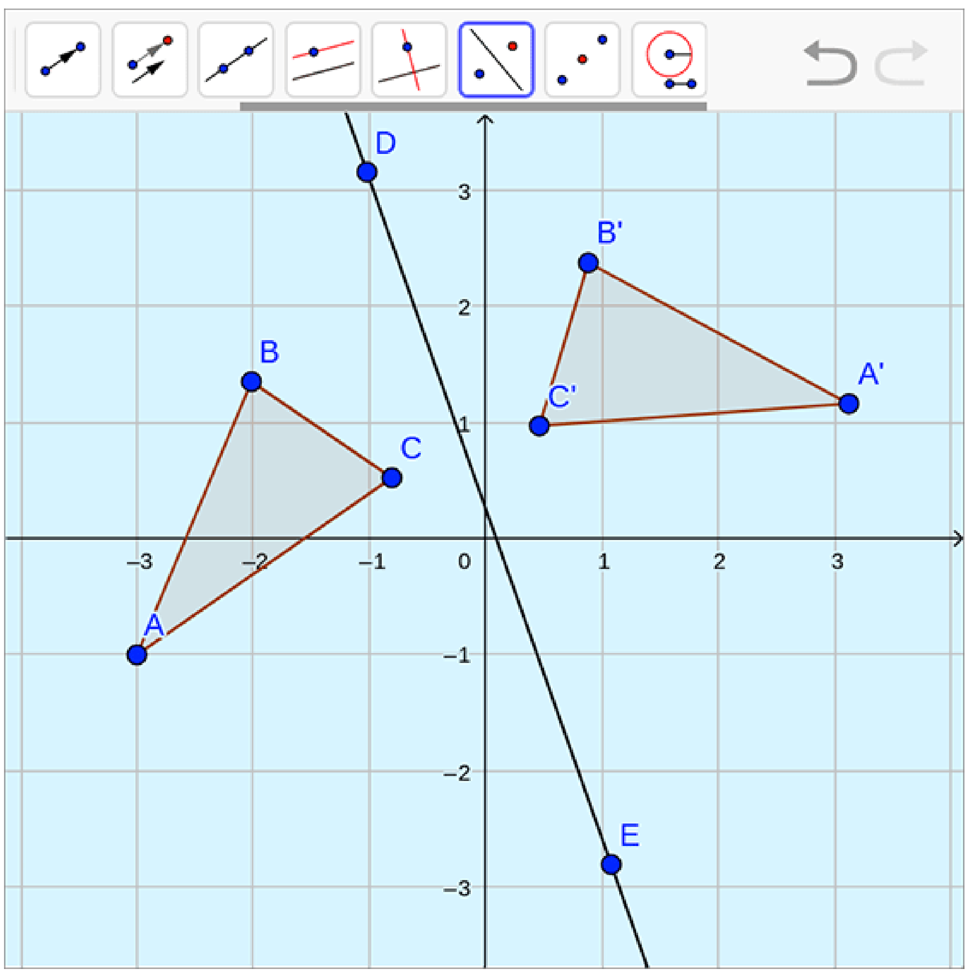
MEDIA

Click image to the left or use the URL below.

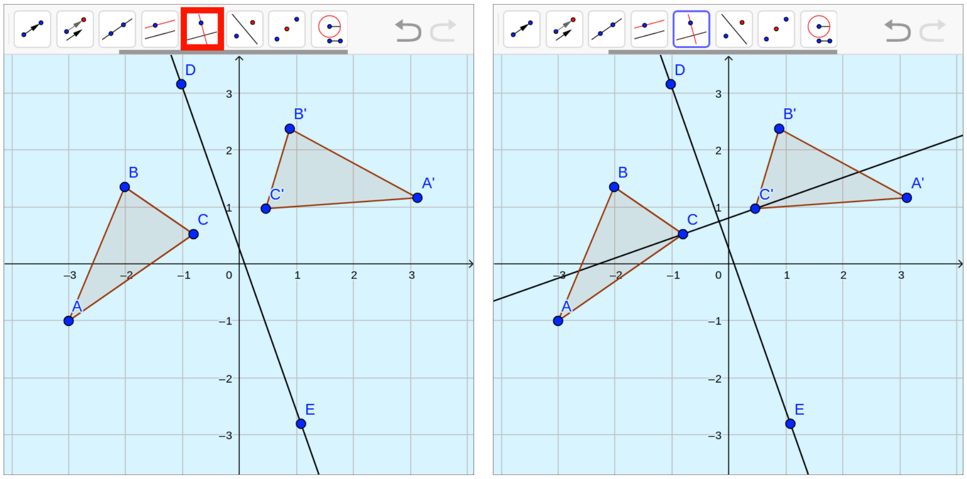
URL: http://www.ck12.org/flx/render/embeddedobject/71068

Reflection Proof

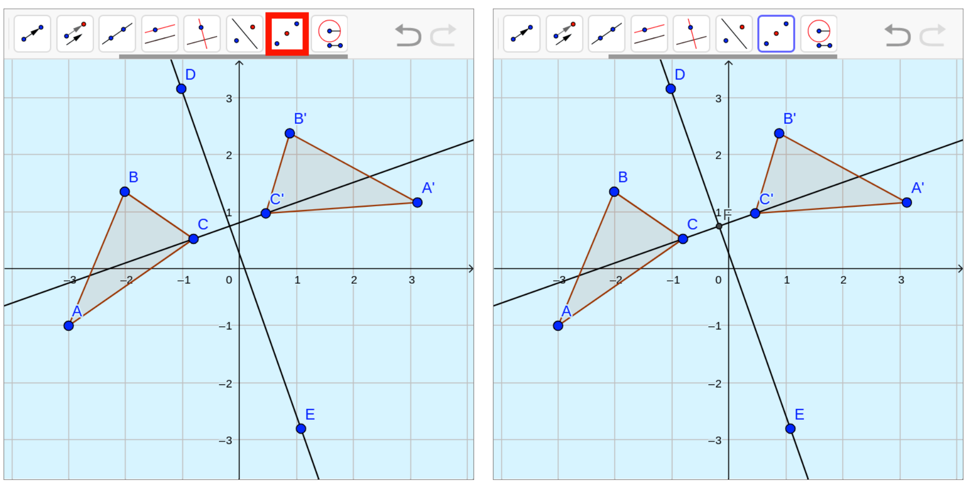
ΔABC is reflected across line $\overleftrightarrow{DE}$ to create $\Delta A'B'C'$. Show that the line of reflection $\overleftrightarrow{DE}$ is the perpendicular bisector of the line segment connecting C with C'.



First, construct a line perpendicular to the line of reflection $\overleftrightarrow{DE}$ through point C. Select the "perpendicular line through point" tool, click on the point C, then click on the line $\overleftrightarrow{DE}$. Verify that this line goes through C'.



Construct the midpoint of the segment connecting C with C' by using the "midpoint" tool, then clicking on the points C and C'. Verify that this point F lies on the line of reflection $\overleftrightarrow{DE}$.

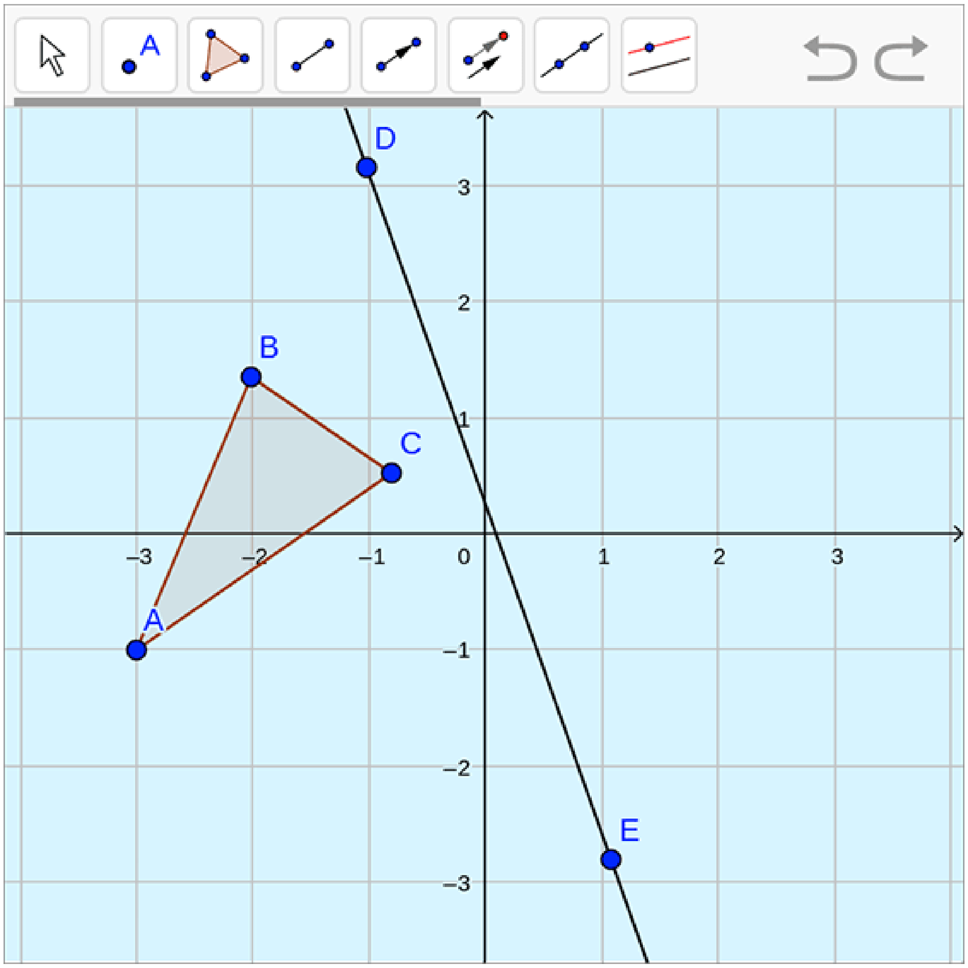


Because the line segment connecting C with C' is perpendicular to the line of reflection $\overleftrightarrow{DE}$ and its midpoint F lies on the line of reflection $\overleftrightarrow{DE}$, the line of reflection is the perpendicular bisector of this segment.

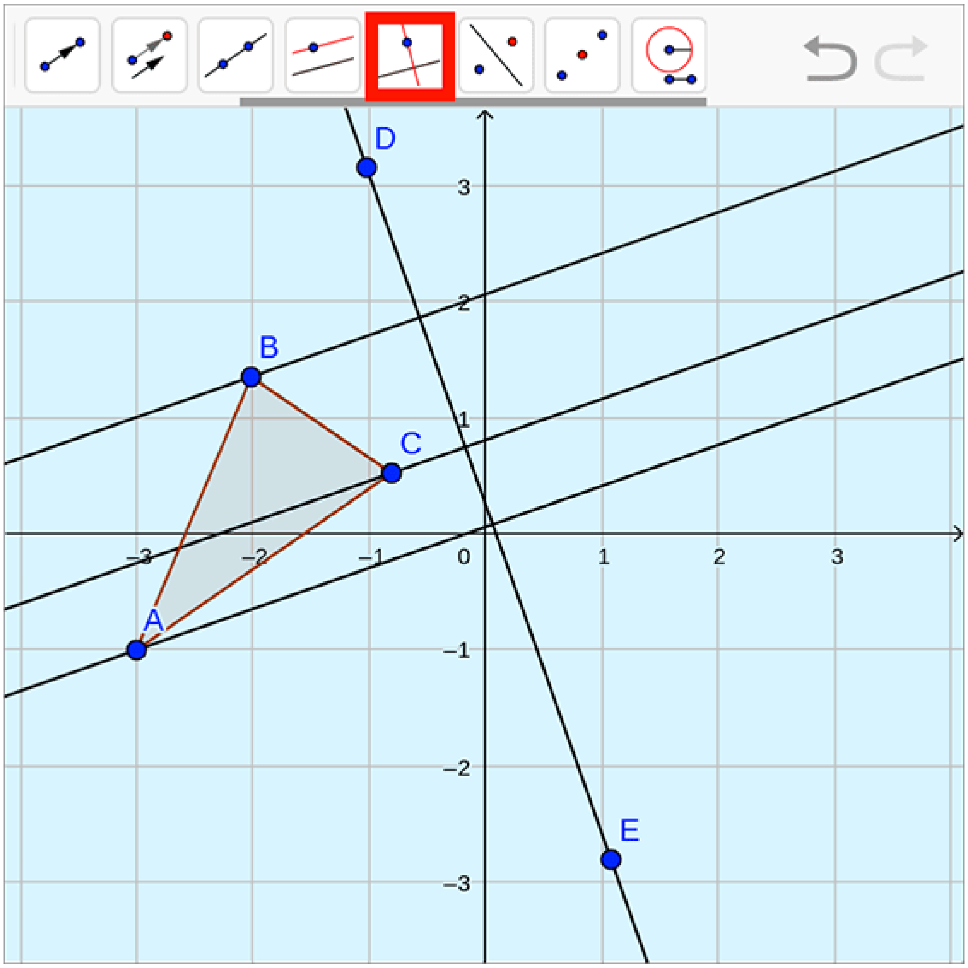
Geometry Software Reflections

Use geometry software to perform a reflection without the dedicated tool.

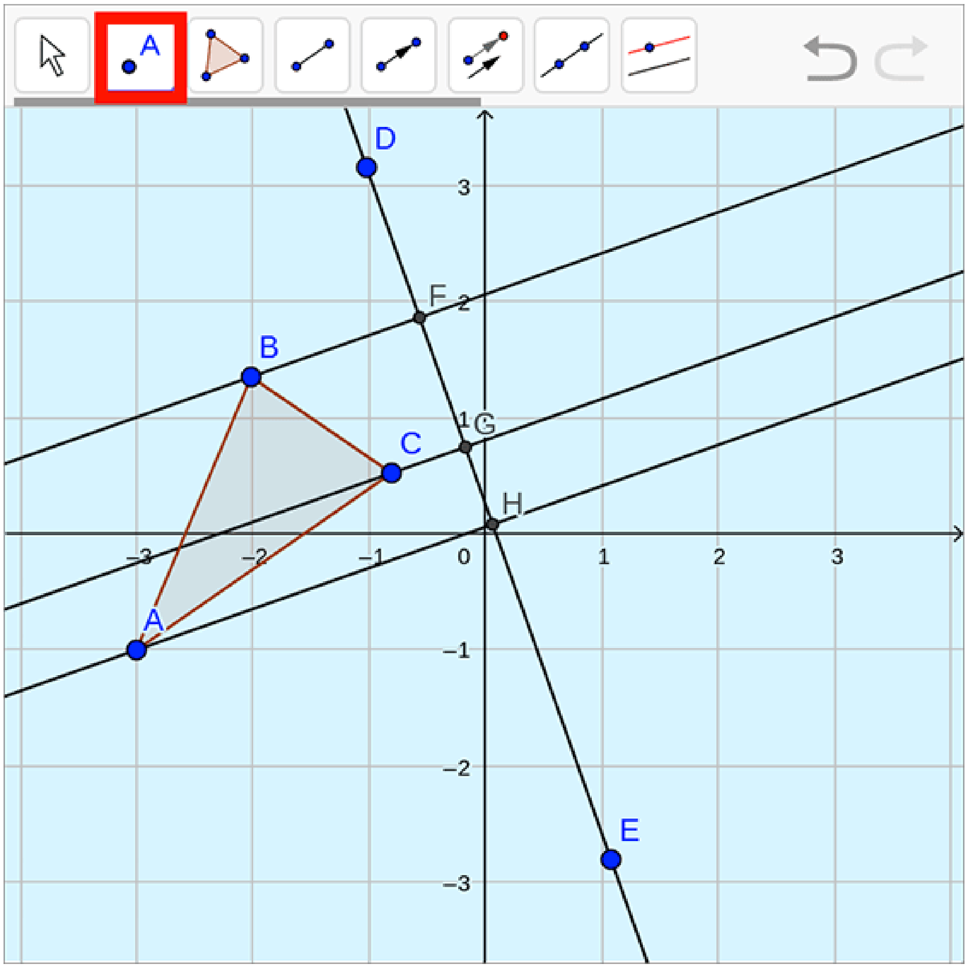
Reflect ΔABC across the line $\overleftrightarrow{DE}$ without using the “reflect” tool.



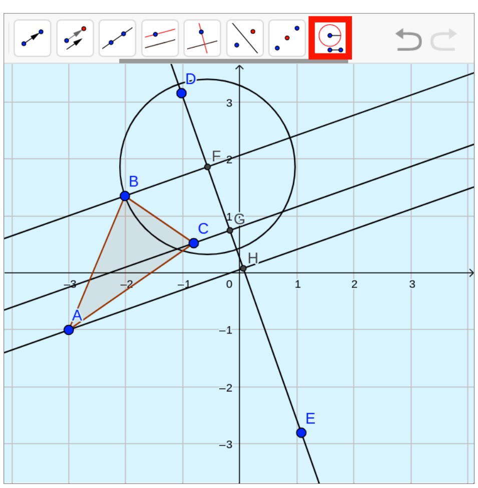
First, using the "perpendicular line through point" tool, construct lines perpendicular to the line of reflection $\overleftrightarrow{DE}$ through each of the three points that define ΔABC.

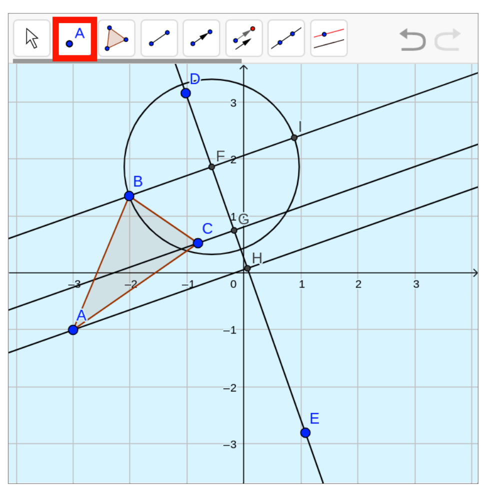


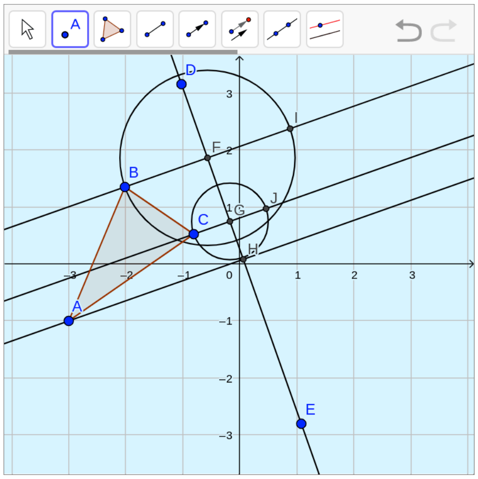
Next, select the "point" tool in the menu, then click on the canvas to construct the points of intersection F, G, and H for the perpendicular lines and line of reflection $\overleftrightarrow{DE}$.

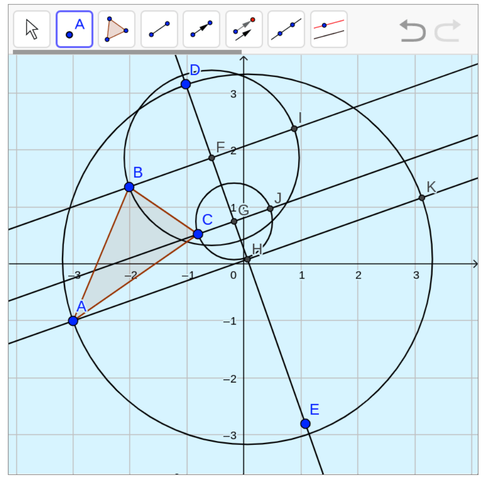


Select the "compass" tool, click on the point B, then double click on the point F. Next, select the "point" tool and construct the point of intersection I of the line $\overleftrightarrow{BF}$ and the circle centered at F. Repeat for the vertices C and A to construct the points J and K respectively.

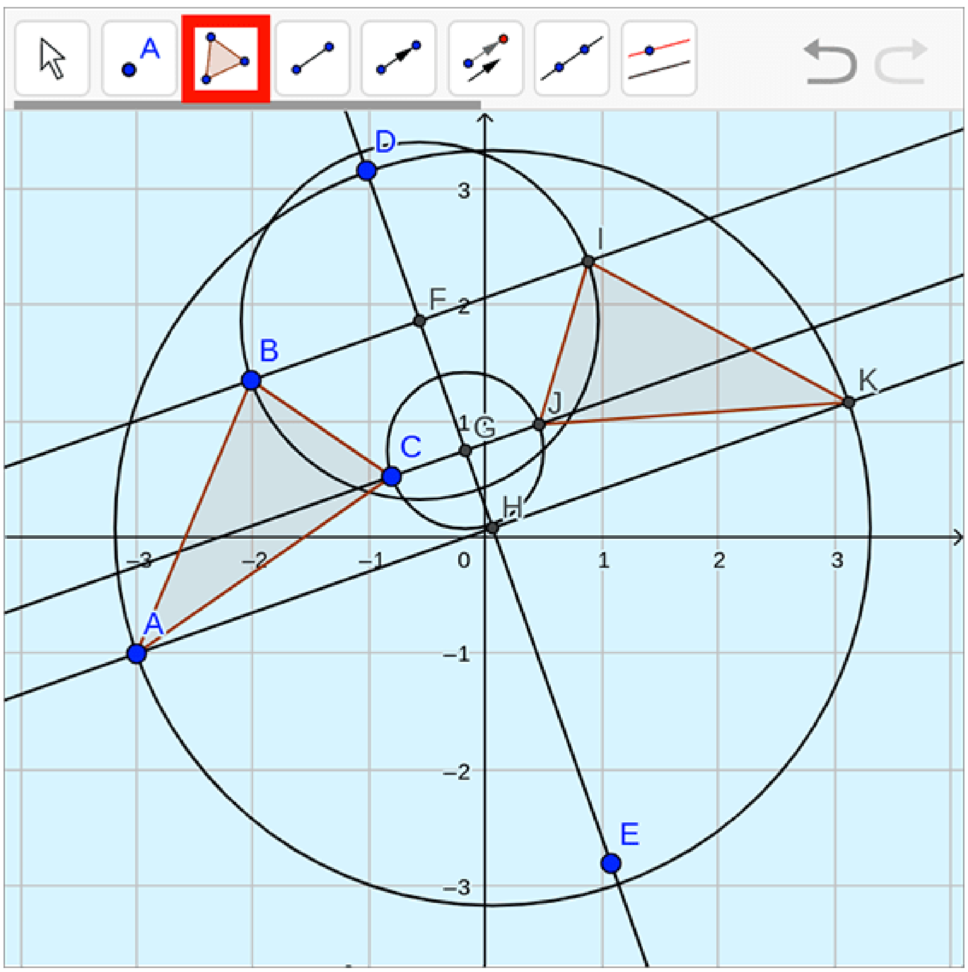








Using the "polygon" tool, connect the points I, J, and K to form the triangle image.



Review

1. Create a polygon in interactive geometry software. The provided interactive found earlier in this lesson is a great resource.

2. Create a line of reflection that is horizontal.

3. Reflect a polygon across the line of reflection using the reflect button.

4. Create another line of reflection that is at a slant.

5. Reflect a polygon across a slant line of reflection without using the reflect button. Construct perpendicular lines and copy and paste segments as shown in the "Reflection Proof" section from the lesson above.

6. Use the reflect button to reflect a polygon across a slant line of reflection. Use this method to verify that your prior answer was correct.

7. Create a third line of reflection that passes through the polygon.

8. Reflect a polygon across a line of reflection that passes through the polygon by using the reflect button.

9. Reflections are rigid transformations which means that distance is preserved. Verify that distance has been preserved by using the software to measure sides of an original polygon and its images after reflections. *Select "distance or length" from one of the drop down menus. Then, click on each line segment that you want to measure to see its length.*

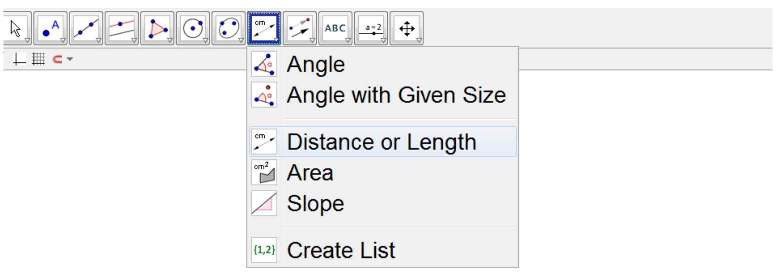


FIGURE 3.38

10. Reflections are rigid transformations which means that angles are preserved. Verify that angles have been preserved by using the geometry software to measure two corresponding angles. *Select "angle" from the same drop down menu the 'distance or length' tool used previously. Then, specify the angle you want to measure by clicking on the three points you would use to name the angle. You must click on the points in clockwise order for it to measure the correct angle.*

11. Construct a circle using geometry software. The provided interactive will work well for this.

12. What types of lines of reflection would create an image that is in the same location as the original circle? Try reflecting your circle across two of those lines to check your ideas.

13. Could all reflections of circles also have been translations? Explain.

14. Create a line of reflection that will cause the image of the circle to be to the right of the original circle.

15. Explore how you might be able to reflect a circle across a line of reflection without using the reflect button by using perpendicular lines and copying and pasting segments.

16. If a polygon is reflected across a line, is it possible to use a translation to map the image after reflection to the original? Experiment. Are there any polygons for which this is possible? Explain.

17. Reflect a point A across a line. Create a segment connecting pre-image point A to its image point. Label the image point B. Create any point C on the line of reflection. Complete polygon ABC by connecting the three points. State the type of polygon created and describe its characteristics.

18. True or False: The segment connecting a point and its image after reflection is bisected by the line of reflection. Any point on the line of reflection is equidistant from the endpoints of the segment. Explain.

19. Prove that a pair of points that are equidistant from any two points on a line are equidistant from the line.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.5.

3.6 Reflection Symmetry

Learning Objectives

Here you will learn about reflection symmetry.

Reflection Symmetry

- A shape has symmetry if it is indistinguishable from its transformed image.
- A shape has reflection symmetry if there is a line through the center of the shape that you can reflect across without the shape appearing to move at all.
- This line of reflection is called a line of symmetry.

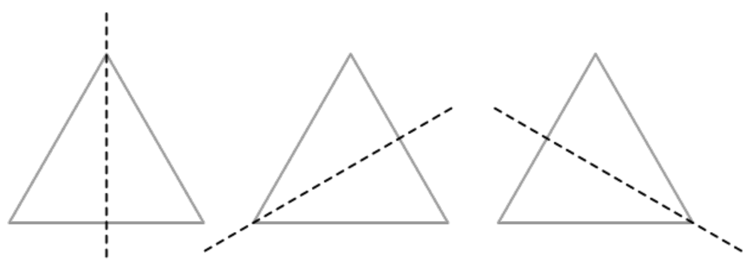


FIGURE 3.39

In other words, a line of symmetry is a line that divides a figure into two mirror images. The figure is mapped onto itself by a reflection in this line. Some figures have one or more lines of symmetry, while other figures have no lines of symmetry.

A rectangle is an example of a shape with reflection symmetry. A line of reflection through the midpoints of opposite sides will always be a line of symmetry.

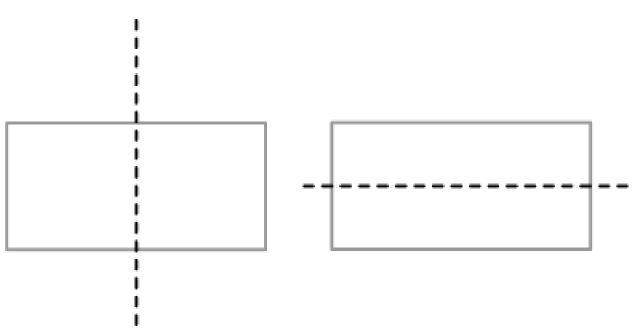
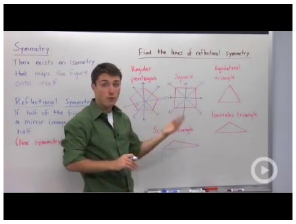


FIGURE 3.40

A rectangle has two lines of symmetry. You can imagine folding the rectangle along each line of symmetry and each half of the rectangle would match up perfectly. Remember that a shape has to have *at least one* line of symmetry for it to be considered a shape with reflection symmetry.



MEDIA
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/71070

Recognizing Reflection Symmetry

Let's consider a regular hexagon as an example.

- A regular hexagon has all its sides congruent and each of the angles measure 120°.
- A regular hexagon has a number of lines of reflection: three along the lines joining the midpoints of its opposite sides and three along the diagonals.
- Thus, a hexagon has reflection symmetry.

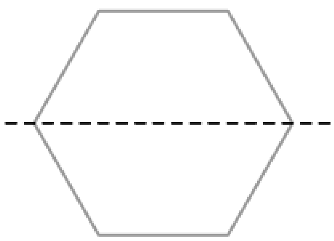


FIGURE 3.41

MEDIA
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URL: http://www.ck12.org/flx/render/embeddedobject/223437

MEDIA
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/223438

Identifying Lines of Symmetry

A square is an example of a shape with reflection symmetry.

- In a square, all sides are congruent and each angle is a right angle.
- There are four lines of reflection that carry the square onto itself. These lines of reflection will always be the lines of symmetry.
- Two lines of symmetry separate the square into two equal rectangles, whereas the other two separate the square into two equal triangles.

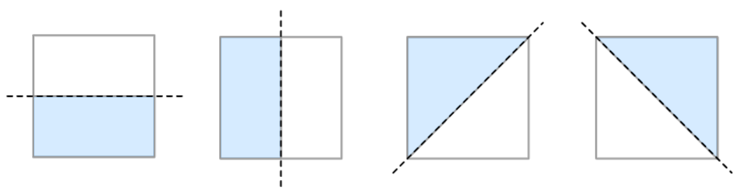


FIGURE 3.42

Identify the line of reflection symmetry in the given figure.

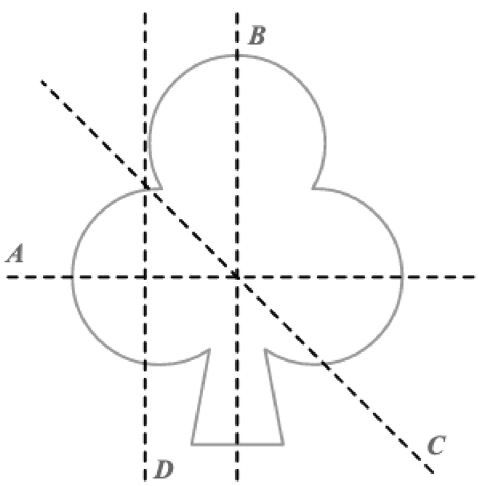


FIGURE 3.43

_____ is the line of reflection symmetry.

Reflection Symmetry in Trapezoids

A generic trapezoid will not have reflection symmetry.

- An **isosceles** trapezoid will have reflection symmetry because the line connecting the midpoints of the bases will be a line of symmetry.

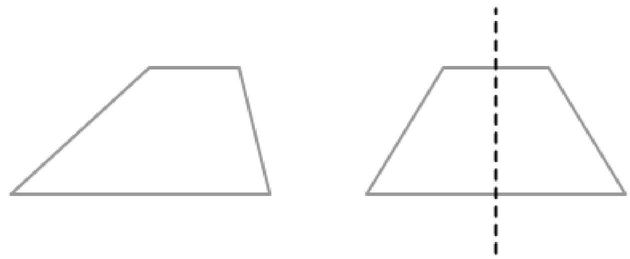


FIGURE 3.44

Reflection Symmetry

Explore the reflection symmetry of the selected shape. Click the blue arrow to activate the figure, then drag the cursor to draw as many lines of symmetry as are given for each shape.

- Rectangle: 2
- Triangle: 3
- Pentagon: 5
- Circle: ? How many can you find?

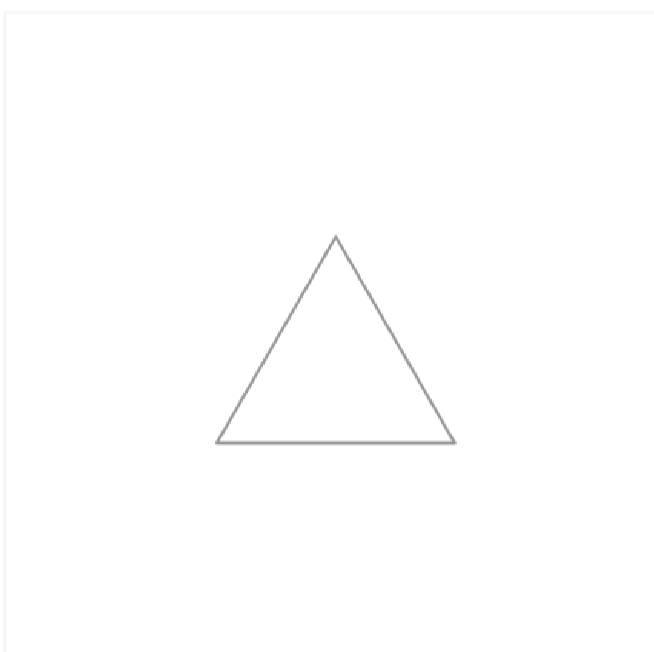


FIGURE 3.45

Examples

Example 1

What happens when you reflect the regular pentagon below across line f? Why is the line of reflection in this case called a line of symmetry?

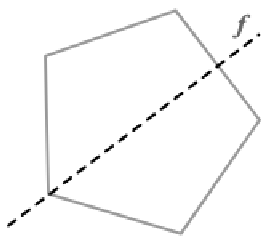


FIGURE 3.46

When you reflect the regular pentagon below across line f, the pentagon will look exactly the same. Without labeled points, it will be impossible to tell the difference between the original pentagon and its image. This means that the pentagon has reflection symmetry. Line f is a line of symmetry because it's the line across which the pentagon can be reflected without visible change.

Example 2

Do the capital letters below have reflection symmetry? If so, state how many and where these lines of symmetry are.

a.

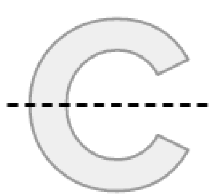


FIGURE 3.47

The capital C has a line of reflection going horizontally through its center, and it looks the same on the top and the bottom, so we say that **C has a horizontal line of symmetry.**

b.

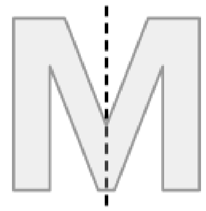


FIGURE 3.48

The capital M has a line of reflection going through its center where the M is the same on the left and right. Because the line of reflection is vertical, we say **M has a vertical line of symmetry.**

c.

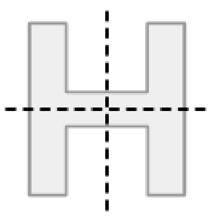
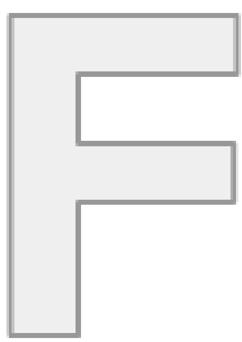


FIGURE 3.49

The H has two lines of reflection - the vertical line through its center and the horizontal line through its center are both lines of reflection, so **H has both a horizontal and a vertical line of symmetry.**



d.

No, the capital **F does not have any line of reflection** - you can't split it into two halves that are exactly the same on both sides of a line.

CK-12 PLIX Reflection Symmetry

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/reflection-symmetry/plix/Seafloor-for-Reflection-Symmetry-5464fd698e0e0802a76df18d

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Review

1. What does it mean for a shape to have symmetry?

2. What does it mean for a shape to have reflection symmetry?

3. What do you think it would mean for a shape to have translation symmetry? Can you think of any shapes or objects with translation symmetry?

For each of the following shapes, state whether or not it has reflection symmetry. If it does, state how many lines of symmetry it has and describe where the lines of symmetry are.

4. Equilateral triangle

5. Isosceles triangle

6. Scalene triangle

7. Parallelogram

8. Rhombus

9. Regular pentagon

10. Regular hexagon

11. Regular 12-gon

12. Regular n-gon

13. Circle

14. Kite

15. In order to have reflection symmetry, must a polygon have at least two sides that are the same length? Explain.

16. Give examples of objects with reflection symmetry in nature.

17. Does every regular polygon have at least one line of symmetry? How many does an equilateral triangle have? A square? A pentagon? Experiment. (In interactive geometry software, in the dropdown for polygon creation, one can create a regular polygon of any number of desired sides. One can also create a perpendicular bisector of a segment, and an angle bisector.) How does the number of lines of symmetry change as the number of sides on the regular polygon changes? Why? How do the characteristics of these lines of symmetry change as the number of sides increase? Why?

18. Construct a line, then three non-collinear points on one side. Then reflect the points across a line. Connect these points to form a polygon which has reflection symmetry. Use this construction to explain the meaning of reflection symmetry and the characteristics of objects which have it.

19. If a figure with reflection symmetry is reflected across a line that does not intersect it, is there a translation and

rotation that will map the image to the original? Experiment and discuss. Is this different from polygons that do not have reflection symmetry? Why or why not?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.6.

3.7 Rotations

Learning Objectives

Here you will learn about rotations.

Rotation

A rotation is one example of a rigid transformation.

- Rotation is a geometric transformation that involves rotating a figure a certain number of degrees about a fixed point.
- A positive rotation is counterclockwise and a negative rotation is clockwise.
- Precisely speaking: A rotation of $t°$about a given point O takes each point on a shape P and moves it to P' such that P' is on the circle with center O and radius $\overline{OP}$ and $\angle POP' = t°$.
- Informally: To rotate a shape, move each point on the shape the given number of degrees around a circle centered on the point of rotation. Make sure each new point is the same distance from the point of rotation as the corresponding original point.

Rotation about a point

Below, the parallelogram has been rotated 90°counterclockwise about point O to create a new parallelogram (the image).

- Notice point D rotates the specified 90 degrees around a circle centered on point O, so that D' is the same distance from O as D. This process is the same for all four points in the original parallelogram.

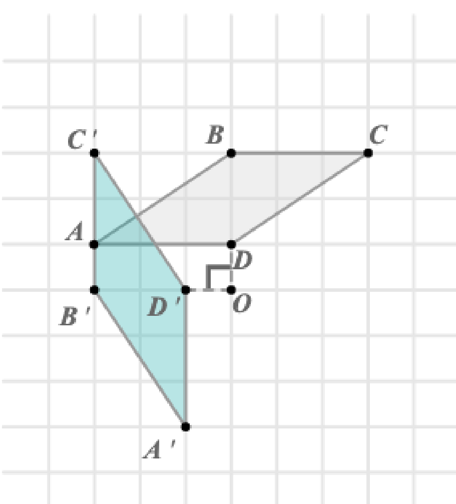


FIGURE 3.50

The location of point O **is important!**

The parallelogram below was also rotated 90°counterclockwise about point O, but point O is in a different location.

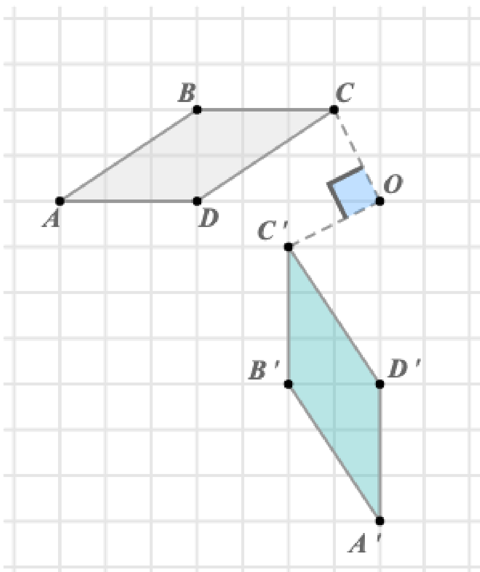


FIGURE 3.51

There is a circle with center O that passes through each pair of corresponding points.

Below, you can see the angle between each point and its image is the same, centered on point O.

Point C is rotated 90 degrees around O to be C', point D is rotated to be D', and so on.

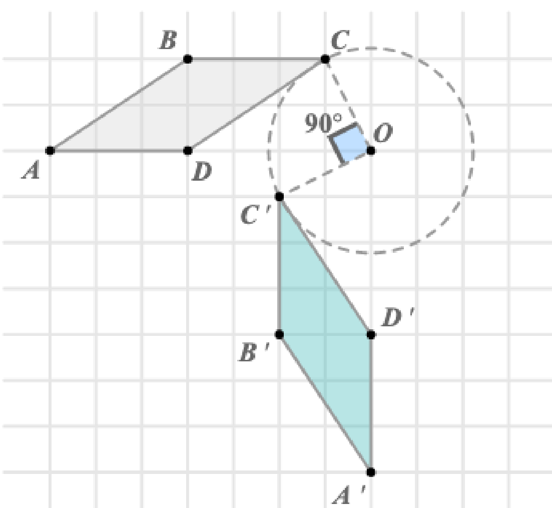


FIGURE 3.52

When describing a rotation, it is important to consider three pieces of information.

1. The center of rotation (point O in the figure above). This is the fixed point about which each point turns (the center of each circle passing through corresponding points).
2. The direction of rotation. You can rotate clockwise or counterclockwise.

3. The degree of rotation. This tells you how much you are turning.

   a. If the degrees are **positive**, the rotation is performed **counterclockwise**.
   b. If the degree measure is **negative**, the rotation is **clockwise**.
   c. Remember that a full circle is 360°.

Rotation

Select a shape with the dropdown, and then click or tap on the blue arrow to explore the rotation of the selected shape about the center of rotation O.

Observe the angle of rotation, the direction of rotation and how the point being rotated moves along a circle centered at O.

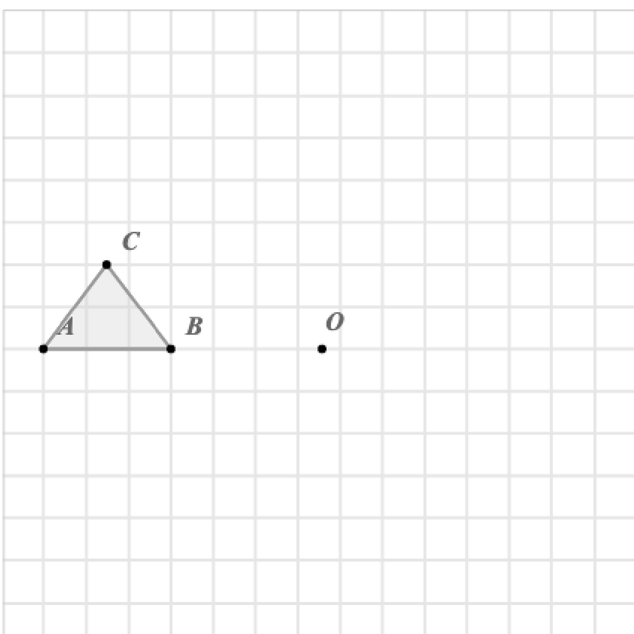


FIGURE 3.53

Rotations in the coordinate plane

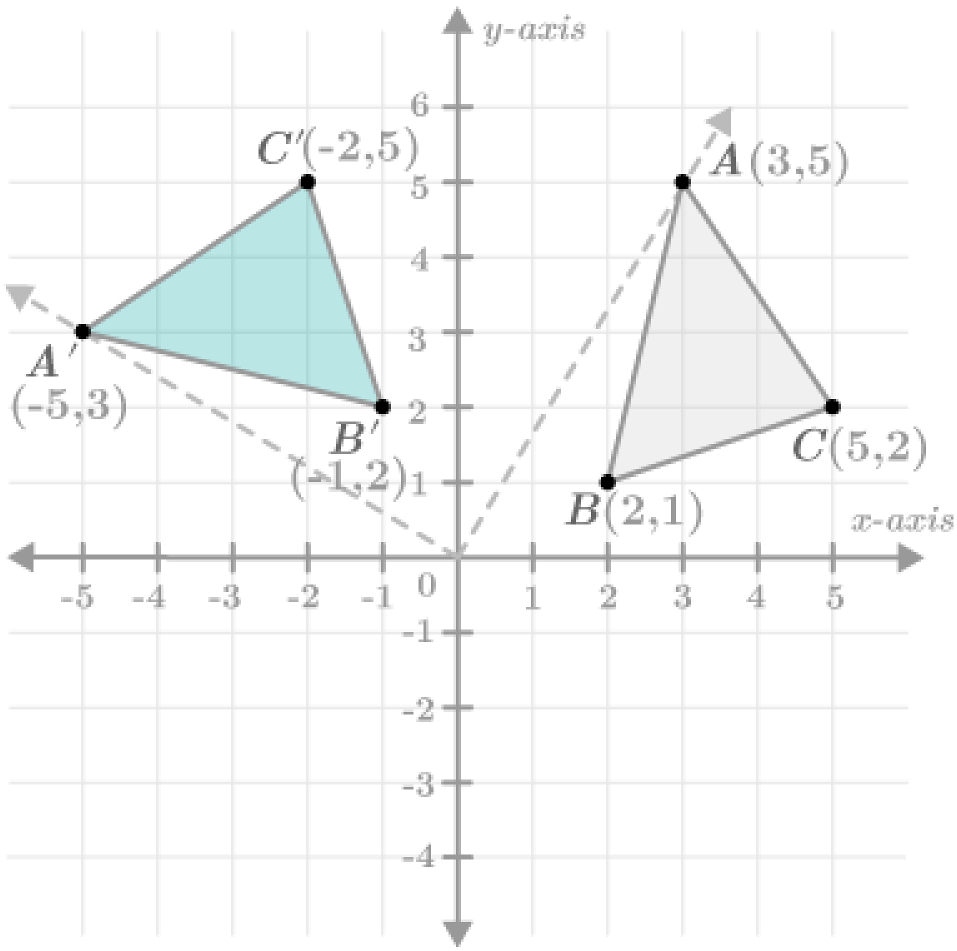
Although a figure can be rotated any number of degrees, the rotation will often be a common angle such as $90^\circ, 180^\circ$, or 270°.

Keep in mind that if the number of degrees are positive, the figure will rotate counter-clockwise and if the number of degrees are negative, the figure will rotate clockwise. While most rotations will be centered at the origin, the figure can rotate around any given point, indicated in the problem.

Rotation of 90 degrees on coordinate axes.

Centered at origin.

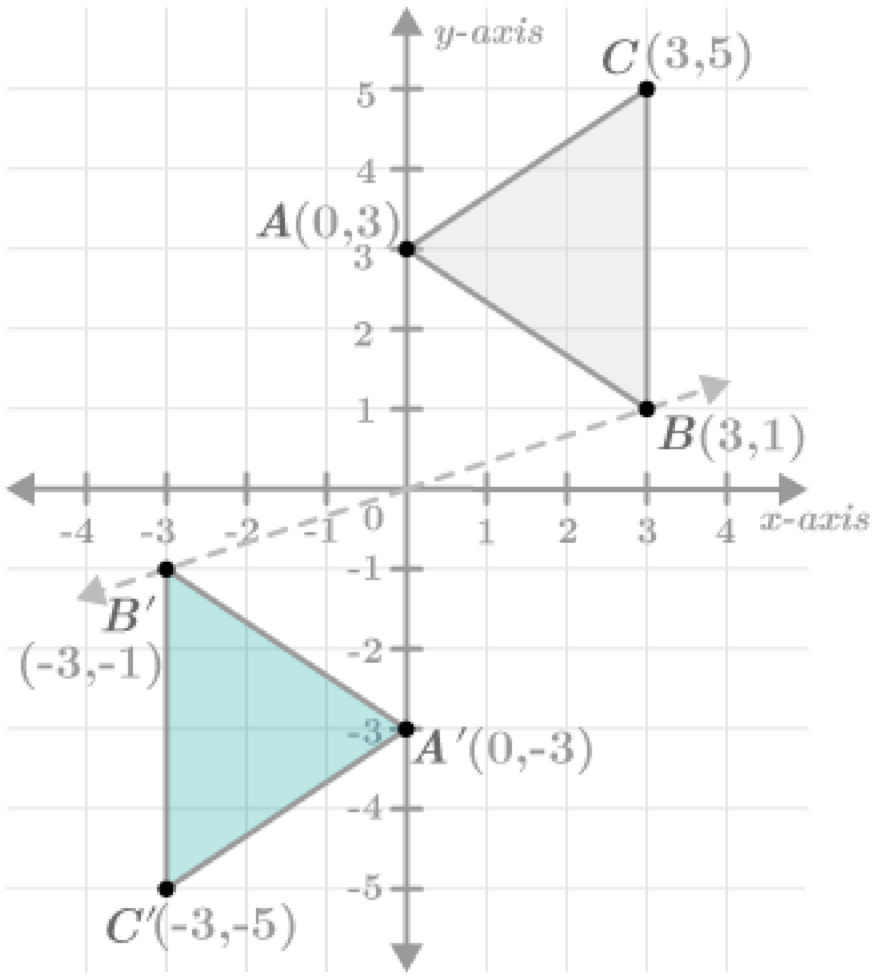
$(x,y) \rightarrow (-y,x)$



Rotation of 180 degrees on coordinate axes.

Centered at origin.

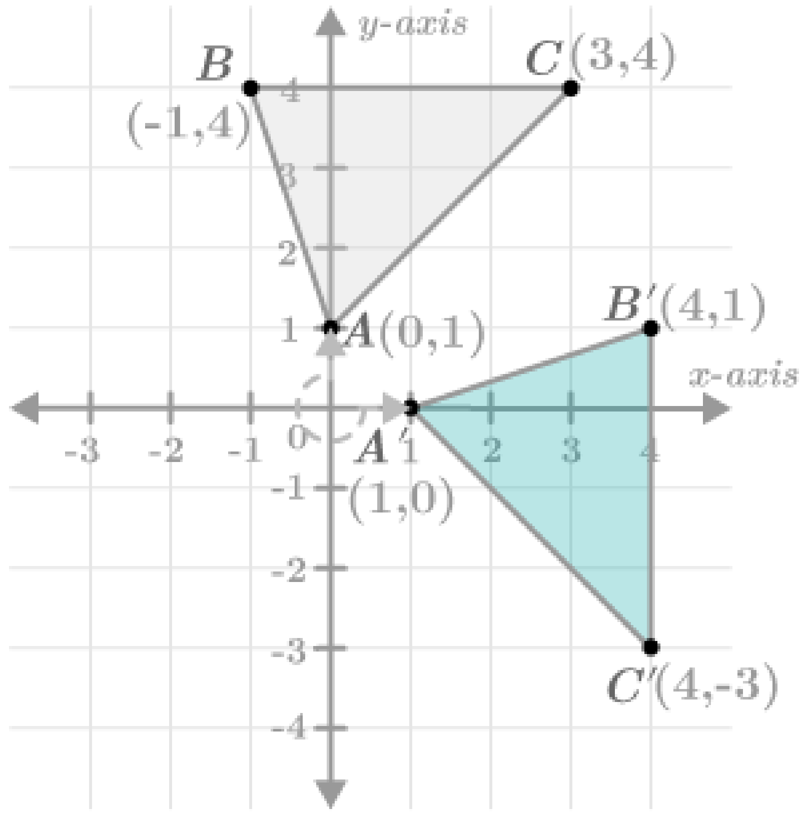
$(x,y) \rightarrow (-x,-y)$



Rotation of 270 degrees on coordinate axes.

Centered at origin.

$(x,y) \rightarrow (y,-x)$

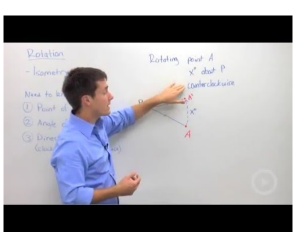


CK-12 PLIX: Reflecting Figures

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/reflecting-figures/plix/Translations-Rotations-and-Reflections-Rotate-the-Triangle-54f602a9da2cfe572242a0ea



MEDIA

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URL: http://www.ck12.org/flx/render/embeddedobject/71072

Identifying and Describing Rotations

1. Describe the rotation below.

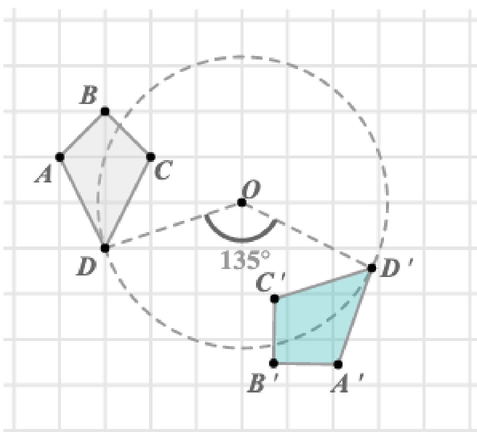


FIGURE 3.54

This is a 135°counterclockwise rotation about point O.

You could also say this is a $225^\circ(360^\circ - 135^\circ)$ clockwise rotation about point O. Once the rotation has taken place, there is no way to know the direction of rotation.

2. Is the following transformation a rotation?

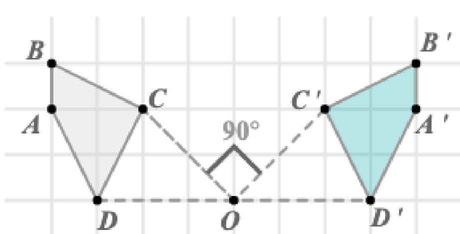


FIGURE 3.55

No, while $\angle DOD'$ is 180°, the angles connecting other pairs of points and point O are not 180°.

This transformation is actually a reflection across a line through point O, perpendicular to $\overline{DD'}$.

3. Rectangle $ABCD$ is rotated 90°counterclockwise about the origin. Draw the image of this rotation. How are the points of the original rectangle connected to the points of the image?

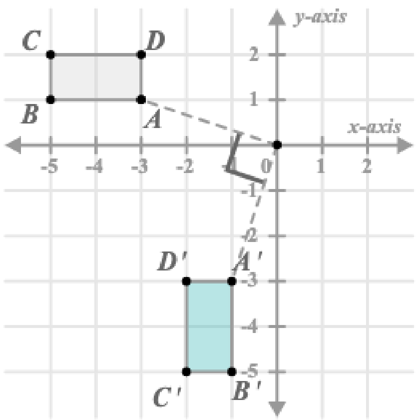


FIGURE 3.56

Pick one point, such as point A, and draw a line segment connecting it to the origin. Because perpendicular lines form 90°angles, to rotate 90°, find a line perpendicular to the line segment connecting A to the origin. *Remember that perpendicular lines have opposite reciprocal slopes.* A' will be on the perpendicular line, the same distance from the origin as A was.

Once you have A', you can build the rectangle around it. A was at point (-3, 1) and A' is at point (-1, -3).

- **In general, a** 90°**counterclockwise rotation about the origin takes** (x,y) **and maps it to** $(-y,x)$.

MEDIA

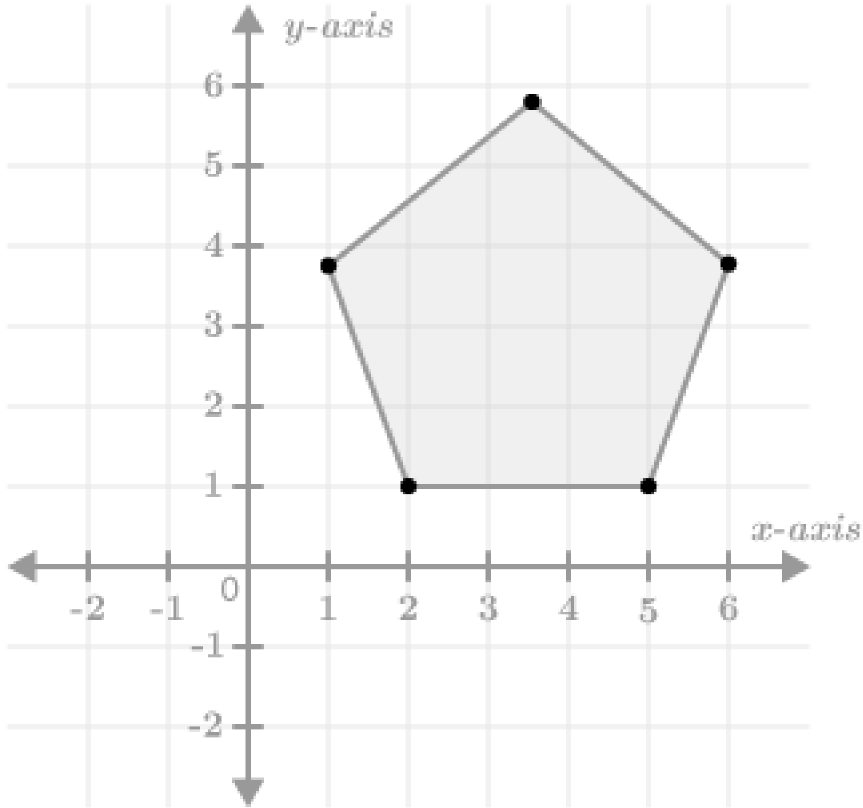
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/223443

Examples

Example 1

You want your friend to rotate the pentagon 90°. What are the two other pieces of information you need to tell your friend to ensure that she rotates the pentagon the way you are imagining?



To make sure your friend correctly rotates the pentagon 90°, you need to also tell her the center of rotation and the direction of rotation.

Example 2

Describe the rotation below.

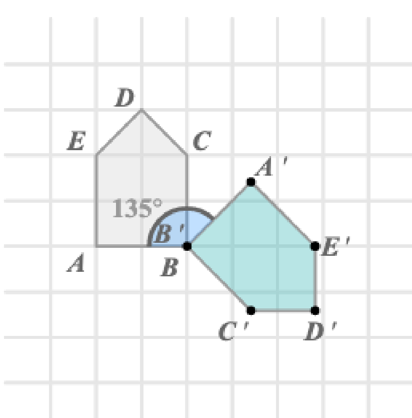


FIGURE 3.57

This is a 135°clockwise rotation about point B.

You could also say this is a 225°(360° − 135°) counterclockwise rotation about point B.

Example 3

Rotate the rectangle 180°counterclockwise about the origin. How do the points of the original rectangle compare to their corresponding points on the image? How would your answer be different if instead you rotated 180°clockwise about the origin?

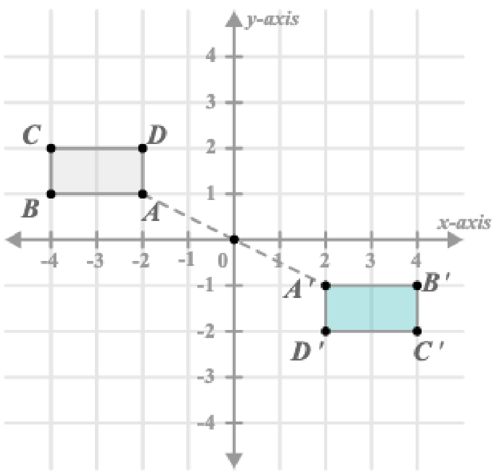


FIGURE 3.58

There would be no difference. A rotation 180°clockwise is the same as a rotation 180°counterclockwise.

You can see there is a straight line (180 degrees) passing through the origin and connecting each pair of corresponding points. The result is that the signs of each term in the ordered pairs changes: $(x,y) \rightarrow (-x,-y)$.

Review

Describe each of the following rotations in TWO ways.

1.

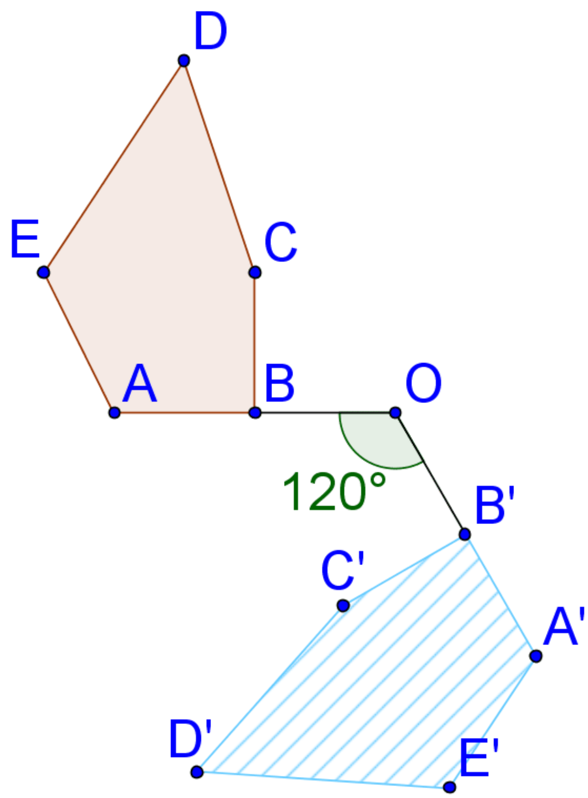


FIGURE 3.59

2.

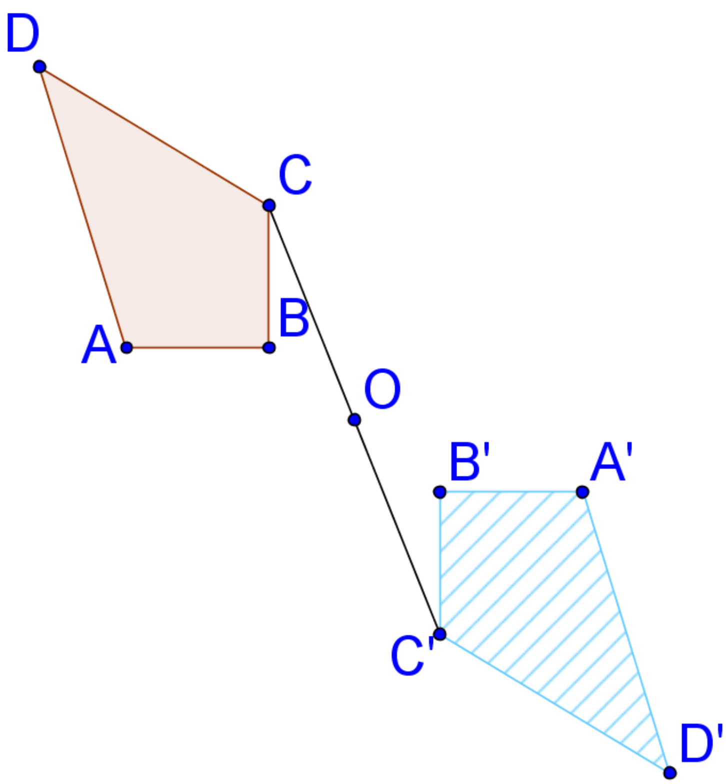


FIGURE 3.60

3.

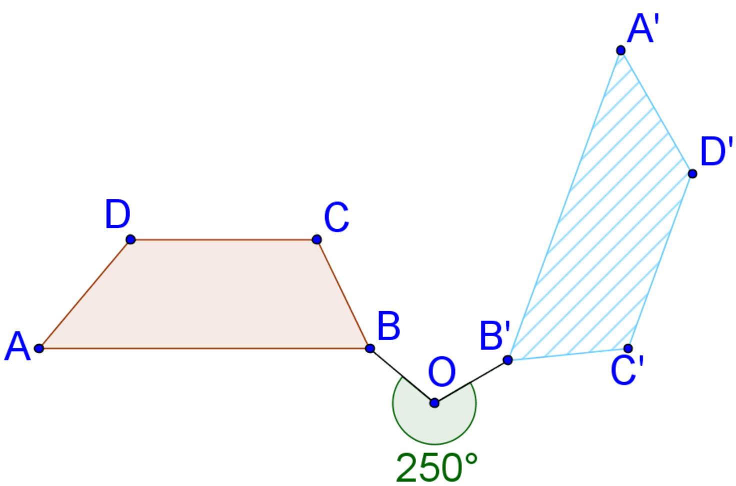


FIGURE 3.61

4. What does a 360°rotation look like?

5. Rotate the rectangle below 90°clockwise about the origin.

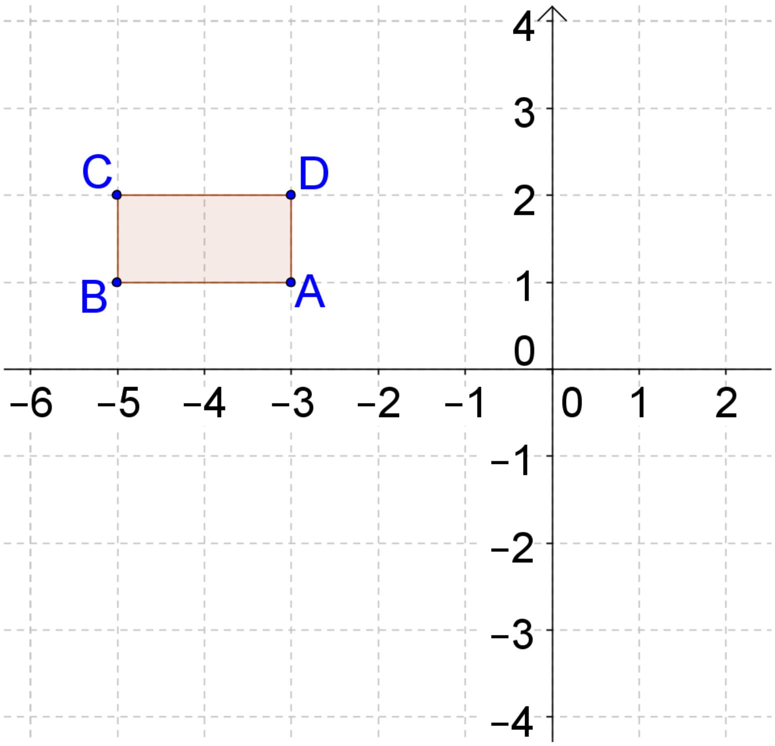


FIGURE 3.62

6. Use the previous example to help you describe what happens to a point (x,y) that is rotated 90°clockwise about the origin.

7. (3, 2) is rotated 90°clockwise about the origin. Where does it end up?

8. (3, 2) is rotated 180°clockwise about the origin. Where does it end up? *Hint: Use Guided Practice #2 for help.*

9. (3, 2) is rotated 90°counterclockwise about the origin. Where does it end up? *Hint: Use Example C.*

10. What do circles have to do with rotations?

11. ΔABC is rotated 130°clockwise about point O to create $\Delta A'B'C'$. Describe two connections between the points B, O, and B'.

For 12-15, consider ΔABC below.

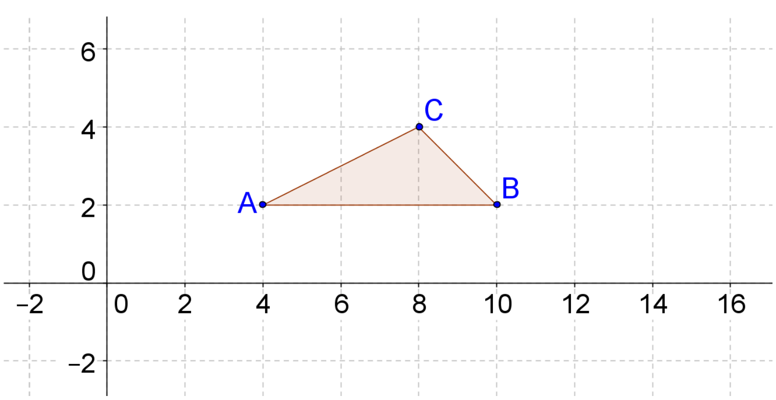


FIGURE 3.63

12. Rotate ΔABC 90°counterclockwise about point B.

13. Rotate ΔABC 90°counterclockwise about point A.

14. Rotate ΔABC 90°counterclockwise about point C.

15. Could you have used the rule that a 90°counterclockwise rotation takes (x,y) to $(-y,x)$ to perform the previous three rotations? Explain.

16. The diagram below shows a rotation of the brown triangle to the green triangle. It shows a sequence of reflections that also maps the brown triangle to the green triangle. Explore the diagram further and explain the relationship between rotations and reflections shown there.

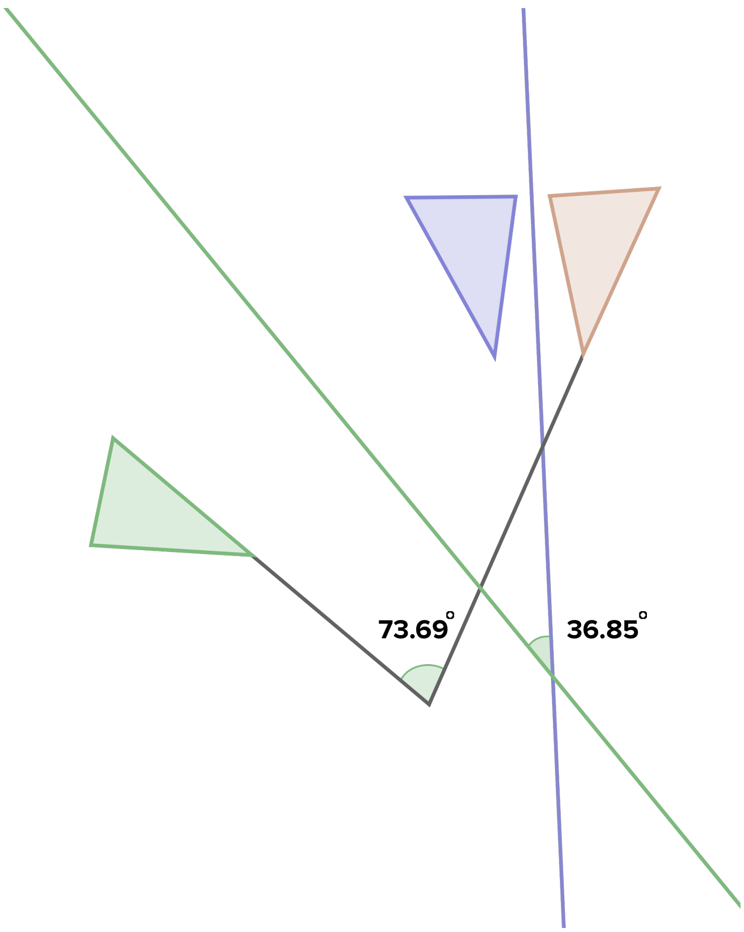


FIGURE 3.64

17. What rotations map every polygon to itself? Explain. What rotations maps a line to a parallel line? Explain. What rotations map a line to itself? Explain.

18. A given polygon A maps to polygon B by rotation. Is it possible to map A to B by translation? Why or why not? Is it possible to map A to B by a single reflection? Why or why not?

19. Describe and compare the translations, reflections, and rotations that map lines to themselves. Describe and compare the translations, reflections, and rotations that map angles to themselves. Describe and compare the translations, reflections, and rotations that map any polygon to itself.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.7.

3.8 Geometry Software for Rotations

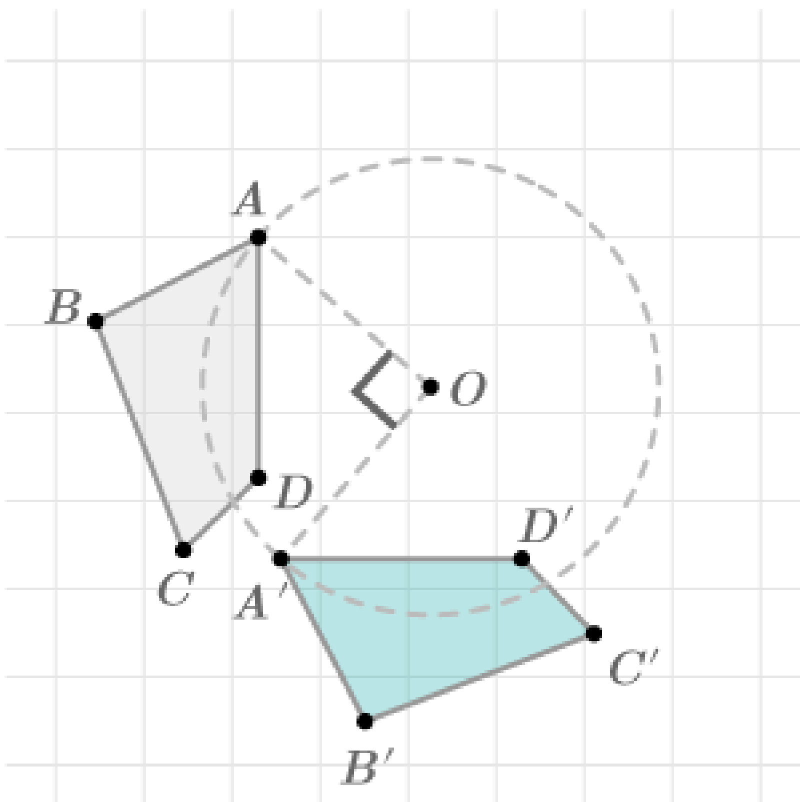
Learning Objectives

Here you will learn how to perform rotations using geometry software.

Rotations with a Grid

Recall that a rotation is one example of a rigid transformation. A rotation of $t°$about a given point O takes each point on a shape P and moves it to P' such that P' is on the circle with center O and radius $\overline{OP}$ and $\angle POP' = t°$.

Below, quadrilateral $ABCD$ has been rotated $90°$counterclockwise about point O to create new quadrilateral $A'B'C'D'$ (the image).



Rotation Without a Grid

When you are working on a grid (or graph paper), it's possible to perform rotations of $90°, 180°$, or $270°$by using slopes to find perpendicular lines. But what if the grid is not there, or the rotation is not a multiple of $90°$? Then it is not as easy to do the rotation because there are no grid lines as a guide.

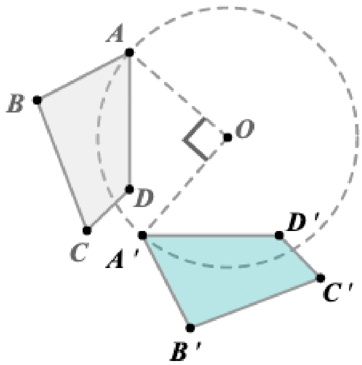


FIGURE 3.65

To perform a rotation without a grid, you need to:

1. Construct a circle with center O through one of the points that define the shape.
2. Construct a segment connecting that point that defines the shape with point O.
3. Construct an angle of the given number of degrees for the rotation from this segment in the direction specified. The endpoints for this angle should be on the circle previously drawn.
4. Repeat the above three steps for all the points that define the shape.
5. Connect the endpoints to form the rotated image.

Doing this by hand requires careful construction of circles and angles using a compass and a protractor. Geometry software simplifies this process, because geometry software commonly has a “rotate” tool. There are many different examples of geometry software that are free to download.

Rotation Using Geometry Software

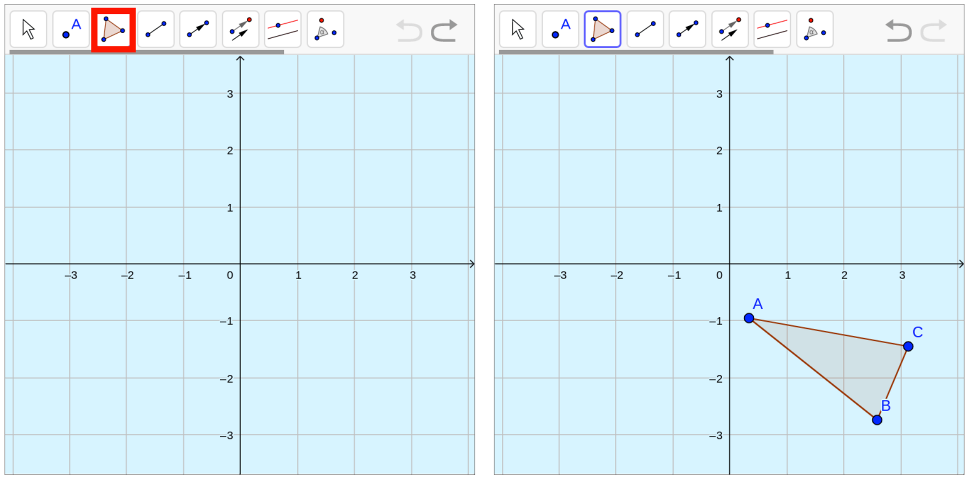
Follow the steps to perform the rotation of a polygon in the interactive.

MEDIA

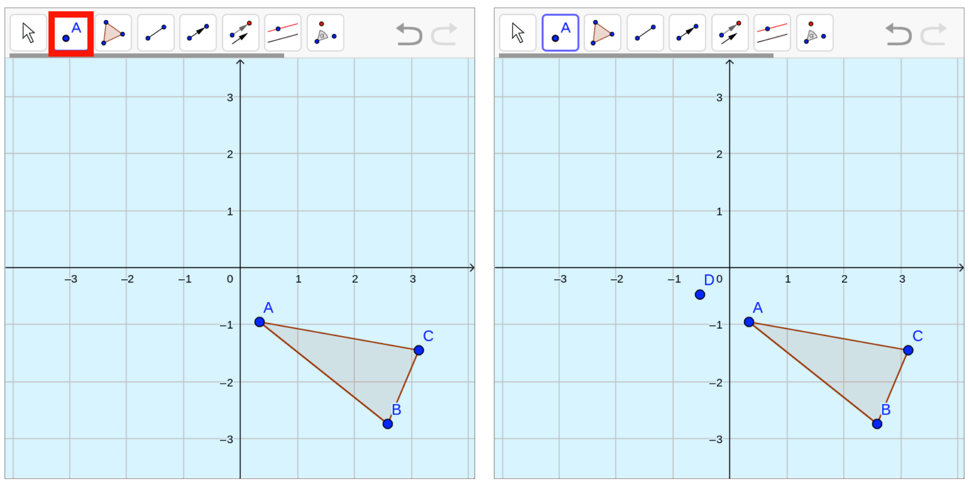
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/237085

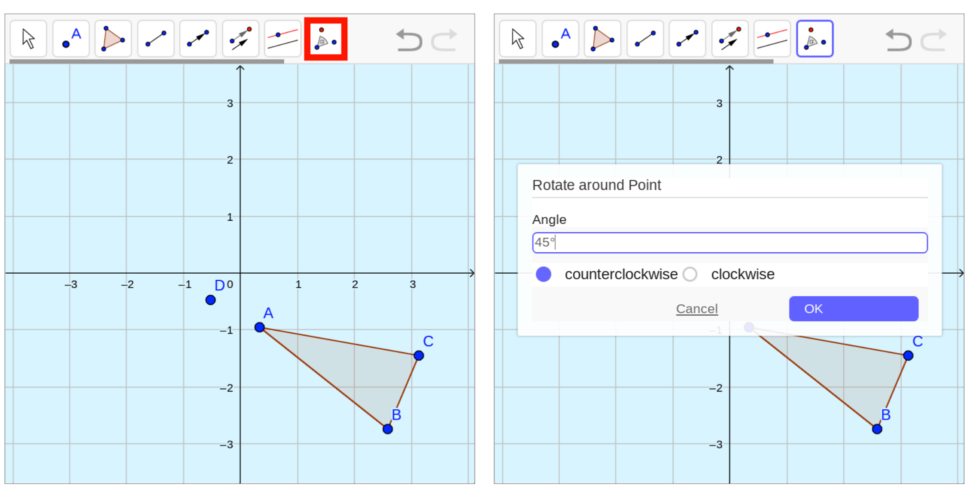
To perform a rotation, first select the "polygon" tool and create your polygon.

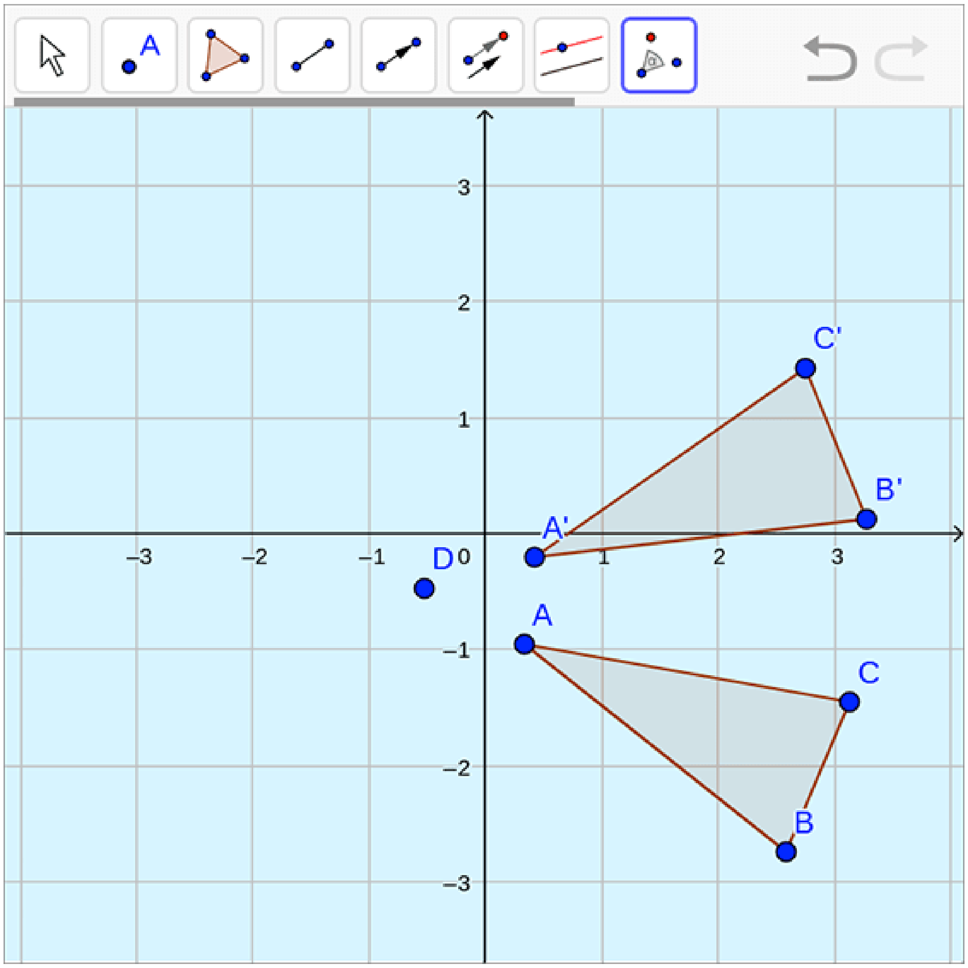


Next, use the "point" tool to create the point you will rotate the polygon about (this could be one of the points that defines the polygon, but may also be elsewhere).

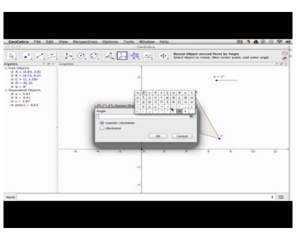


Now, rotate the polygon about the point by clicking on the “rotate” tool, then the shape and then the center of rotation point. Specify the number of degrees and direction of rotation in the window that pops up.





Note that the points defining the image are labeled with prime notation. At this point you can move point D to redefine the center of rotation or change your original shape and your image will move/change accordingly. *Make sure to select the cursor button before attempting to move points.*



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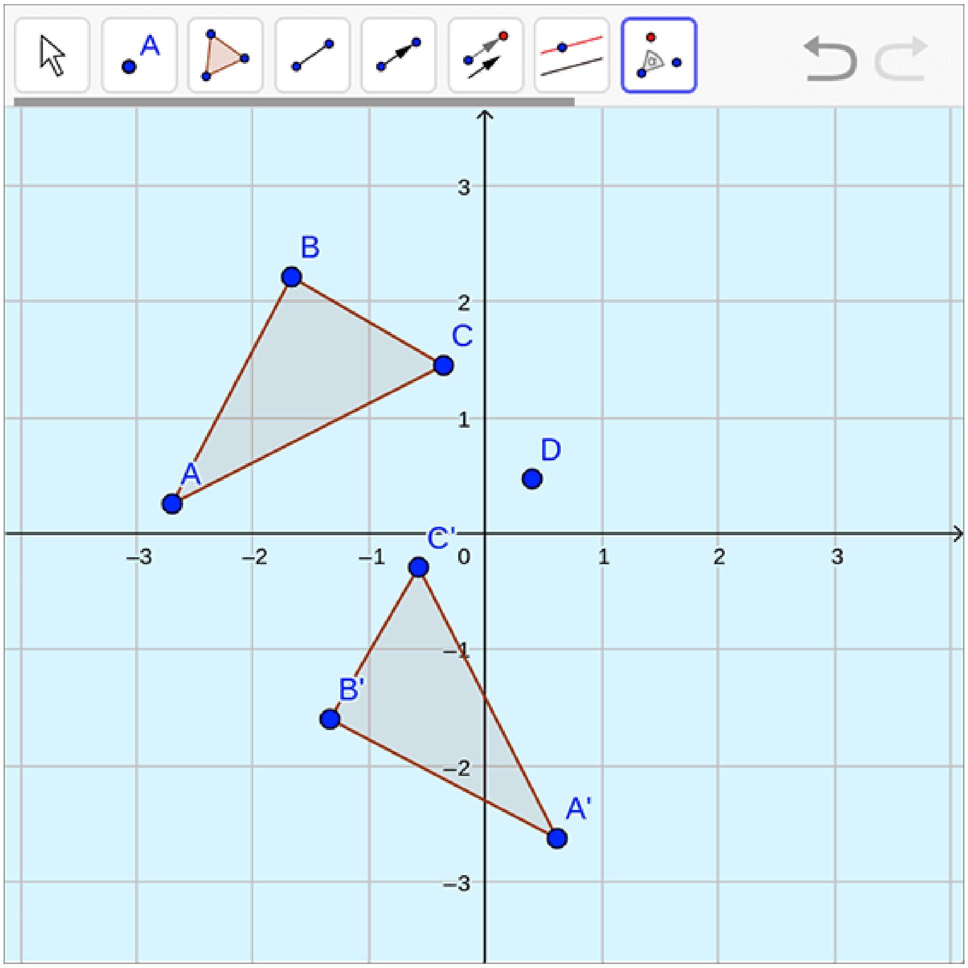
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/71074

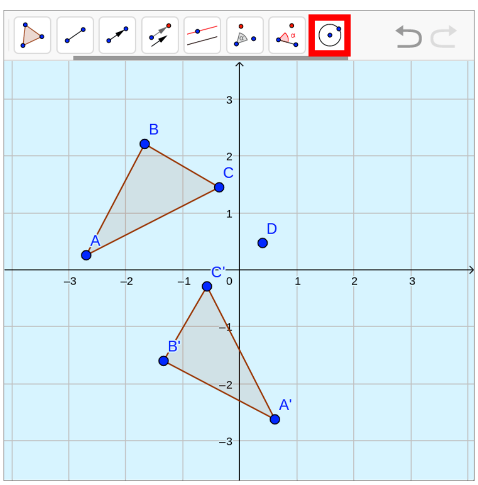
Rotation Proof

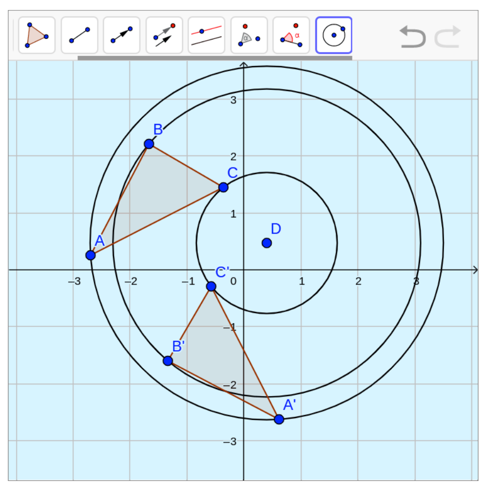
ΔABC is rotated 90°counterclockwise around point D to create $\Delta A'B'C'$.

Show that the center of rotation D is the center of circles containing A and A', B and B', and C and C'.



Construct circles by clicking on the "circle with center through point" tool. Construct each circle by selecting point D and then one vertex of the original triangle.



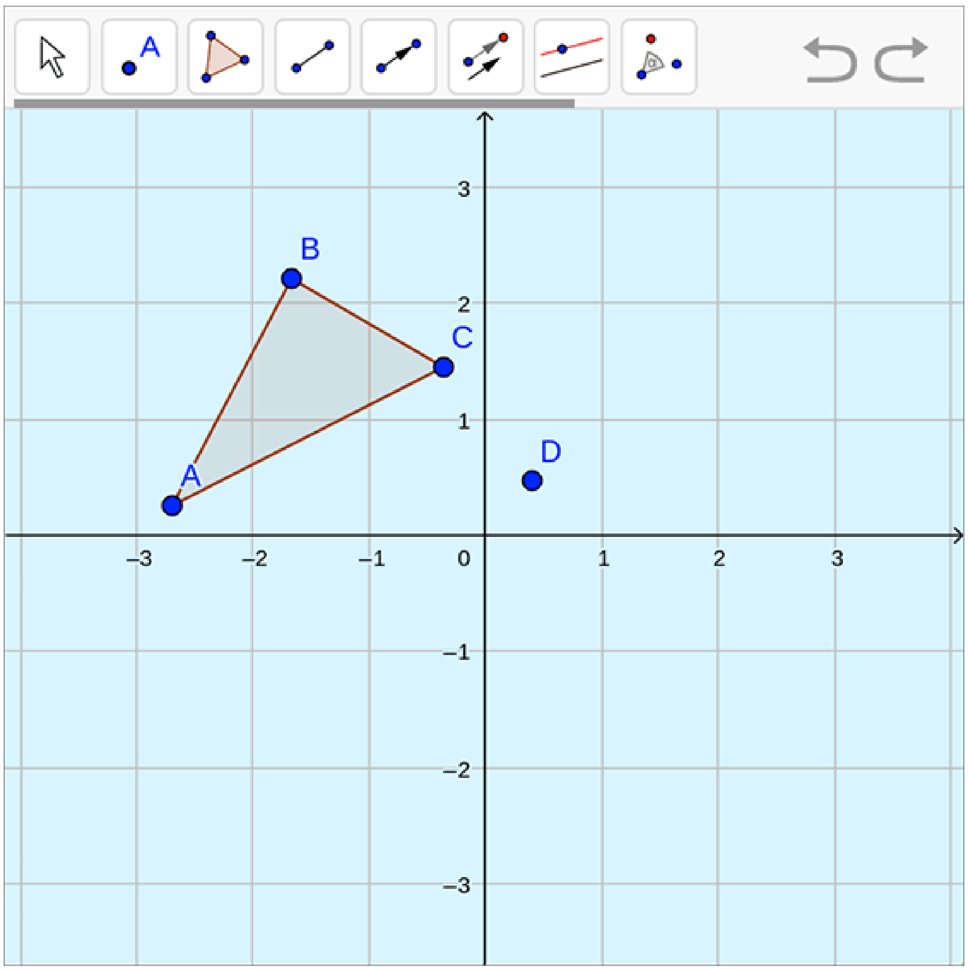


The circle containing point A also contains point A', the circle containing point B also contains point B', and the circle containing point C also contains point C'.

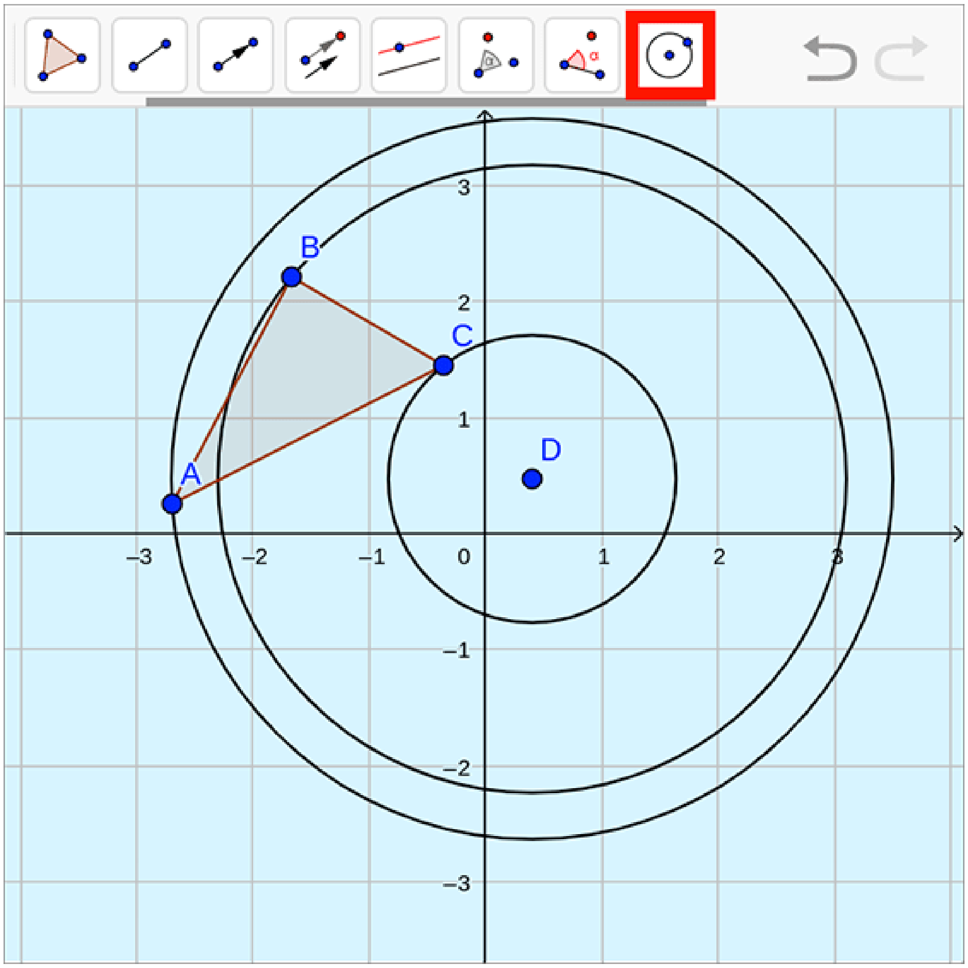
All three circles are centered at point D.

Performing Rotations

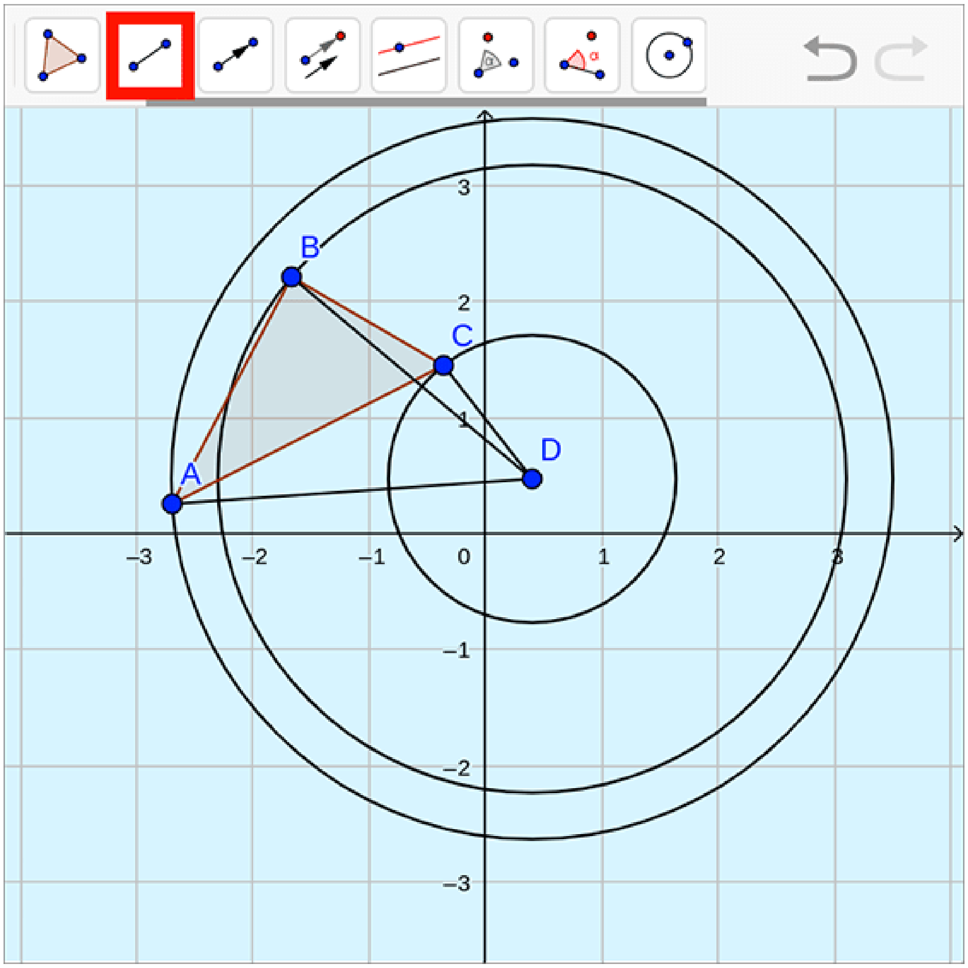
Rotate ΔABC 90°counterclockwise about point D by using the "rotate" tool for each vertex of the triangle.



First, using the "circle with center through point" tool to construct circles with centers at point D that pass through each of the three vertices that define the triangle. Use the process from the previous example problem to construct the circles.

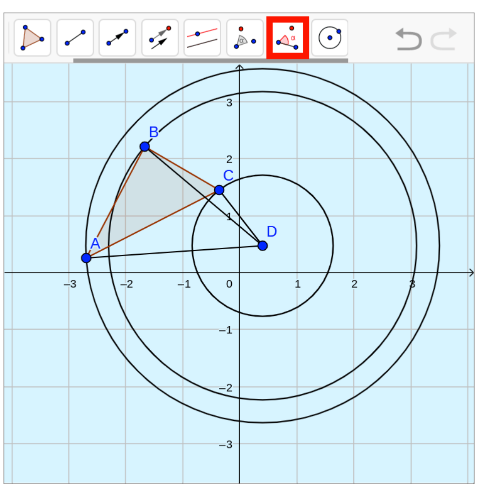


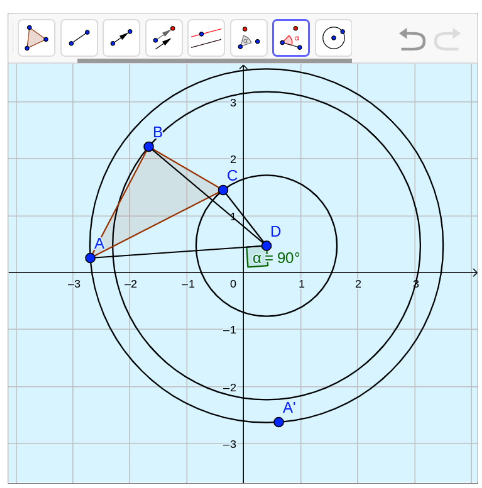
Next, selecting the "segment" tool, click on point D, then on point A. Repeat for vertices B and C to construct segments connecting each vertex with the center D of the circle.



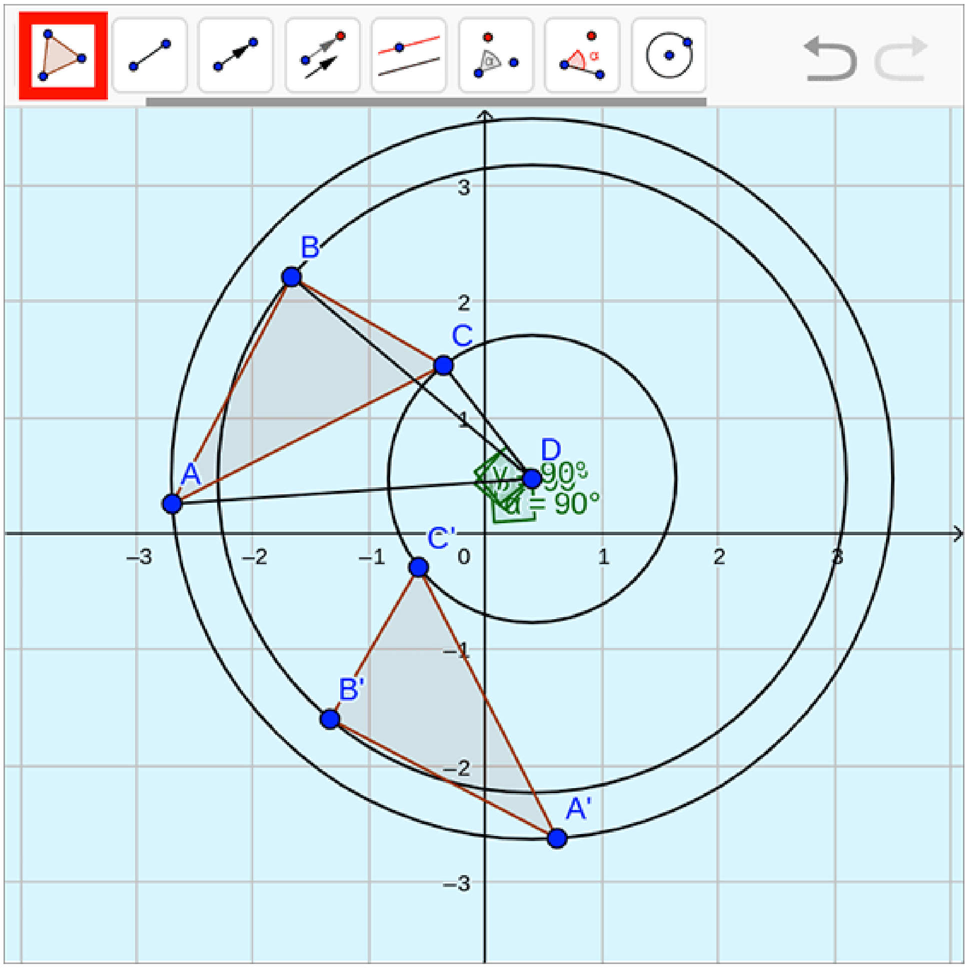
Then, construct a 90°angle from $\overline{AD}$ by selecting the "angle with given size" tool. Select point A, then point D, and then enter 90°counterclockwise in the window that pops up.

A' will appear.





Repeat for each vertex of the triangle. Then, using the "polygon" tool, connect A',B', and C' to form the triangle image.



Note that it isn't actually necessary to first construct the circles and line segments in order to do the rotation in this way. However, they allow you to be confident that your rotation is correct because the image points end up on the same circles as their corresponding points. Also, doing it in this way comes closest to what you would need to do to perform the construction by hand.

Compare and contrast the 'Rotation Proof' and 'Performing Rotation' methods for explored above.

Both methods correctly performed a rotation. The first method was faster. The second method made it more clear that *each* point that defined the triangle was being rotated around a circle with center *D*. The second method shown is closer to the method you would have to use to perform a rotation by hand.

Examples

Example 1

Geometry software has a rotation function that makes it easy to rotate an object about a point. Can you perform a rotation using geometry software without using that function?

> You can perform a rotation using geometry software without the "rotate" tool by constructing circles and angles. This method was shown in the second example titled "Performing Rotations."

Example 2

Try doing the same rotation without using the "rotate" tool by constructing circles and angles as shown in the second example titled "Performing Rotations". How can you verify that you have done this correctly?

Answers will vary depending on what polygon you construct. Look back at the guidance section for help. To check that you have performed the rotation without the "rotate" tool correctly, just make sure that both images ended up in the same place.

Review

1. Create a polygon.

2. Rotate the polygon 100° counterclockwise about one of its own points using the rotate button.

3. Rotate the polygon 260° clockwise about the same point that you chose in #2 using the rotate button. What happened? Why?

4. Rotate the polygon 90° counterclockwise about a point that is not on the polygon.

5. Rotate the polygon 90° counterclockwise about the same point you used in #4 without using the rotate button by constructing circles and angles. Is your rotation correct?

6. Create a regular decagon. *To do this, instead of selecting "polygon", select "regular polygon". Plot the first two points of your polygon, then enter the number of points/sides you want your polygon to have (10).*

7. Where would the center of rotation have to be for it to be possible to rotate the decagon less than 360° and have the image be indistinguishable from the original decagon? Find a way to plot this center of rotation.

8. What's the smallest number of degrees you can rotate the decagon about the point from #7 and have the image be indistinguishable from the original decagon? Test this out.

9. Rotations are rigid transformations which means that distance is preserved. Create a new polygon and rotate it about some point. Verify that distance has been preserved by using your geometry software to measure sides of your original polygon and their images. *Select "distance or length" from one of the drop down menus. Then, click on each line segment that you want to measure to see its length.*

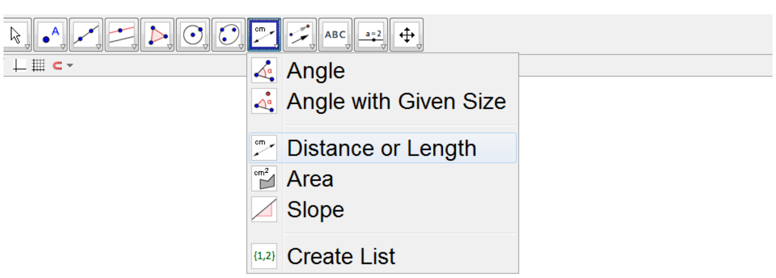


FIGURE 3.66

10. Rotations are rigid transformations which means that angles are preserved. Verify that angles have been preserved by using your geometry software to measure two corresponding angles for your polygon and image from

#9. *Select "angle" from the same drop down menu as in #9. To find what angle you want to measure, click on the three points you would use to name the angle. You must click on the points in clockwise order for it to measure the correct angle.*

11. Construct a circle.

12. Where would the center of rotation have to be for a rotation of any number of degrees to produce an image that is indistinguishable from the original circle? Test this idea.

13. Could all rotations of circles also have been translations or reflections? Explain.

14. Create a center of rotation that is outside of the circle.

15. Explore how you might be able to rotate the circle 135° counterclockwise about the center of rotation without using the rotate button by constructing additional circles and angles.

16. Create a regular hexagon. (In the drop-down menu for create polygon there is an option to create a regular polygon.) What is the measure of each interior angle of the hexagon? Why? Rotate the hexagon this number of degrees around one of the vertices. Repeat this rotation around each vertex. What do you observe? Notice that the rotations covered the plane without any overlaps or gaps. How is this possible? Are there other regular polygons this would work for? If so, which ones, and why? Experiment.

17. Look at the triangle below. Describe the relationship of $\angle 1, \angle 2$, and $\angle 3$ to the interior angles of the triangle. Is it possible to place two copies of the triangle such that $\angle 5$ and $\angle 6$ fit perfectly in place of $\angle 2$? Why or why not? Is this also true for $\angle 3$? Why or why not? Is it possible to put one copy of the triangle such that $\angle 4$ fits perfectly in place of $\angle 1$? Why or why not?

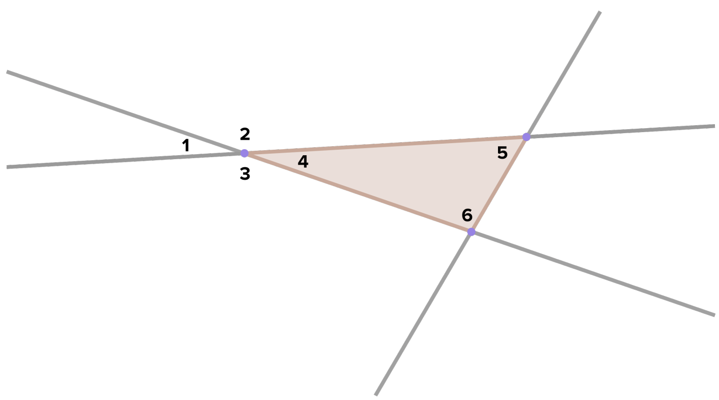


FIGURE 3.67

18. Create a scalene triangle. Call this A. Rotate A 180°around a vertex. Call this image B. Now perform a **translation** of A that maps a side of A to the corresponding side of B. Call this triangle C. Now continue using translations to map any of the existing triangles so that one side coincides with the corresponding side on an existing image. What do you observe? Explain how this is possible in the context of the previous problem.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.8.

3.9 Rotation Symmetry

Learning Objectives

Here you will learn about rotation symmetry and will identify shapes with rotation symmetry.

Rotation Symmetry

A shape has symmetry if it can be indistinguishable from its transformed image.

- A shape has rotation symmetry if there exists a rotation less than 360°that carries the shape onto itself.
- If you can rotate a shape less than 360°about some point and the shape looks like it never moved, it has rotation symmetry.

There are 3 ways to name rotation symmetry.

1. **Order**: The order of rotation symmetry of a geometric figure is the number of times you can rotate the geometric figure so that it looks exactly the same as the original figure before you get back to where you started.
    a. In the lefthand image below, the shape has two positions that are indistinguishable, so it has rotation symmetry of order 2.
    b. On the right, the shape has the positions that are indistinguishable, so it has rotation symmetry of order 3.

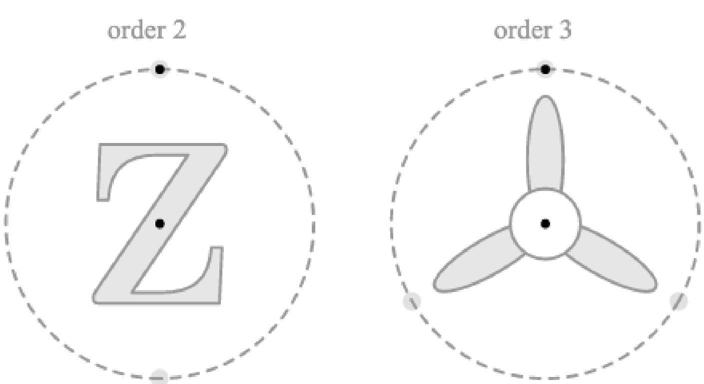


FIGURE 3.68

2. **Fraction of a turn:** If you look at how far you turn the shape to get it to look the same, you can think of that as a fraction of how far you go to get all the way around.
    a. On the left, you turn $\frac{1}{2}$ way around, so it has $\frac{1}{2}$ turn symmetry.
    b. On the right, it turns $\frac{1}{3}$ of the way around to look like itself again, so it has $\frac{1}{3}$ turn symmetry.

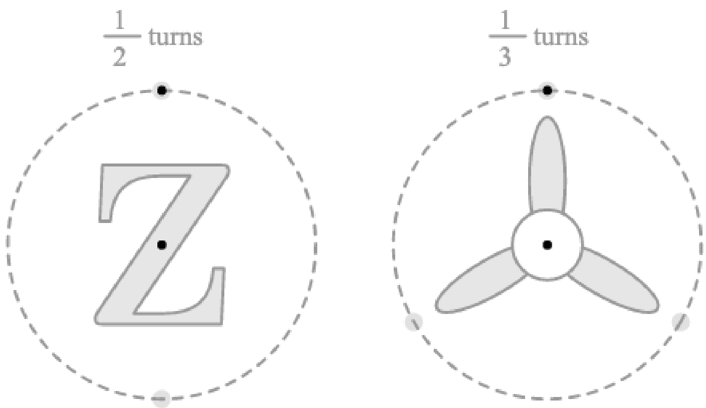


FIGURE 3.69

3. **Angle:** The angle of rotation symmetry is the smallest angle the figure can be rotated to coincide with itself. So the half turn becomes half of $360^\circ = 180^\circ$, and the one-third turn becomes a third of $360^\circ = 120^\circ$.

   a. The shape on the left has 180°rotation symmetry.
   b. The shape on the right has 120°rotation symmetry.

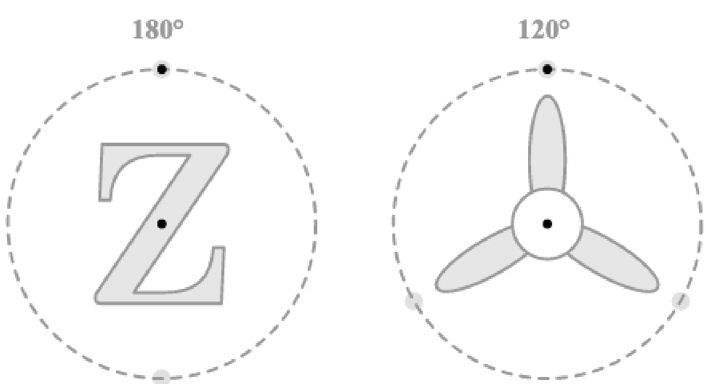


FIGURE 3.70

Identifying Rotation Symmetry

A rectangle is an example of a shape with rotation symmetry.

- A rectangle can be rotated 180°about its center and it will look exactly the same and be in the same location. The only difference is the location of the named points.
- A rectangle has half-turn symmetry, and therefore is order 2.

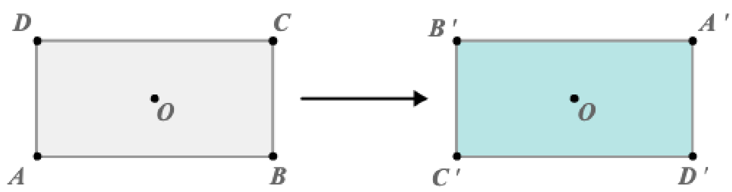
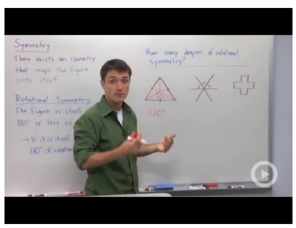


FIGURE 3.71



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/71075

Does a square have rotation symmetry?

Yes, a square can be rotated 90°counterclockwise (or clockwise) about its center and the image will be indistinguishable from the original square.

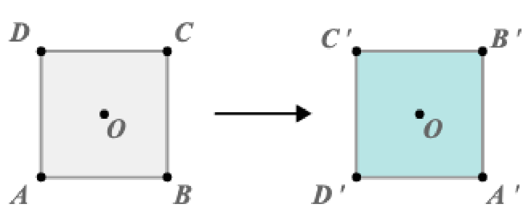

FIGURE 3.72

Identifying Angles of Rotation

How many angles of rotation cause a square to be carried onto itself?

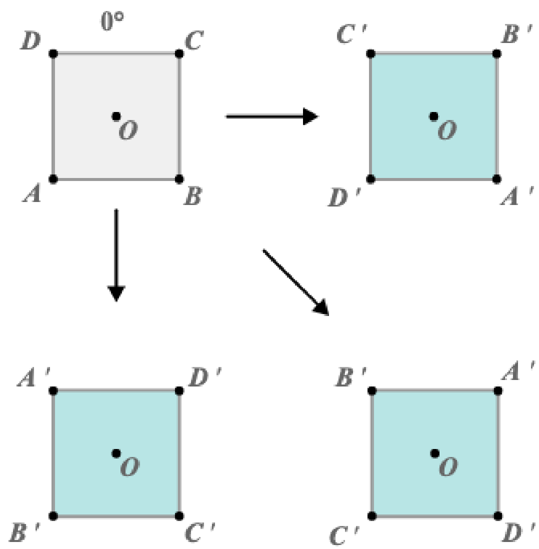

FIGURE 3.73

- Rotations of 90°, 180°, and 270° *in either direction* will all cause the square to be carried onto itself.
- A square has quarter-turn rotation symmetry, and so has an order of 4.

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Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/223448

Rotation Symmetry of a Trapezoid

Consider rotation symmetry in trapezoids.

- In a generic trapezoid as well as isosceles trapezoid, there is no rotation symmetry.
- A trapezoid must be rotated a full 360°to again appear in its original position.

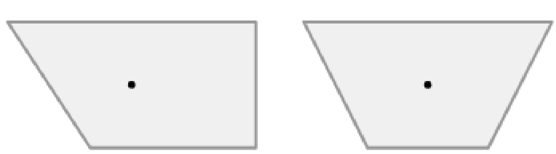
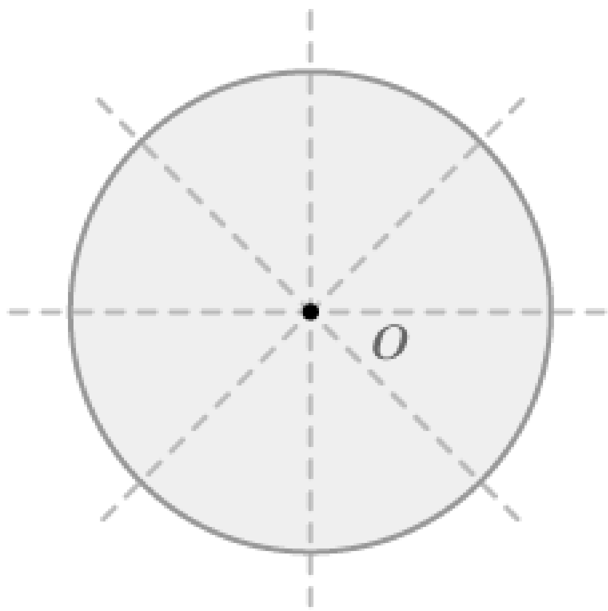


FIGURE 3.74

Rotation Symmetry of a Circle

Rotate a circle about its center O through **any angle** and it fits onto itself.

- A circle has rotation symmetry around the center for every angle.
- A circle has an unlimited number of angles of symmetry and the order of its rotation is infinite.



Rotation Symmetry

Drag the cursor to rotate the polygon and observe the angle of rotation.

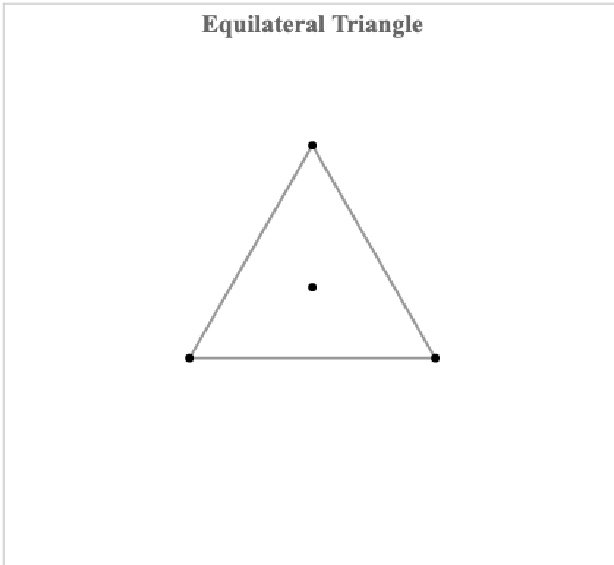


FIGURE 3.75

Examples

Example 1

What happens when you rotate the regular pentagon below 72°clockwise about its center? Why is 72°special in this case?

When you rotate the regular pentagon 72°about its center, it will look exactly the same. This is because the regular pentagon has rotation symmetry, and 72°is the minimum number of degrees you can rotate the pentagon in order to carry it onto itself.

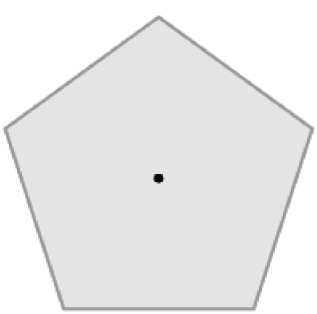


FIGURE 3.76

Example 2

Does each capital letter below have rotation symmetry? If so, state the angles of rotation that carry the letter onto itself.

a. Capital letter N

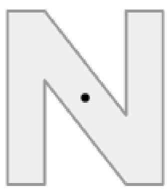


FIGURE 3.77

Yes, it does have rotation symmetry. It can be rotated 180°.

b. Capital letter S

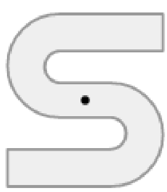


FIGURE 3.78

Yes, it does have rotation symmetry. It can be rotated 180°.

c. Capital letter H

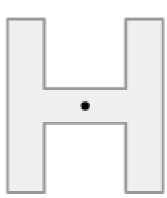


FIGURE 3.79

Yes, it does have rotation symmetry. It can be rotated 180°.

d. Capital letter B

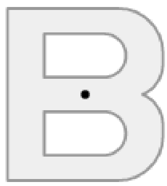


FIGURE 3.80

No, it does not have rotation symmetry.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/rotation-symmetry/plix/Shapes-for-Rotational-Symmetry-546556738e0e08226537d97e

Review

1. What does it mean for a shape to have symmetry?
2. What does it mean for a shape to have rotation symmetry?
3. Why does the stipulation of less than 360° exist in the definition of rotation symmetry?

For each of the following shapes, state whether or not it has rotation symmetry. If it does, state the number of degrees you can rotate the shape to carry it onto itself.

4. Equilateral triangle
5. Isosceles triangle
6. Scalene triangle
7. Parallelogram
8. Rhombus
9. Regular pentagon
10. Regular hexagon
11. Regular 12-gon

12. Regular n-gon
13. Circle
14. Kite
15. Where will the center of rotation always be located for shapes with rotation symmetry?
16. Does every polygon that has rotation symmetry also have reflection symmetry? Why or why not?
17. Does every polygon that has reflection symmetry also have rotation symmetry? Why or why not?
18. Does a line have rotation symmetry? How about reflection symmetry? Explain.
19. Does an angle have rotation symmetry? How about reflections symmetry? Explain?
20. Harold argues that a semi-circle has rotation symmetry, but Jay disagrees. Who is correct and why?
21. Tina argues that our reflections in the mirror show that humans have rotational symmetry, the image in a mirror is a 180°rotation of the original. Brenda disagrees. Who is correct and why?
22. Vanessa wants a house with a floor plan that has reflection symmetry, while Jessica prefers one that has rotation symmetry. Draw sketches of each. Which do you prefer and why?
23. Many animals, plants, or other objects in the world have reflection or rotation symmetry. Give examples and describe the symmetry in as much detail as possible.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.9.

3.10 Composite Transformations

Learning Objectives

Here you will learn about composite transformations.

A composite transformation (or composition of transformations) is two or more transformations performed one after the other. The following is an example of a translation followed by a reflection. The original triangle is the grey triangle and the image is the blue triangle. The purple triangle shows the intermediate step after the translation has taken place.

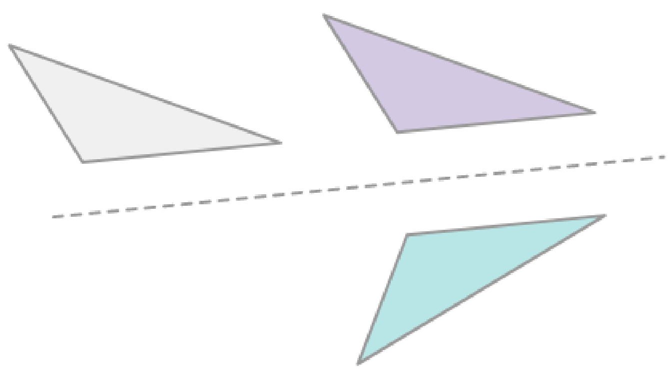


FIGURE 3.81

There is no single transformation that could have replaced the composite transformation above.

Composite transformations follow the order of operations.

- Just as $2 + (3 \times 4)$ is not the same as $(2 + 3) \times 4$, composite transformations must be performed in a specific order to ensure the correct result.
- The two Compare the two methods below to see the difference.

1. Transform the rectangle by **first rotating** the rectangle 90^{o}counterclockwise about the origin and **then translating** it along vector $\vec{v}$.

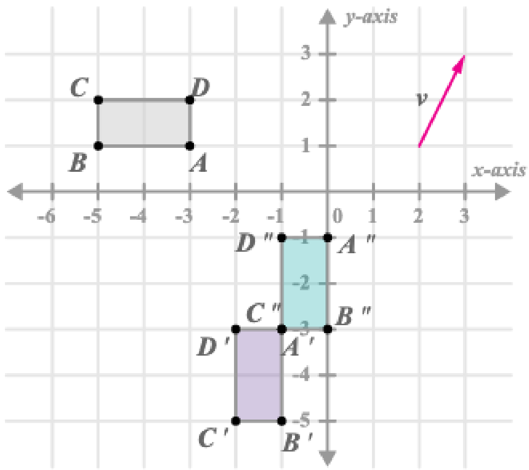


FIGURE 3.82

2. Perform the same transformations in the other order, **first translating** along vector $\vec{v}$, and **then rotating** 90^{o} counterclockwise.

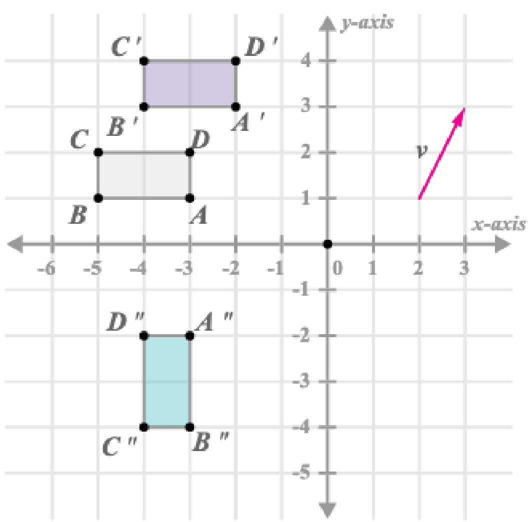


FIGURE 3.83

Notice that the final images are NOT in the same place.

- Transformations are not commutative.
- The order that transformations are performed matters.

Mapping Shapes

Describe a possible sequence of transformations that would carry ΔABC to ΔDEF.

Option 1: A 180°rotation about a point directly in between points C and D.

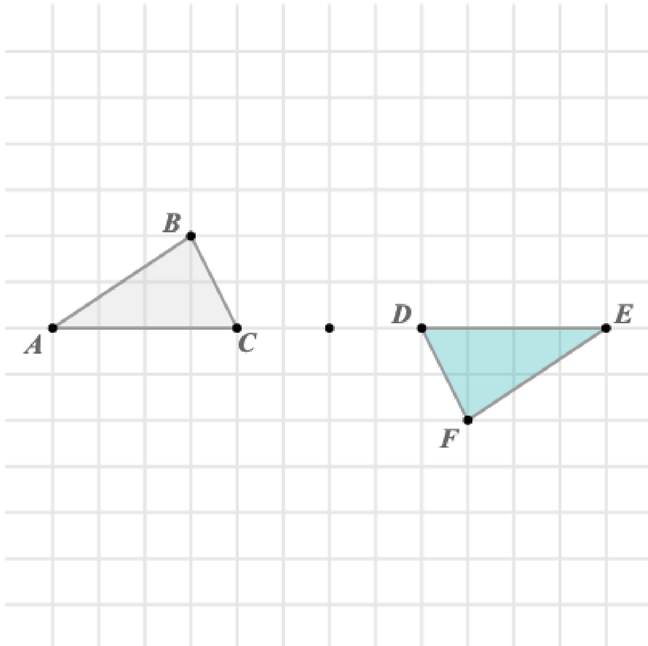


FIGURE 3.84

Option 2: A reflection across AC followed by reflection across a vertical line.

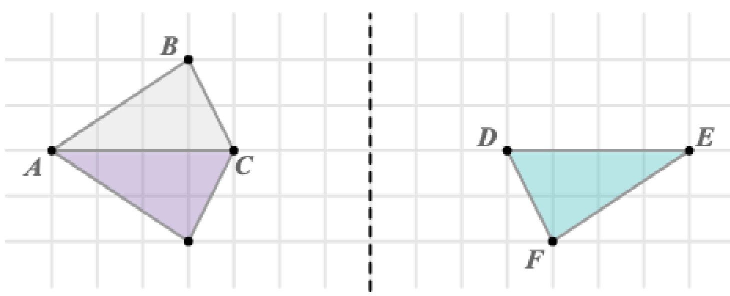


FIGURE 3.85

Option 3: Another possibility is that ΔABC was rotated 180° about point C and then translated to ΔDEF.

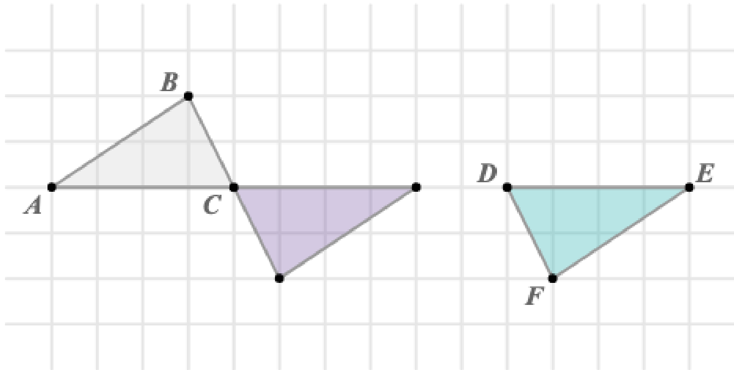


FIGURE 3.86

These are only three possible descriptions of the transformation. Can you think of another?

MEDIA

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Examples

Example 1

A reflection followed by a translation where the line of reflection is parallel to the direction of translation is called a glide reflection or a walk. Why do you think this is?

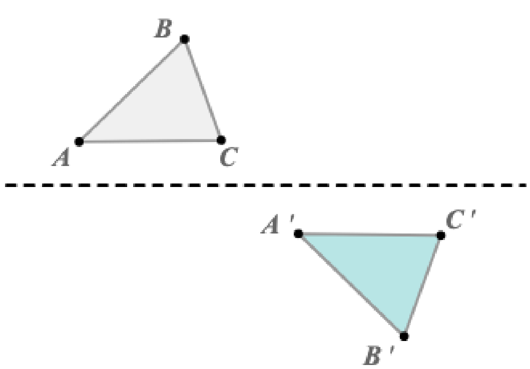
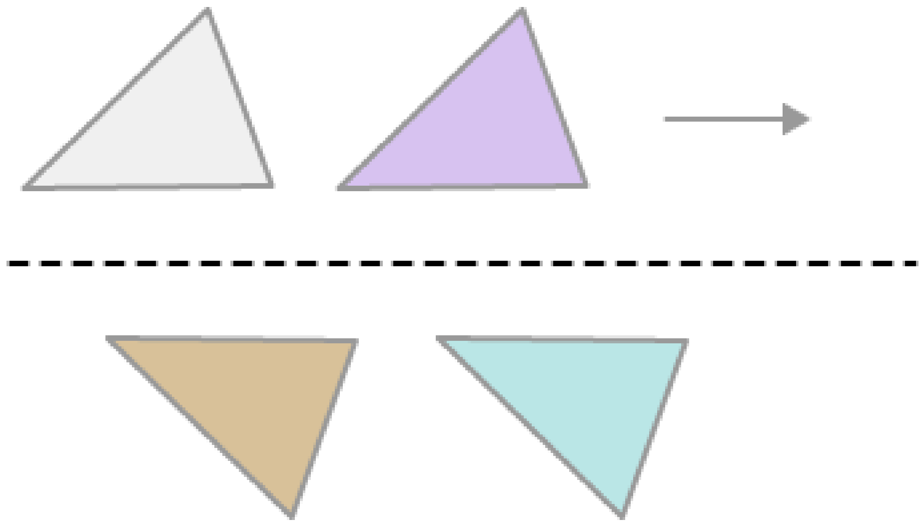


FIGURE 3.87

It is called a glide reflection or a walk because it's as if the shape was reflected and then glided over to a new location. When done repeatedly, the shapes look like footsteps walking.



Example 2

Transform the triangle by first reflecting across $\overline{AC}$ and then across the line perpendicular to $\overline{AC}$ through C. What one transformation could you have performed to get the same result?

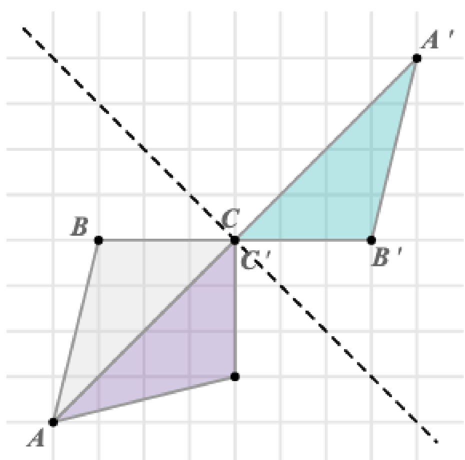


FIGURE 3.88

A 180°rotation about point C would have produced the same result.

Review

1. What is a composite transformation?
2. When doing a composite transformation, does the order in which you perform the transformations matter?
3. Describe a possible sequence of transformations that would carry ΔABC to ΔDEF.

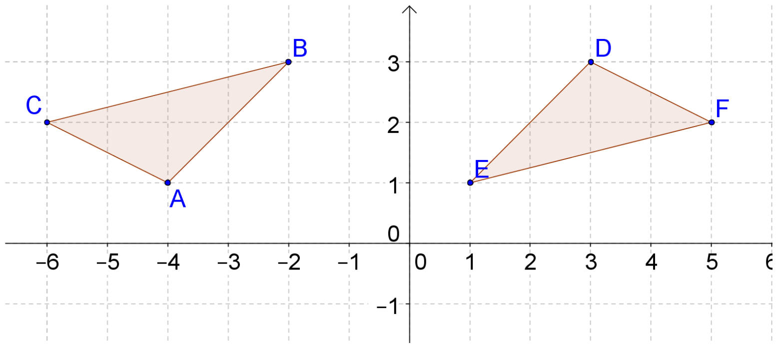


FIGURE 3.89

4. Describe another possible sequence of transformations that would carry ΔABC to ΔDEF.

5. Describe a possible sequence of transformations that would carry ΔGHI to ΔJKL.

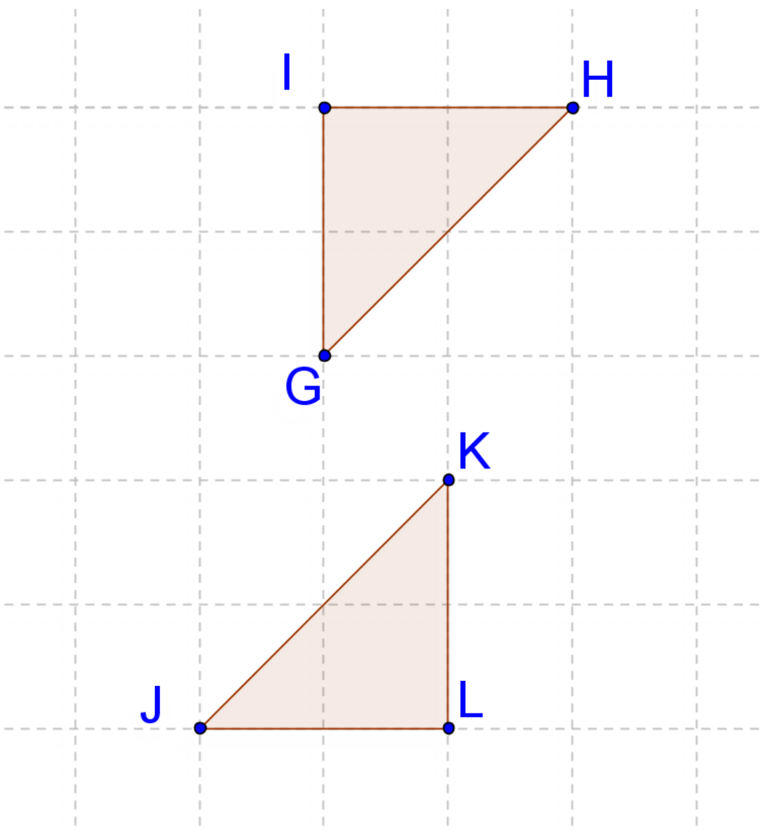


FIGURE 3.90

6. Describe another possible sequence of transformations that would carry ΔGHI to ΔJKL.

7. Describe a possible sequence of transformations that would carry ΔMNO to ΔPQR.

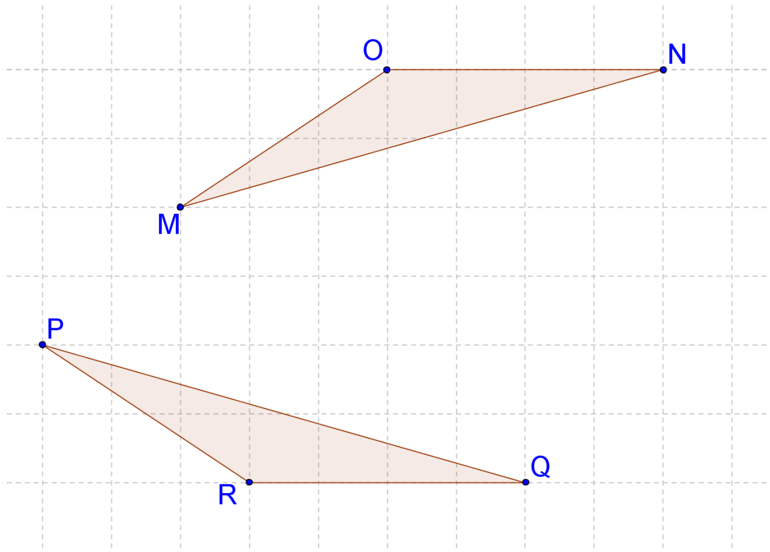


FIGURE 3.91

8. Describe another possible sequence of transformations that would carry ΔMNO to ΔPQR.

9. Construct a polygon on graph paper or with interactive geometry software.

10. Reflect the polygon twice across parallel lines. What one transformation could you have performed to get the same result?

11. Reflect the polygon twice across another set of parallel lines. What one transformation could you have performed to get the same result?

12. Make a conjecture by completing the sentence. Two reflections across parallel lines is the same as a ________-______.

13. Reflect the polygon twice across intersecting lines (not necessarily perpendicular). What one transformation could you have performed to get the same result?

14. Make a conjecture by completing the sentence. Two reflections across intersecting lines is the same as a ______________.

15. Given any translation, is it possible to accomplish the same result through a series of reflections? If so, describe them in detail. Use interactive geometry software to experiment.

16. Given any rotation, is it possible to accomplish the same result through a series of reflections? If so, describe them in detail. Use interactive geometry software to experiment.

17. Describe sequences of transformations that map the triangles in the diagram below to each other.

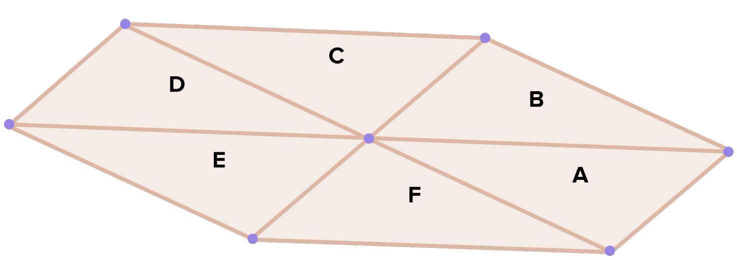


FIGURE 3.92

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.10.

3.11 Dilations

Learning Objectives

Here you will learn about dilations.

When you dilate a line segment, how is the original line segment related to the image?

Dilations

A **transformation** is a function that takes points in the plane as inputs and gives other points as outputs. You can think of a transformation as a rule that tells you how to create new points.

A **dilation** is an example of a transformation that moves each point along a ray through the point emanating from a fixed center point P, multiplying the distance from the center point by a common scale factor, k.

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The figure ΔABC below has been dilated about point P by a scale factor of 2. Notice that P, A, and $A'$are all collinear. Similarly, P, B, and B' are collinear and P, C, and C' are collinear. $PC = 3$ and $PC' = 6$. The scale factor of this dilation is 2 because $\frac{PC'}{PC} = \frac{6}{3} = 2$. If you calculate PA, PA', PB and PB' you will find that $\frac{PA'}{PA} = \frac{PB'}{PB} = 2$ as well.

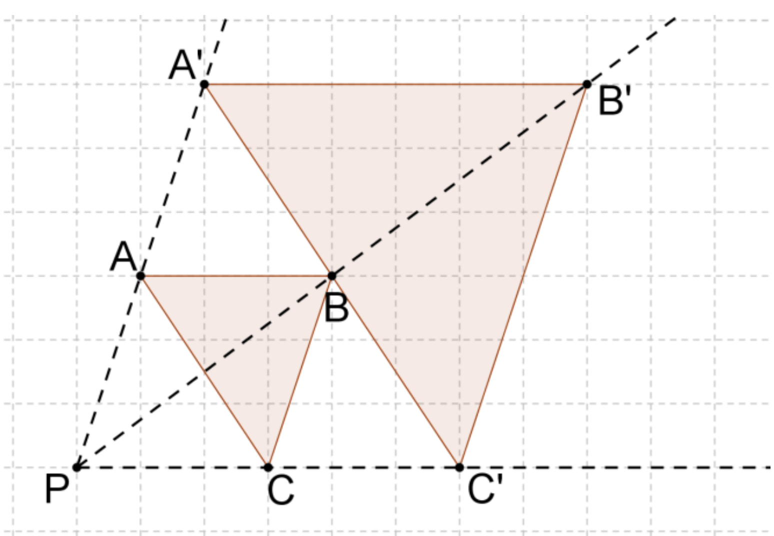

FIGURE 3.93

Triangle ABC has been dilated about point P by a scale factor of 2.

Note that a dilation is not a **rigid transformation** because it does not preserve distance. In the dilation above, $\Delta A'B'C'$ is larger than ΔABC. Dilations do, however, preserve angles. A shape and its image after a dilation will be **similar**, meaning they will be the same shape but not necessarily the same size.

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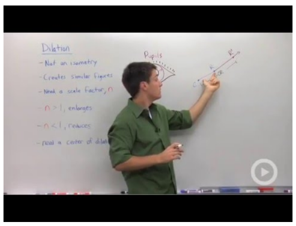

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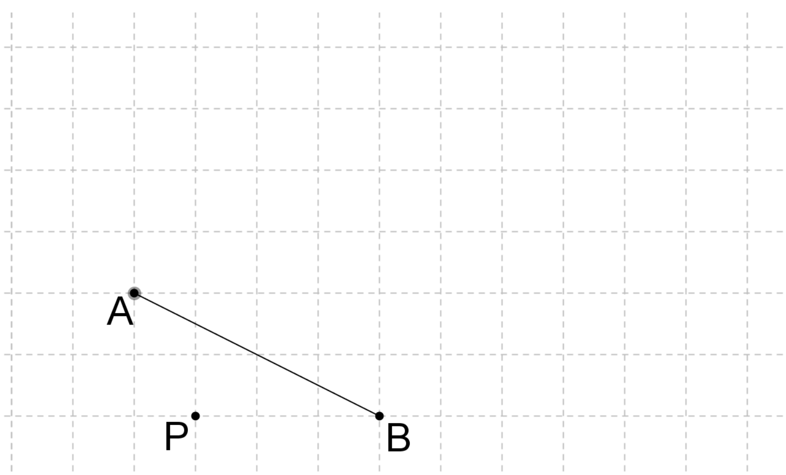
Let's take a look at a few dilation problems.

1. A shape is dilated by a scale factor of $\frac{1}{2}$. How does the image relate to the original shape?

If the scale factor is less than 1, the image will be smaller than the original shape.

2. Dilate the line segment below about point P by a scale factor of 3. Make at least two conjectures about how $\overline{AB}$ relates to $\overline{A'B'}$.



To dilate the line segment, draw a ray starting at point P through each end point. Use the grid lines to help you find points on these rays that are three times the distance from point P as the original endpoints were.

Two conjectures you might make are that $A'B' = 3AB$ or $\overline{A'B'} \parallel \overline{AB}$

3. Show that $A'B' = 3AB$ and $\overline{A'B'} \parallel \overline{AB}$ for the dilation in the previous problem.

To find the lengths of the segments, create right triangles with the segments as their hypotenuses.

4. Use the Pythagorean Theorem to find the lengths of the hypotenuses.

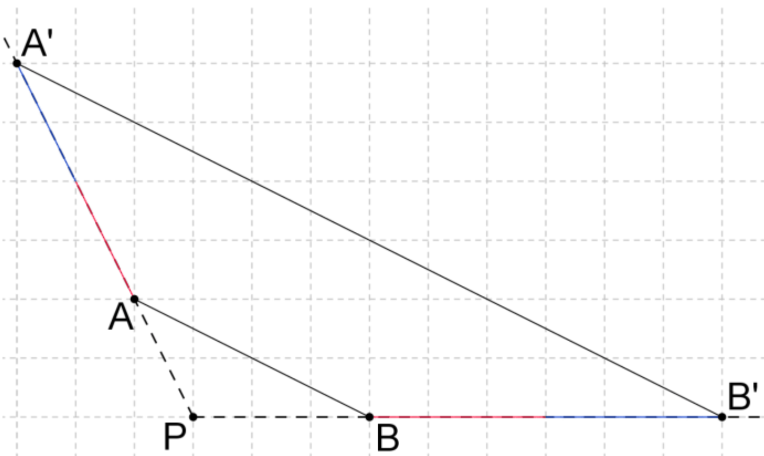


FIGURE 3.94
To dilate the line segment, draw a ray starting at point P through each and point.

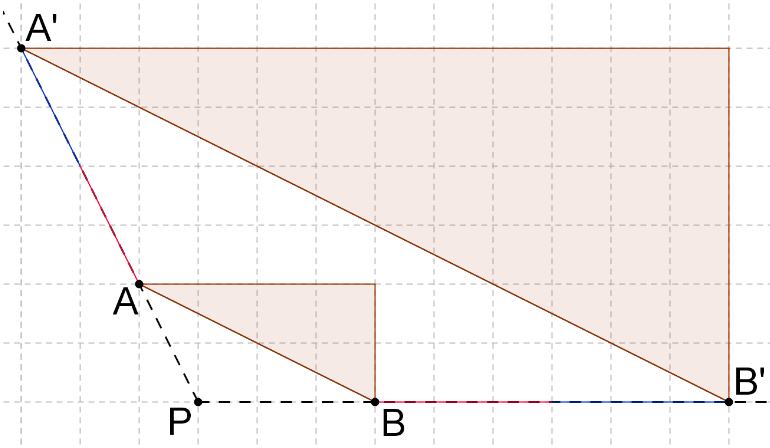


FIGURE 3.95
To find the lengths of the segments, create right triangles with the segments as their hypotenuses.

- $AB^2 = 4^2 + 2^2 \rightarrow AB^2 = 20 \rightarrow AB = \sqrt{20} = 2\sqrt{5}$
- $A'B'^2 = 12^2 + 6^2 \rightarrow A'B'^2 = 180 \rightarrow A'B' = \sqrt{180} = 6\sqrt{5}$

Therefore, $A'B' = 3AB$.

Two line segments are parallel if they have the same slope.

- Slope of $\overline{AB} : -\frac{1}{2}$
- Slope of $\overline{A'B'} : -\frac{1}{2}$

Therefore, $\overline{A'B'} \parallel \overline{AB}$.

CK-12 PLIX Interactive

MEDIA

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Examples

Example 1

Earlier, you were asked how is the original line segment related to the image when you dilate a line segment.

When you dilate a line segment, the original line segment will always be parallel to (or on the same line as) the image. Also, if the length of the original line segment is L and the scale factor is k, the length of the image will be kL.

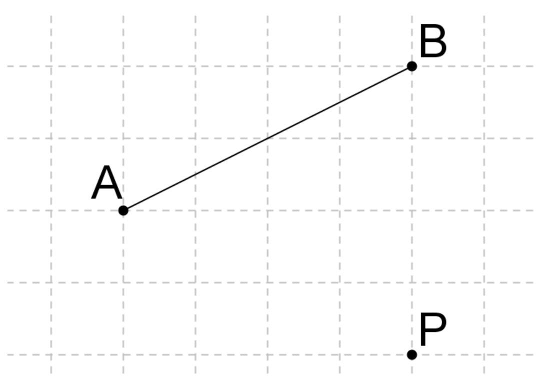
Example 2

A shape is dilated by a scale factor of 1. How does the image relate to the original shape?

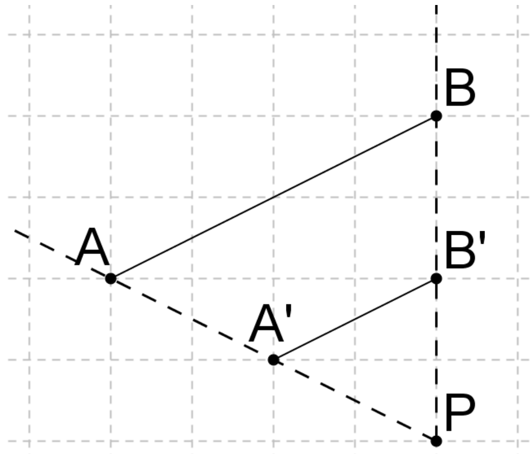
If the scale factor is 1, the shape does not change size or move at all. The image will be equivalent to the original figure.

Example 3

Dilate the line segment below about point P by a scale factor of $\frac{1}{2}$.



The distance from P to A' should be half the distance from P to A. Similarly, the distance from P to B' should be half the distance from P to B.



Notice that A' is the midpoint of $\overline{PA}$ and B' is the midpoint of $\overline{PB}$.

Example 4

Using your answer to #2, show that $A'B' = \frac{1}{2}AB$ and $\overline{A'B'} \parallel \overline{AB}$.

Use the Pythagorean Theorem to compare the lengths of the two segments.

- $AB^2 = 4^2 + 2^2 \to AB^2 = 20 \to AB = \sqrt{20} = 2\sqrt{5}$
- $A'B'^2 = 2^2 + 1^2 \to A'B'^2 = 5 \to A'B' = \sqrt{5}$

Therefore, $A'B' = \frac{1}{2}AB$.

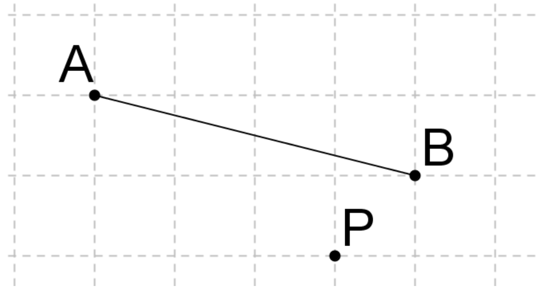
Two line segments are parallel if they have the same slope.

- Slope of $\overline{AB}$: $\frac{1}{2}$
- Slope of $\overline{A'B'}$: $\frac{1}{2}$

Therefore, $\overline{A'B'} \parallel \overline{AB}$.

Review

1. Describe how to perform a dilation.
2. Explain why a dilation is not an example of a rigid transformation.
3. True or false: angle measures are preserved in a dilation.
4. A shape is dilated by a scale factor of $\frac{3}{2}$. How does the image relate to the original shape?
5. In general, if $k > 1$ will the image be larger or smaller than the original figure?
6. In general, if $k < 1$ will the image be larger or smaller than the original figure?
7. Dilate the line segment below about point P by a scale factor of 2.

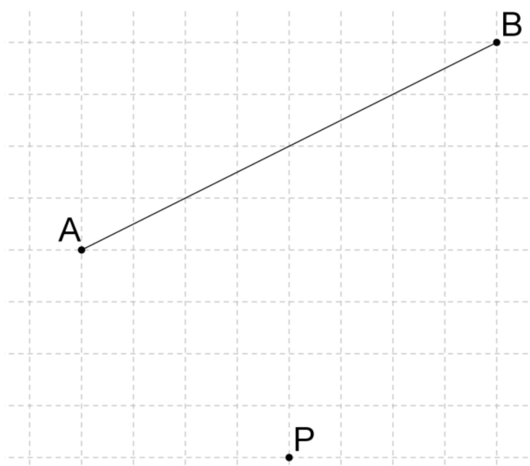


8. Using your answer to #7, show that $A'B' = 2AB$.

9. Using your answer to #7, show that $\overline{A'B'} \parallel \overline{AB}$.

10. If one of the points of your figure IS the center of dilation, what happens to that point when the dilation occurs?

11. Dilate the line segment below about point P by a scale factor of $\frac{1}{4}$.



12. Using your answer to #11, show that $A'B' = \frac{1}{4}AB$.

13. Using your answer to #11, show that $\overline{A'B'} \parallel \overline{AB}$.

You can perform dilations using interactive geometry software just like you can perform other transformations. Start by creating your figure and the point for your center of dilation. Then, select "Dilate an Object from Point by Factor", then your figure, and then the center of dilation.

Enter the scale factor into the pop up window and your figure will be dilated.

14. Create a triangle with interactive geometry software and dilate it about the origin by a scale factor of 2.

15. Dilate the same triangle about a different point by a scale factor of 2.

16. Compare and contrast the two images from #13 and #14.

17. Below is a diagram of a center of dilation C and a point P. Create your own version of this diagram. The length of $\overline{CP}$ is 1. Dilate point P around center C by a scale factor of 2 to get to point P'. (You can use a compass to do this by capturing the length of $\overline{CP}$ and making an arc from P that crosses the ray on the opposite side from C.) What is the length of $\overline{PP'}$? What is the ratio $\frac{CP'}{CP}$? If we dilated point P by a scale factor of 3, how would these lengths and ratios change?

18. Sketch a line containing the points A and B, with a center of dilation not on the line. This part of the diagram is shown below. Dilate the line by a scale factor of 2. What is the image of the line after dilation? What is its relationship with the original line? Does the same observation apply for a ray? Explain.

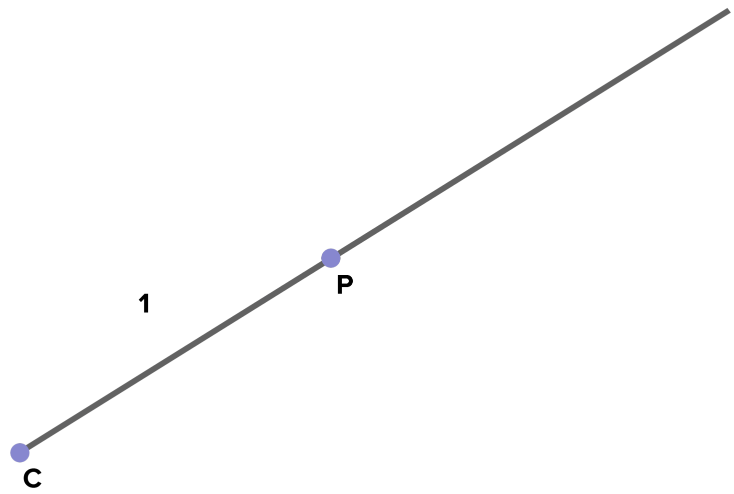


FIGURE 3.96
A diagram of point P and center of dilation C

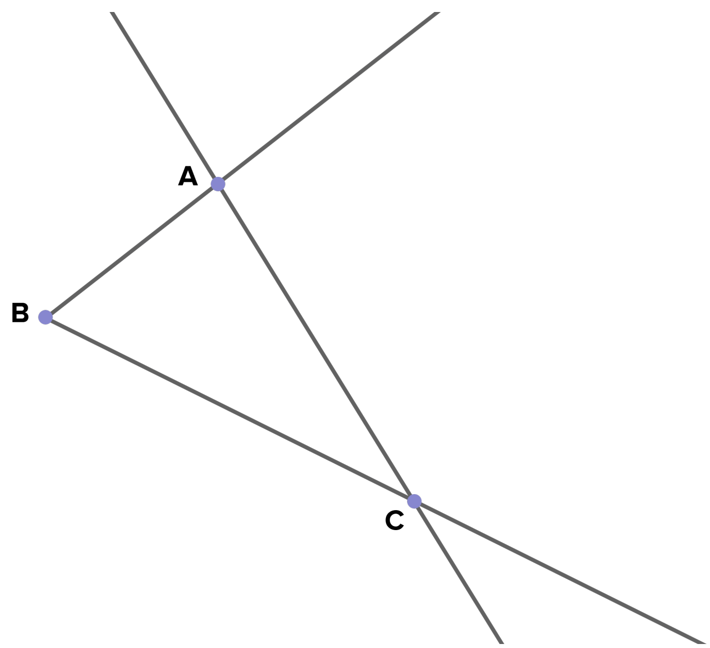


FIGURE 3.97
A line containing points A and B, with a center of dilation not on the line. Can you dilate the line by a scale factor of 2?

19. Sketch $\angle ABC$ and a center of rotation not on the angle. Dilate the angle by a scale factor of your choice. How does the image of the angle after translation compare to the original angle? Why? Prove it.

20. Sketch a triangle and a center of dilation not on the triangle as shown below. Draw straight lines from the center of dilation through each vertex as shown. Dilate the triangle by a scale factor of 2. What are the similarities and differences between the image and the original? Dilate by a scale factor of 3. Measure the angles and compare with the original. Find the ratio of the lengths of corresponding sides by measuring and dividing, then summarize your results.

21. Using interactive geometry software, dilate a segment around a point on the segment by a scale factor of 2. Explain the results. Place the center of dilation outside the segment, but on the line. Explain the results.

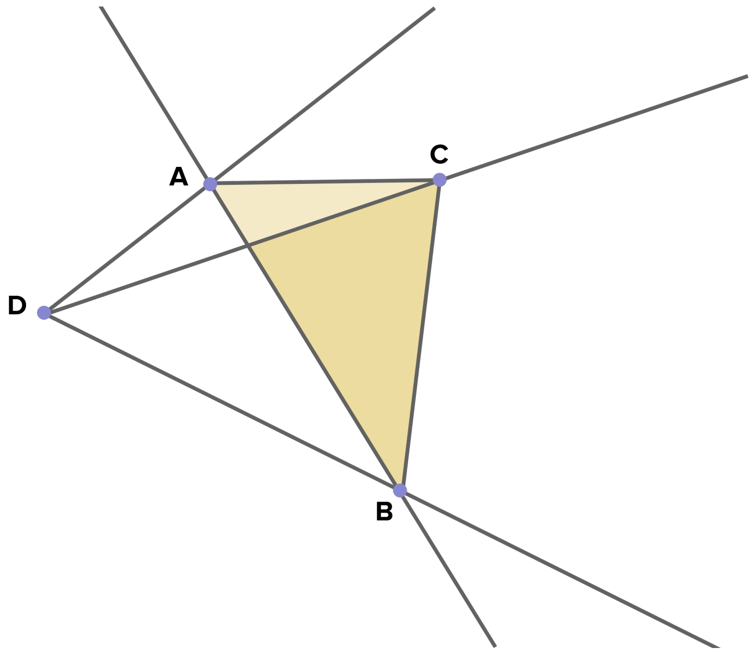


FIGURE 3.98
A triangle with a center of dilation not on the triangle.

22. Using interactive geometry software, dilate a segment around a center of dilation not on the line containing the segment. Are the segments parallel? Why or why not? Construct rays from the center of dilation through the endpoints of segment. Identify two triangles. Describe the ratios of the sides in the two triangles. Explain these results.

23. The triangle below has undergone a sequence of dilations. Describe the sequence. Find a single dilation that would transform the original into the image in one step. Explain how you found the center of dilation. What's the scale factor? Explain. Try your own version of this experiment on paper or with interactive geometry software.

FIGURE 3.99
A triangle undergoing a sequence of dilations.

24. Graph a polygon on the coordinate plane. Dilate it around the origin by a scale factor of 2. How will the coordinates of each point change? Explain. On another set of coordinate axes graph a polygon and dilate it around a point not on the origin. Follow these steps: Translate the object and its center of dilation so that the center is on the origin. Perform the dilation. Move the resulting image and center of dilation so that the center is back to its original

location. Explain how this can be accomplished without graphing if you are given the coordinates of the center and the vertices of the polygon.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.1.

3.12 References

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CHAPTER 4 Triangle Congruence

Chapter Outline

4.1 Triangles

Learning Objectives

Here you will review different types of triangles and properties of triangles.

Triangles can be classified by their sides and by their angles.

Classification Based on Sides

When classifying a triangle by its sides, you should look to see if any of the sides are the same length. If no sides are the same length, then it is a scalene triangle. If two sides are the same length, then it is an isosceles triangle. If all three sides are the same length, then it is an equilateral triangle. You can show that two sides are the same length by drawing tick marks through the middles of the sides. Sides with a corresponding number of tick marks are the same length.

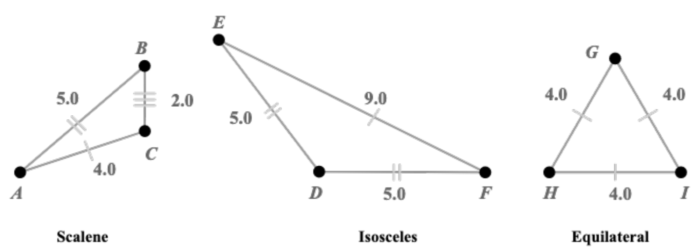

FIGURE 4.1

Classification Based on Angles

When classifying a triangle by its angles, you should look at the size of the angles. If there is a right angle (a 90°angle), then it is a right triangle. If the measures of all angles are less than 90°, then it is an acute triangle. If the measure of one angle is greater than 90°, then it is an obtuse triangle.

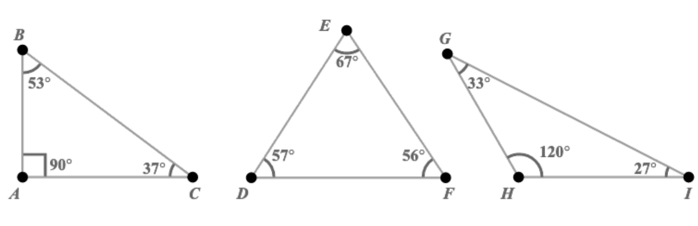

FIGURE 4.2

The sum of the measures of the interior angles of any triangle is 180°. If the three angles of a triangle are all the same, then the triangle is an equiangular triangle and each angle measure is 60°. Equilateral triangles are always equiangular and vice versa. In fact, the number of sides that are the same length will always correspond to the number of angles that are the same measure.

Can an obtuse triangle sometimes be an equilateral triangle?

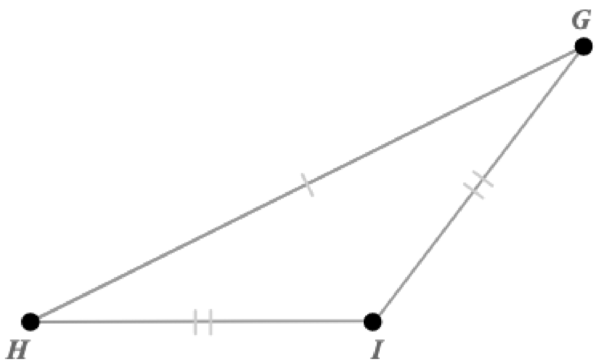

FIGURE 4.3

It is _____ for an obtuse triangle to sometimes be an equilateral triangle.

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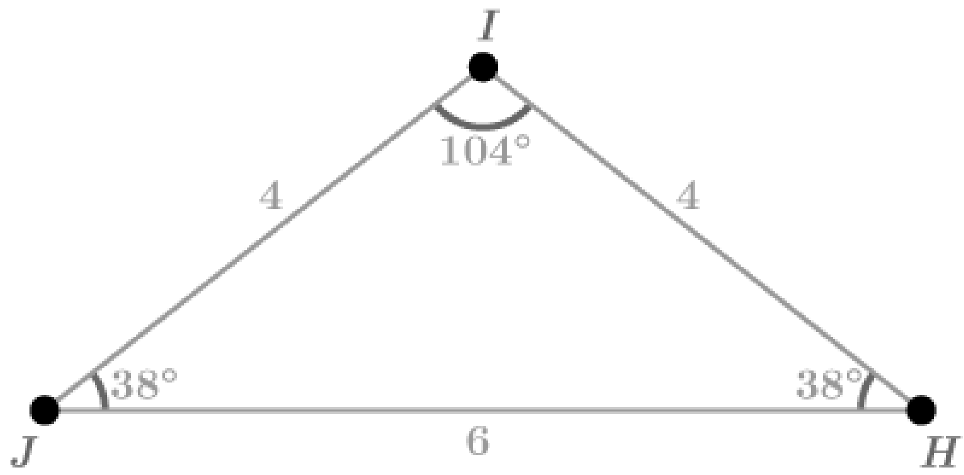
Sample Classifications

Obtuse isosceles triangle



This triangle has two sides that are congruent (the same length), so it is isosceles. More specifically, it also has one angle that is greater than 90°, so it is obtuse.

Scalene right triangle

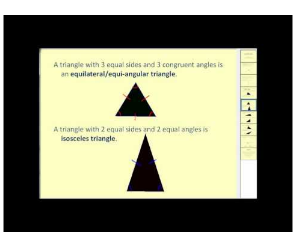
The measures of two angles of a triangle are 30°and 60°. What type of triangle is it?

Let the third angle be 'x'.

We know that the sum of the measures of the interior angles of any triangle is 180°.

$30^\circ + 60^\circ + x = 180^\circ$
$90^\circ + x = 180^\circ$
$x = 180^\circ - 90^\circ$
$x = 90^\circ$

One angle is a right angle, so this is a right triangle. More specifically, all of the angles are different measures (which only happens when all sides are different lengths), so this is a scalene right triangle.



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Examples

Example 1

How is an exterior angle of a triangle related to the interior angles of the triangle? In the triangle below, how is exterior angle $\angle BCE$ related to interior angles $\angle A$ and $\angle B$?

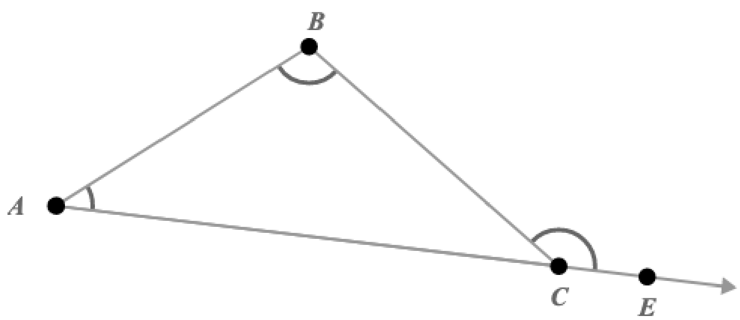


FIGURE 4.4

Answer: The sum of the measures of the interior angles of any triangle is 180°.

In ΔABC,

$$m\angle A + m\angle B + m\angle BCA = 180^\circ$$
$$m\angle ECB + m\angle BCA = 180^\circ [\text{ linear pair }]$$

Since both angle sums are equal to 180 degrees, they are both equal to each other:

$$m\angle A + m\angle B + m\angle BCA = m\angle BCA + m\angle ECB$$
$$m\angle A + m\angle B = m\angle ECB$$

In general, the measure of an exterior angle of a triangle will always be equal to the sum of the measures of the other two interior angles.

Example 2

The measures of two angles of a triangle are $45^\circ$and 45°. What type of triangle is it?

Answer: Let the third angle be 'x'.

We know that the sum of the measures of the interior angles of any triangle is 180°.

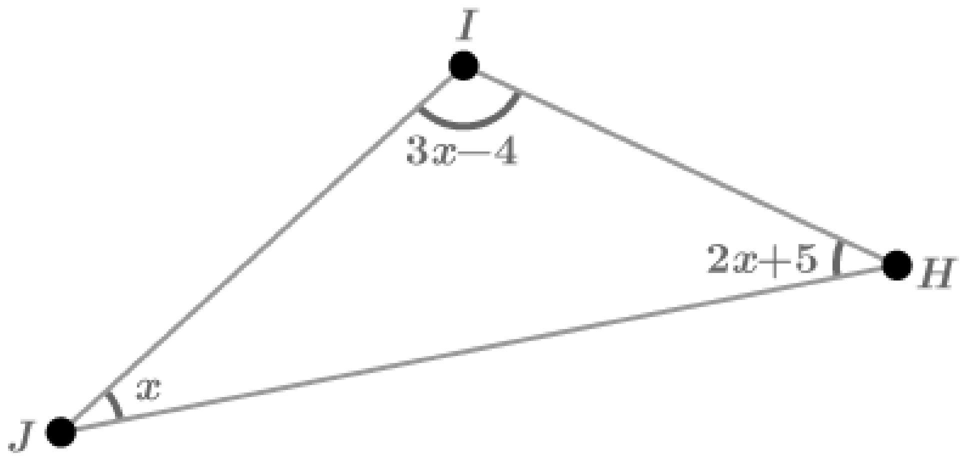
$$45^\circ + 45^\circ + x = 180^\circ$$
$$90^\circ + x = 180^\circ$$
$$x = 180^\circ - 90^\circ$$
$$x = 90^\circ$$

One angle is a right angle, so this is a right triangle. Because two angles are the same measure, two sides must be the same length. Therefore, it is an isosceles triangle.

Example 3

Solve for x (the picture is not drawn to scale).

Find the measure of each angle from ΔHIJ.



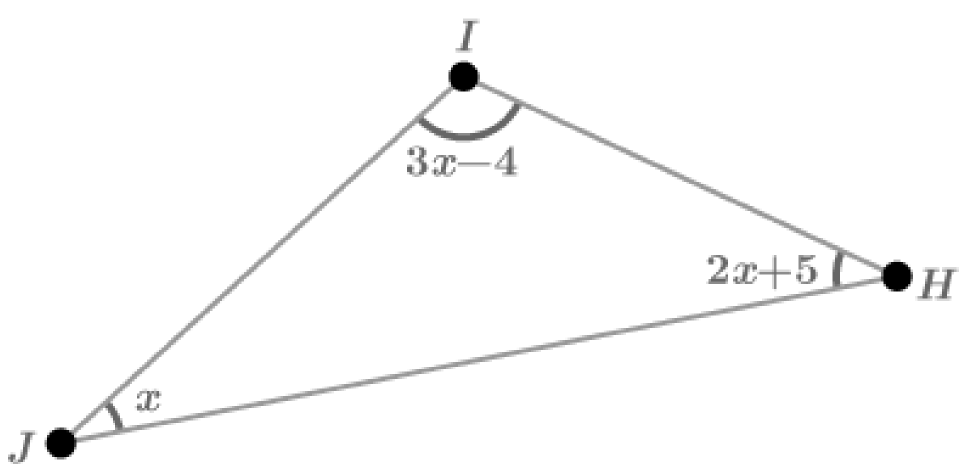
Answer: The sum of the measures of the interior angles of any triangle is 180°.

In ΔHIJ,

$\angle HIJ + \angle IJH + \angle JHI = 180^\circ$
$(3x-4) + x + (2x+5) = 180$
$6x + 1 = 180$
$6x = 180 - 1$
$x = \frac{179}{6} = 29.8$
$m\angle IJH = x = 29.8^\circ$
$m\angle HIJ = 3x - 4 = 3(29.8) - 4 = 85.5^\circ$
$m\angle JHI = 2x + 5 = 2(29.8) + 5 = 64.7^\circ$

Example 4

Which side of ΔHIJ is longest? Which side is shortest? The image is drawn to scale.



Answer: The question states that the image is drawn to scale, and visually, the largest angle is $\angle I$, so the longest side must be the side created by $\angle I$, which is $\overline{JH}$. The shortest side is across from the smallest angle. The smallest angle is $\angle J$, so the shortest side is $\overline{IH}$.

CK-12 PLIX Interactive

PLIX

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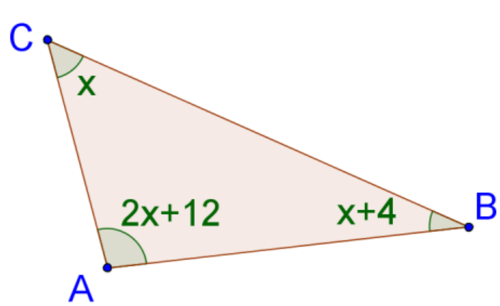
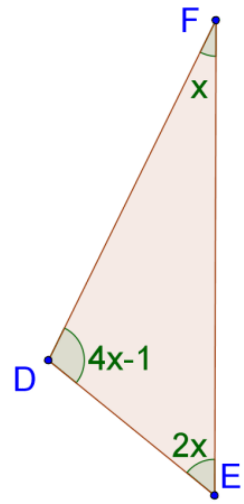
URL: http://www.ck12.org/geometry/classify-triangles-with-side-measurement/plix/Triangles-Classification-by-Side-Lengths-542c88dc5aa4130377018df0

Review

1. What are the three ways to classify a triangle by its sides?
2. What are the four ways to classify a triangle by its angles?
3. Can a right triangle be equiangular? Explain.
4. The measures of two angles of a triangle are 42° and 42°. What type of triangle is it?
5. The measures of two angles of a triangle are 120° and 12°. What type of triangle is it?
6. Solve for x (the picture is not drawn to scale).



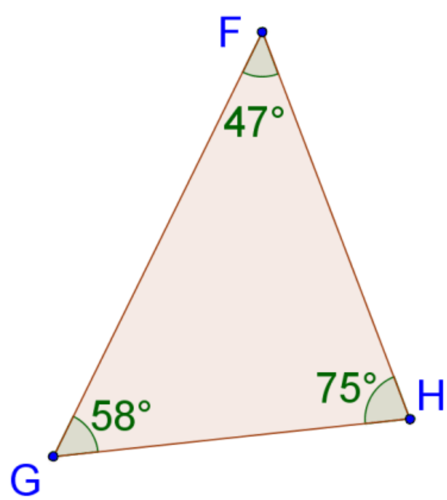
7. Find the measure of each angle for ΔABC in #6.
8. Solve for x (the picture is not drawn to scale).



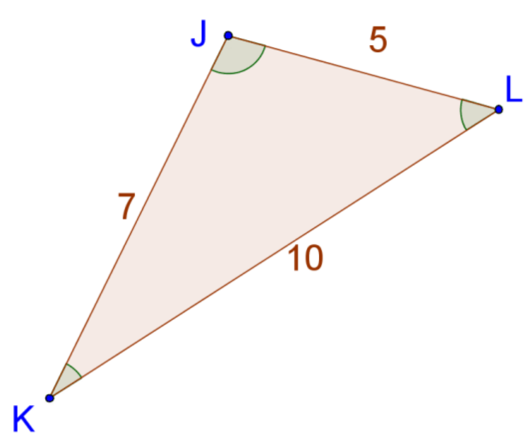
9. Find the measure of each angle for the triangle in #8.

10. Which side ΔDEF of from #8 is the longest? Which side of ΔDEF is the shortest? How do you know?

11. Use the angle measurements to order the sides of the triangle below from shortest to longest.

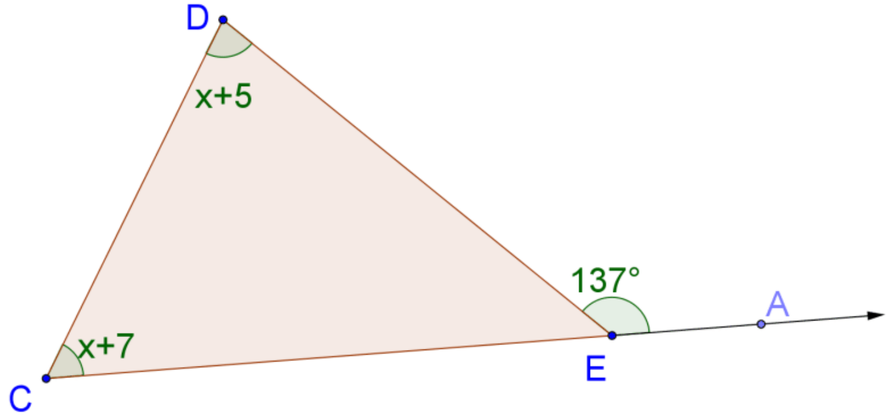


12. Use the side measurements to order the measures of the angles in the triangle below from smallest to largest.

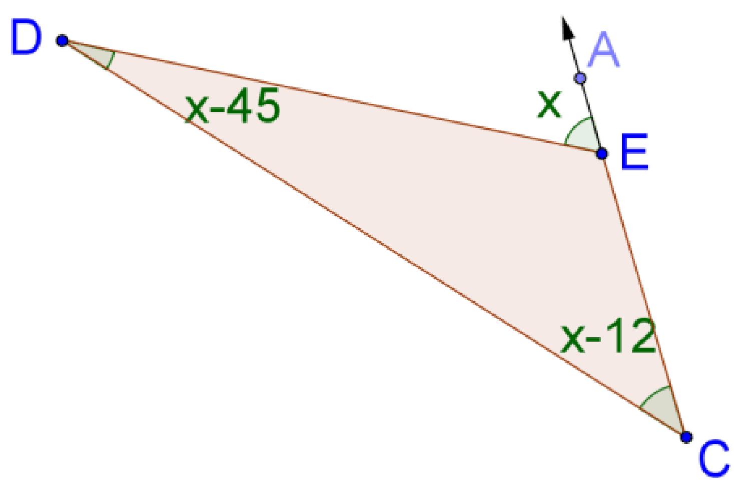


13. One of the exterior angles of a triangle is 100°. What do you know about the interior angles?

14. Solve for x (the picture is not drawn to scale).



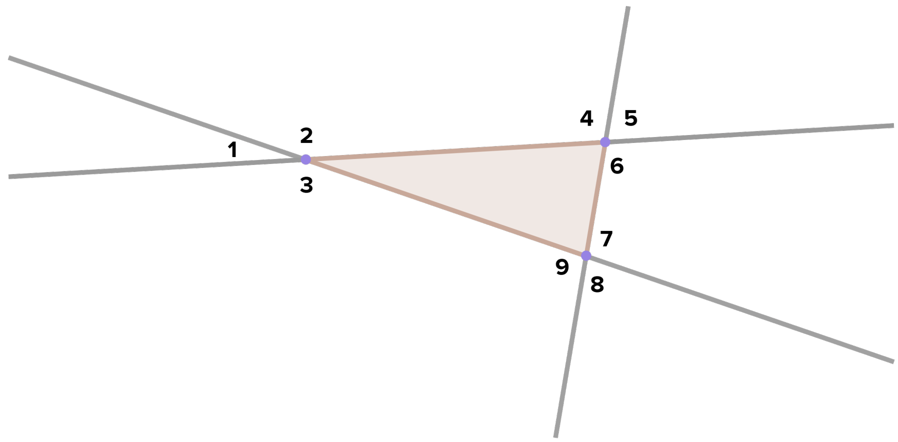
15. Find $m\angle DEC$ (the picture is not drawn to scale).



16. In an equilateral triangle, all the sides are equal in the length and all the angles have equal measure. If we alter one angle of an equilateral triangle, what must happen to the side opposite? Is it always true that the longest side of a scalene triangle is opposite the largest angle? Why or why not?

17. Using a diagram, explain why the measure of an exterior angle of a triangle is the sum of the measures of the two remote interior angles.

18. Frasier argues that every angle labeled below is an exterior angle of the triangle. Is he correct? Why or why not?



19. Draw a triangle with vertex A, along with the two exterior angles whose vertex is A. How many angles have A as a vertex? What is the sum of all these angles? True or False: Each of the angles around vertex A is either the sum of angles of the triangle or equal to an angle of the triangle. Is this also the case for the remaining vertices of the triangle? Explain.

20. Sketch a scalene triangle and make 5 copies, then cut them out. Is it possible to arrange these 6 triangles so that they all share one vertex? Does the resulting arrangement cover a surface without gaps, and without overlapping triangles? What characteristic of the angles around the vertex of a triangle makes this possible? Explain.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.4.

4.2 Definition of Congruence

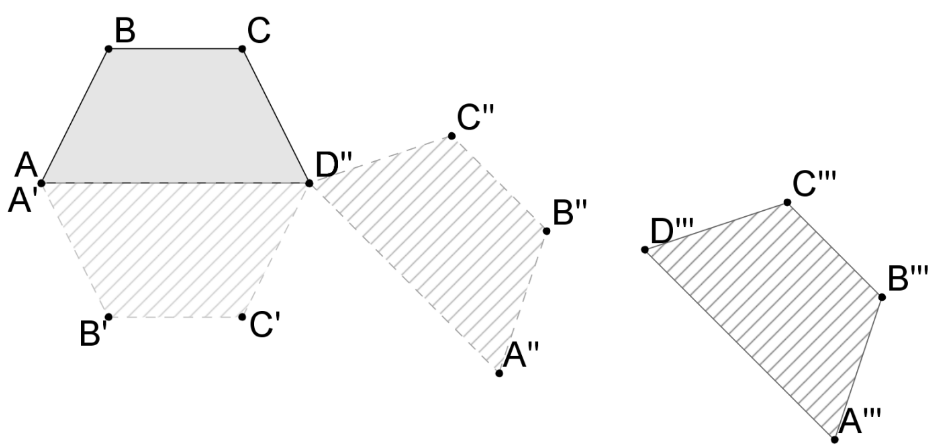
Learning Objectives

Here you will learn what it means for two figures to be congruent.

What does congruence have to do with rigid transformations?

Congruence

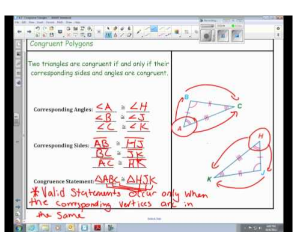
When a figure is transformed with one or more rigid transformations, an image is created that is **congruent** to the original figure. In other words, two figures are **congruent** if a sequence of rigid transformations will carry the first figure to the second figure. In the picture below, trapezoid $ABCD$ has been reflected, then rotated, and then translated. All four trapezoids are congruent to one another.



Recall that rigid transformations preserve distance and angles. This means that **congruent figures** will have corresponding angles and sides that are the same measure and length.

In order to determine if two shapes are congruent, you can:

1. Carefully describe the sequence of rigid transformations necessary to carry the first figure to the second. AND/OR
2. Verify that all corresponding pairs of sides and all corresponding pairs of angles are congruent.



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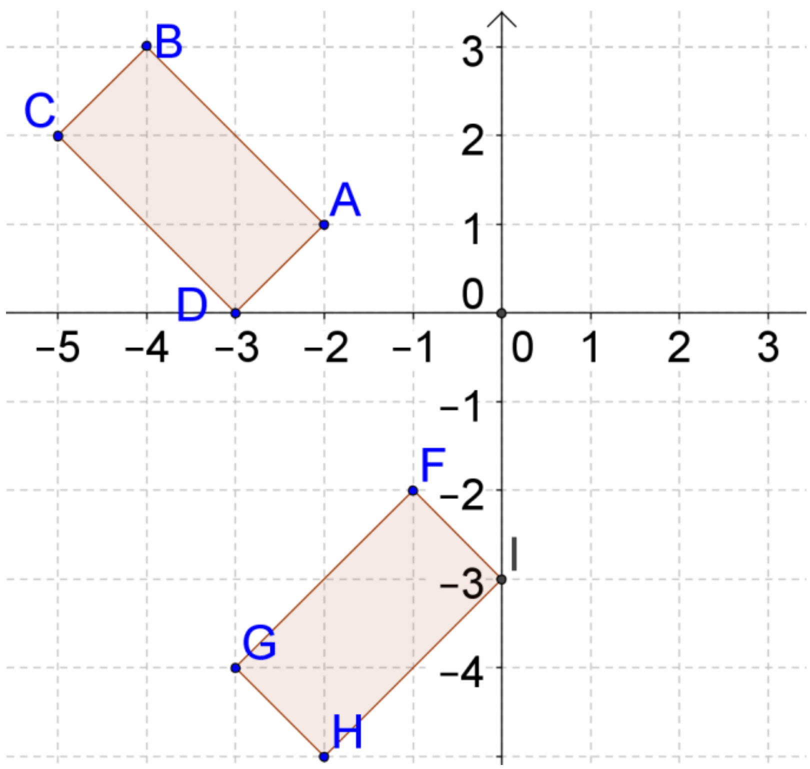
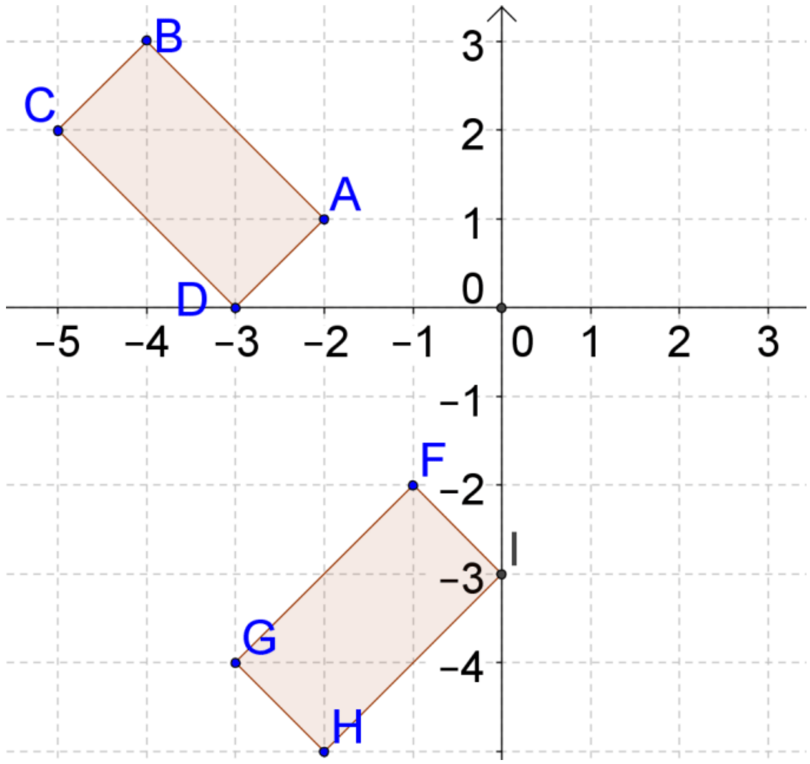
1. Are the two rectangles congruent? Explain.



One way to determine whether or not the rectangles are congruent is to consider if transformations to rectangle $ABCD$ would produce rectangle $FGHI$. Just from looking at the rectangles it appears that if rectangle $ABCD$ were rotated 90° counterclockwise about the origin it would produce rectangle $FGHI$. To verify this, you can check the points and notice that $(x,y) \rightarrow (-y,x)$ for rectangle $ABCD$ to rectangle $FGHI$, so this is in fact a 90° counterclockwise rotation about the origin.

Because a rigid transformation on rectangle $ABCD$ produces rectangle $FGHI$, the two rectangles are congruent.

2. Give another explanation for why the two rectangles from #1 are congruent.



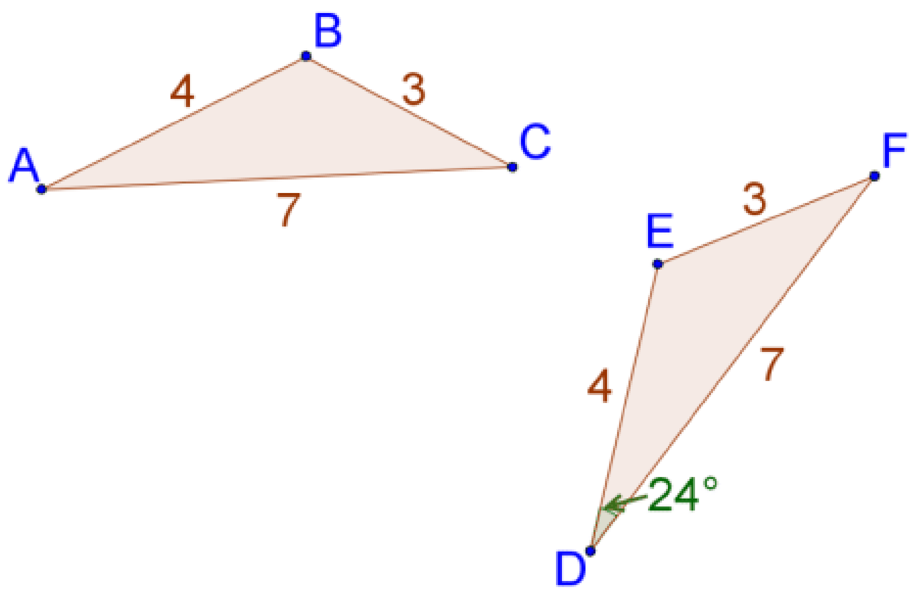
To verify that the rectangles are congruent, you could also verify that all corresponding angles and sides are congruent. Notice that the slopes of each line segment making up the rectangles is either +1 or -1. All adjacent sides have opposite reciprocal slopes and are therefore perpendicular. This means that all angles are 90°. All pairs of angles are congruent since all angles are 90°. To find the length of the line segments, you can use the Pythagorean Theorem (which is the same as the distance formula).

- $AD = BC = FI = GH = \sqrt{1^2+1^2} = \sqrt{2}$, so $\overline{AD} \cong \overline{FI}$ and $\overline{BC} \cong \overline{GH}$
- $CD = BA = FG = IH = \sqrt{2^2+2^2} = 2\sqrt{2}$, so $\overline{CD} \cong \overline{HI}$ and $\overline{AB} \cong \overline{FG}$

Because all corresponding pairs of sides are congruent and all corresponding pairs of angles are congruent, the rectangles are congruent.

Drawing Conclusions

The triangles below are congruent. What does that tell you about $\angle A$?



Because the triangles are congruent, corresponding sides and angles are congruent. By looking at the sides, you can see that $\angle A$ corresponds to $\angle D$, because both of these angles are in between the sides of lengths 4 and 7. Since $\angle D$ is 24°, $\angle A$ must also be 24°.

Graphing and Translating

Graph square $S(1,2), Q(4,1), R(5,4)$ and $E(2,5)$. Find the image after the translation $(x,y) \rightarrow (x-2, y+3)$. Then, graph and label the image.

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/223454

The translation notation tells us that we are going to move the square to the left 2 and up 3.

Examples

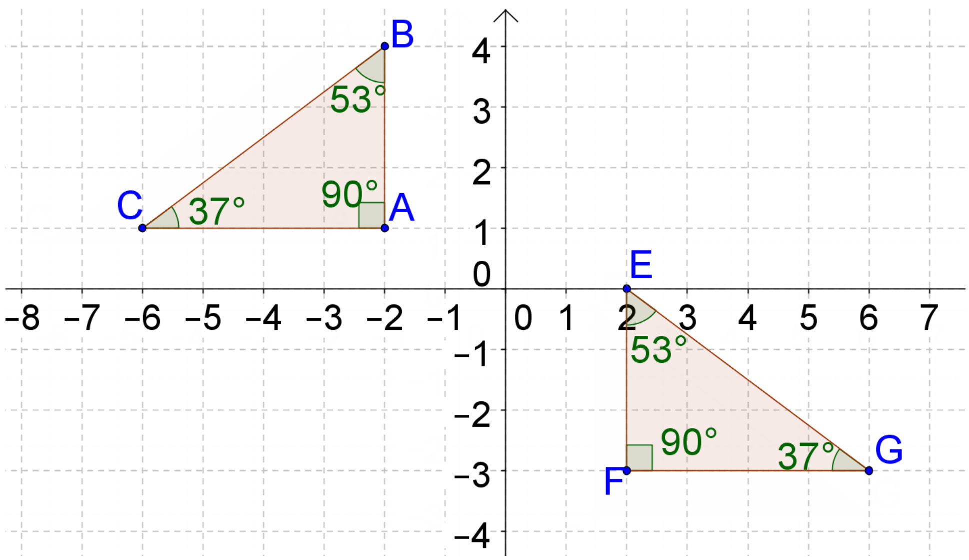
Example 1

Earlier, you were asked what does congruence have to do with rigid transformation.

Rigid transformations create congruent figures. You might think of congruent figures as shapes that "look exactly the same", but congruent figures can always be linked to rigid transformations as well. If two figures are congruent, you will always be able to perform a sequence of rigid transformations on one to create the other.

Example 2

Are the two triangles congruent? Explain.



ΔABC can be reflected across the y-axis and then translated over one unit to the right and down four units to create ΔEFG. Therefore, the triangles are congruent.

Example 3

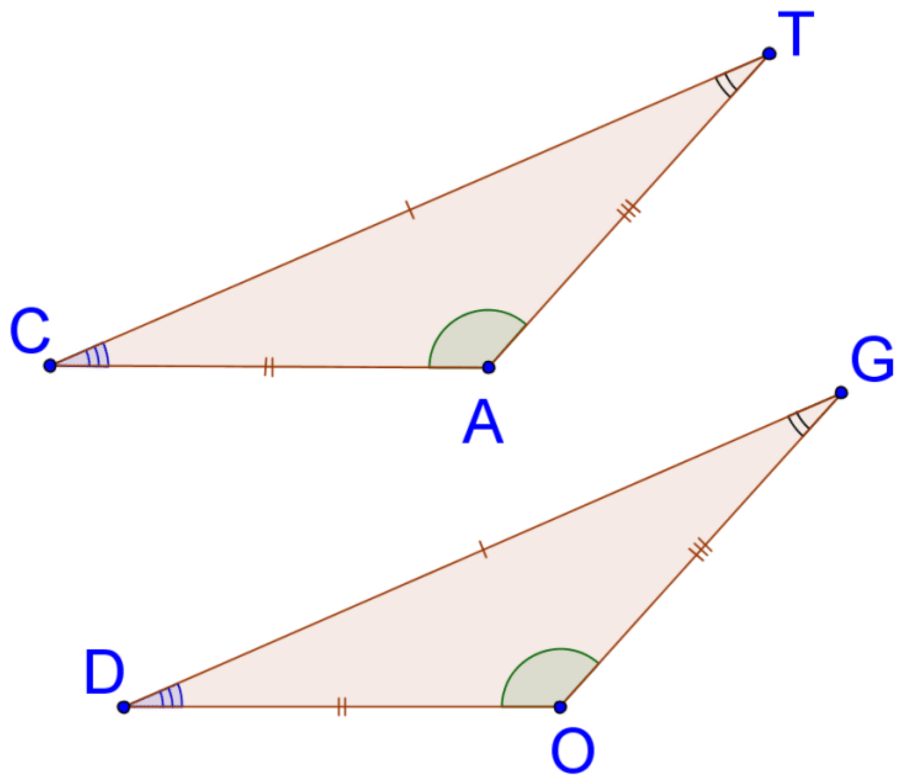
Give another explanation for why the two triangles from #1 are congruent.

You can see that $\angle A \cong \angle F$, $\angle C \cong \angle G$, $\angle B \cong \angle E$. You can also see that from A to B is 3 units and from E to F is 3 units so $\overline{AB} \cong \overline{EF}$. Similarly, from A to C is 4 units and from F to G is 4 units so $\overline{AC} \cong \overline{FG}$. Using the 3, 4, 5 Pythagorean triple you know that both $\overline{BC}$ and $\overline{EG}$ must be 5 units, so $\overline{BC} \cong \overline{EG}$. Because all pairs of corresponding angles and sides are congruent, the triangles are congruent.

Example 4

The symbol for congruence is $\cong$. $\Delta ABC \cong \Delta DEF$ means "triangle ABC is congruent to triangle DEF". The order of the letters matters. When you say $\Delta ABC \cong \Delta DEF$ it means that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. Suppose $\Delta CAT \cong \Delta DOG$. Draw a picture that matches this situation.

Remember that to denote that two sides are congruent, you can either mark them as being the same length (e.g., each 7 units), or use corresponding tick marks. It works the same way with angles. Corresponding angle markings mean congruent angles.



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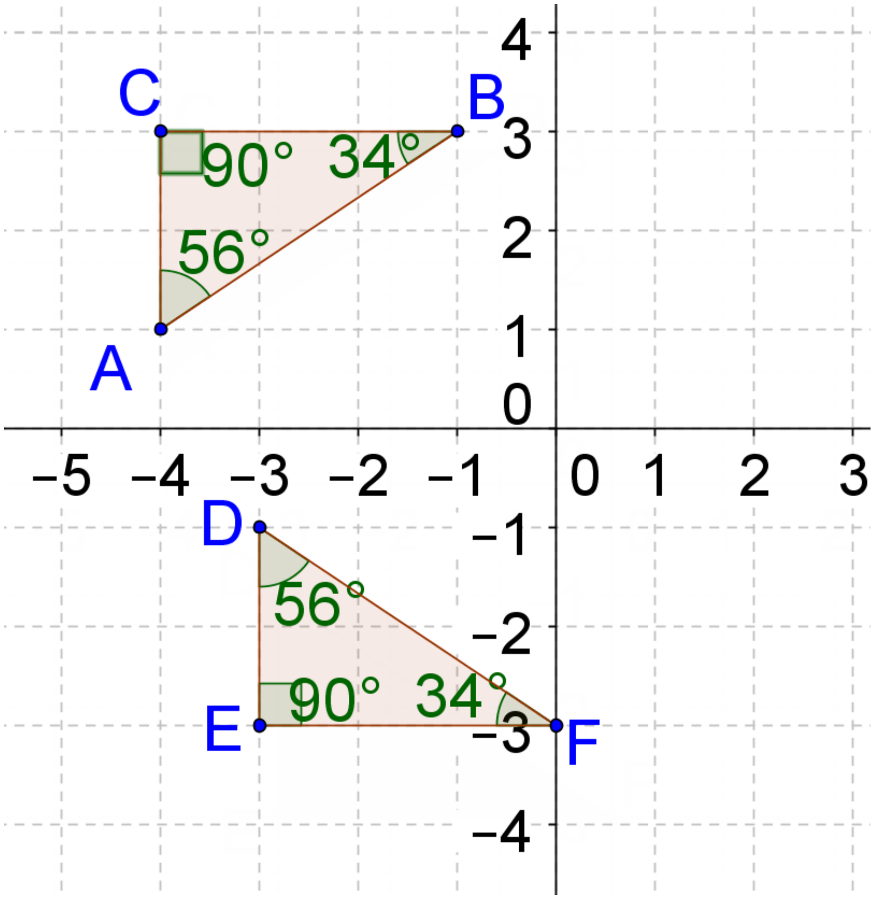
URL: http://www.ck12.org/geometry/triangle-congruence/plix/CPCTC-Congruent-Triangles-5357f19a5aa41321fed42f67

Review

Use the triangles below for #1 - #3.

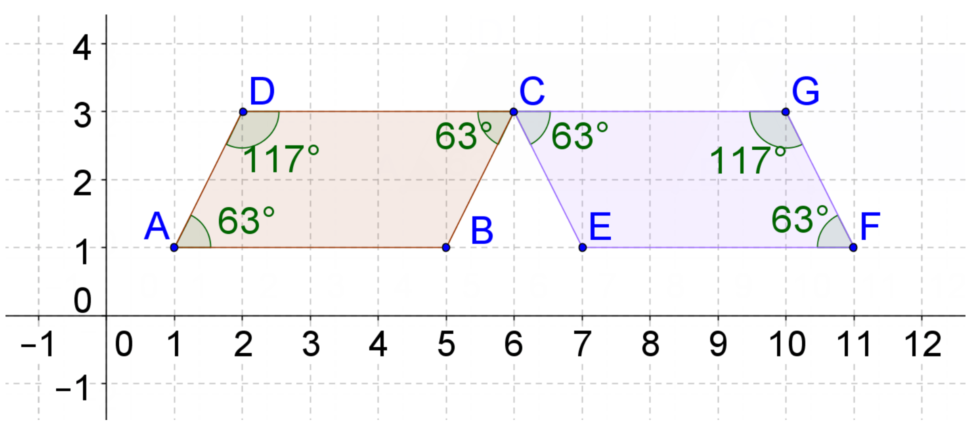


1. Explain why the triangles are congruent in terms of rigid transformations.

2. Explain why the triangles are congruent in terms of corresponding angles and sides.

3. Use notation like $\Delta CAT \cong \Delta DOG$ to state how the triangles are congruent. *Note that there are multiple correct ways to write this!*

Use the parallelograms below for #4 - #6.



4. Explain why the parallelograms are congruent in terms of rigid transformations.

5. Explain why the parallelograms are congruent in terms of corresponding angles and sides.

6. Use notation like $ABCD \cong A'B'C'D'$ to state how the parallelograms are congruent. *Note that there are multiple correct ways to write this!*

$\Delta MRG \cong \Delta KPS$

7. Draw a picture that matches this situation.

8. $\angle R \cong \angle$_____

9. $\overline{RG} \cong$ _____

10. $\overline{SK} \cong$ _____

11. $m\angle M = 60^\circ$ and $m\angle S = 20^\circ$. What does this tell you about $m\angle R$?

12. ΔDLP is reflected across the $x-axis$, then rotated 90° clockwise to create ΔMRK. How are the two triangles related?

13. Why will rigid transformations always produce congruent figures? Could non-rigid transformations also produce congruent figures?

14. If you know that all pairs of corresponding angles for two triangles are congruent, must the triangles be congruent? Explain and provide a counterexample if relevant.

15. If you know that two pairs of corresponding angles and all pairs of corresponding sides for two triangles are congruent, must the triangles be congruent? Explain and provide a counterexample if relevant.

16. Describe a series of rigid motion transformations which map polygon A to A”. Is it convincing that A” is congruent to A? Why or why not? In order to confirm this conclusion, what additional information would one need to know?

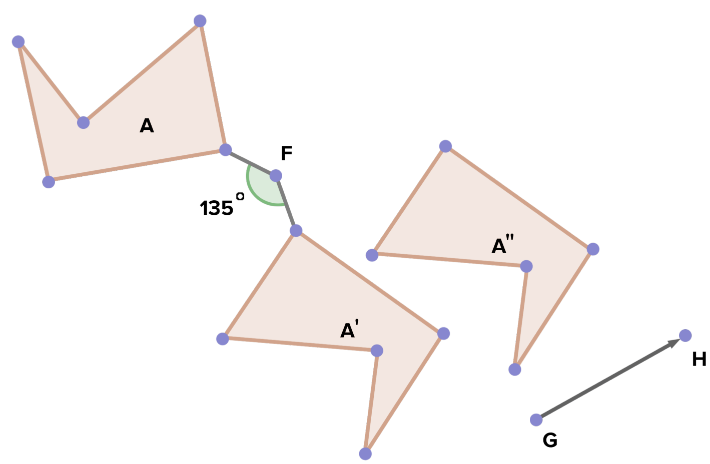


FIGURE 4.5

17. Describe a series of rigid motion transformations which map polygon P to P”’. Is it convincing that P”’ is congruent to P? Why or why not? What additional information would one need to know in order to confirm this conclusion?

18. A figure is rotated around a point, reflected across a line, translated along a vector, then rotated around another point. Is the image congruent to the original? Why or why not?

19. The translation shown below is **not** a rotation, reflection or translation. Is there a sequence of translations that will map T to its image, T”? Why or why not?

20. If a polygon undergoes a series of transformations such that the image has the corresponding sides and angles of equal measure to those of the original, can it be said that the original and image are congruent? Why or why not?

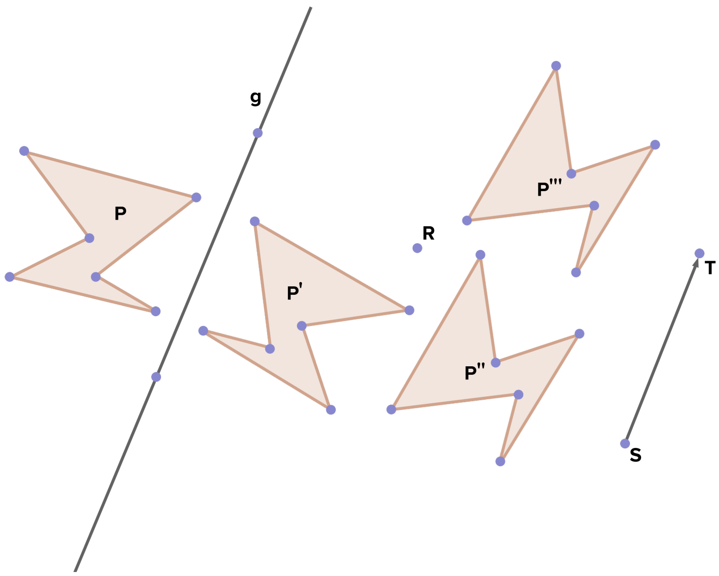


FIGURE 4.6

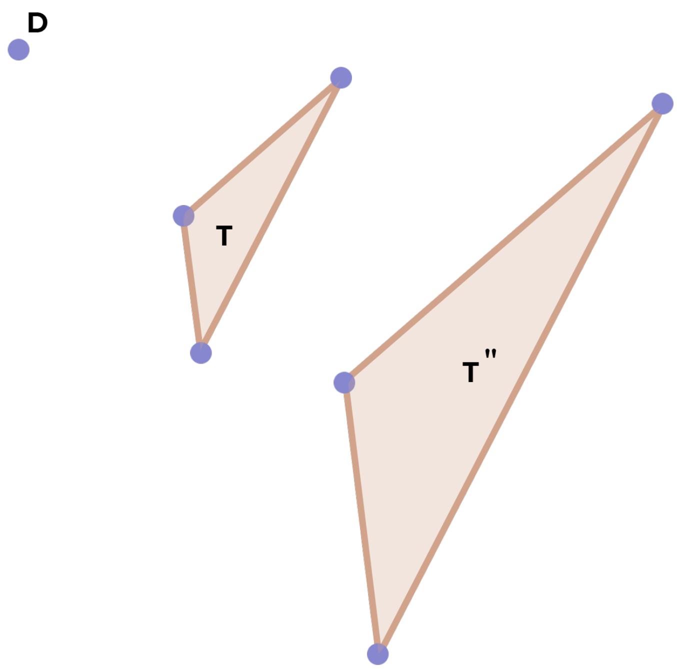


FIGURE 4.7

21. If the image of a polygon after transformation features at least one corresponding length or angle of different measure, can it be said that the image and original are congruent? What is known about the transformations that is applied to the original in this case?

22. Graph a triangle in the coordinate plane such that no sides are vertical or horizontal, and identify the coordinates. Perform a reflection across any given line, and then a translation by any given vector. Be sure to choose transformations that that ensure the coordinates of the image can be determined with certainty. Is the image of this sequence

of transformations is congruent to the original? Why or why not? Confirm results by using the distance formula to compare the corresponding sides of the image and the original, and by measuring the angles with a compass.

23. Graph a right triangle in the coordinate plane such that one leg is vertical. Reflect the triangle across any given line. Be sure to choose a line that ensures the coordinates of the image can be determined with certainty. Is the image of this reflection congruent to the original? Why or why not? Compare corresponding side lengths with the distance formula. How does the slope of the hypotenuse of the image compare with that of the original? Is there a relationship between the slope of the line and the measures of the acute angles?

24.Given the triangles in the image, is there a rigid motion transformation that will map $\overline{AB}$ to $\overline{A'B'}$? If so, describe it. Will the same transformation map $\overline{BC}$ to $\overline{B'C'}$? Why or why not? Are these triangles congruent? Why or why not?

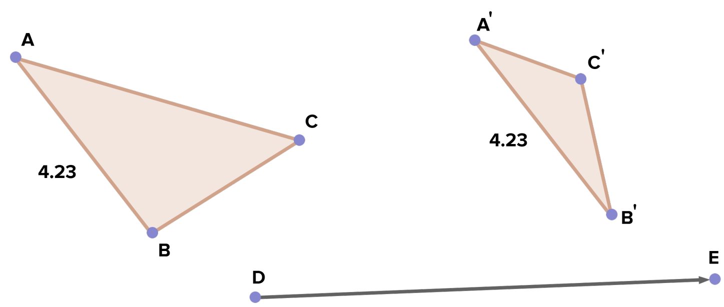


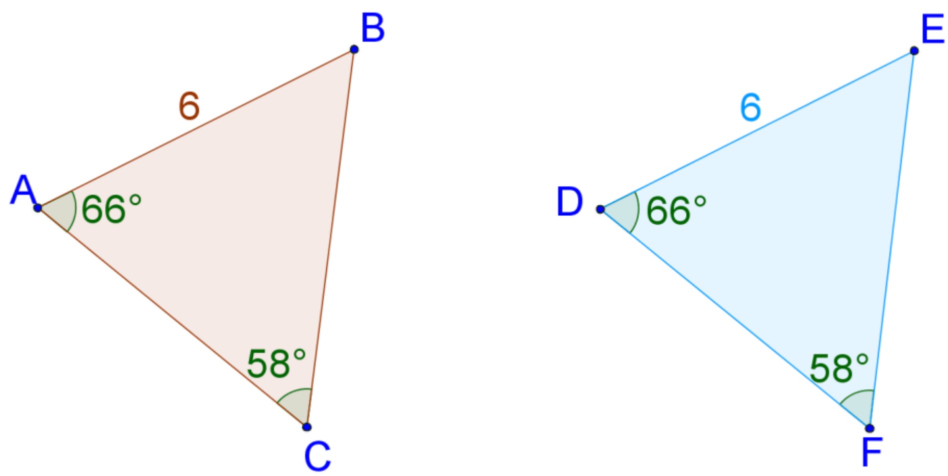
FIGURE 4.8

4.3 ASA and AAS Triangle Congruence

Learning Objectives

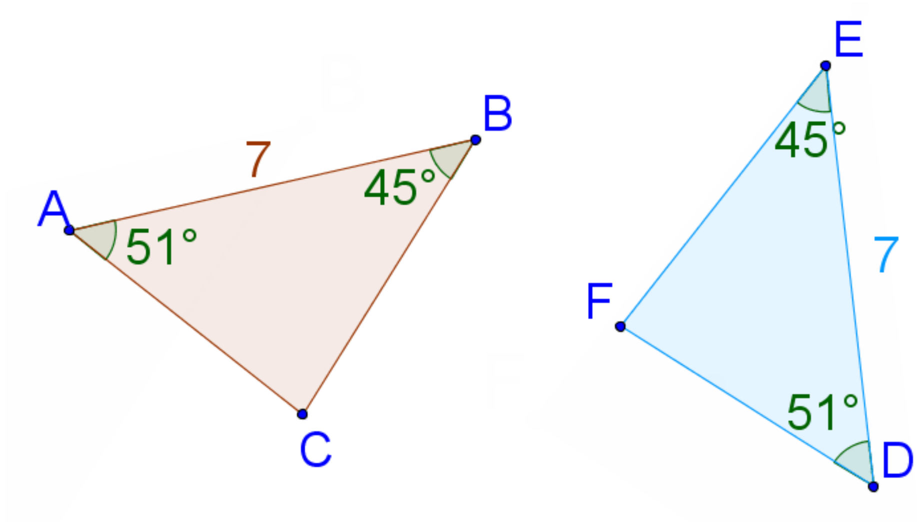
Here you will explore the ASA and AAS criterion for triangle congruence.

The information for the triangles below looks to be "AAS". How could you use "ASA" to verify that the triangles are congruent?



ASA and AAS Triangle Congruence

If two triangles are **congruent** it means that all corresponding angle pairs and all corresponding sides are congruent. However, in order to be sure that two triangles are congruent, you do not necessarily need to know that all angle pairs and side pairs are congruent. Consider the triangles below.



In these triangles, you can see that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\overline{AB} \cong \overline{DE}$. The information you know about the congruent corresponding parts of these triangles is an **a**ngle, a **s**ide, and then another **a**ngle. This is commonly referred to as "angle-side-angle" or "ASA".

The ASA criterion for triangle congruence states that if two triangles have two pairs of congruent angles and the common side of the angles in one triangle is congruent to the corresponding side in the other triangle, then the triangles are congruent.

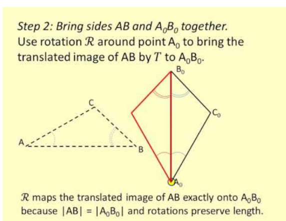
In the examples, you will use rigid transformations to show why the above ASA triangles must be congruent overall, even though you don't know the lengths of all the sides and the measures of all the angles.

"Angle-angle-side" or "AAS" is another criterion for triangle congruence that directly follows from ASA.

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/223459



MEDIA

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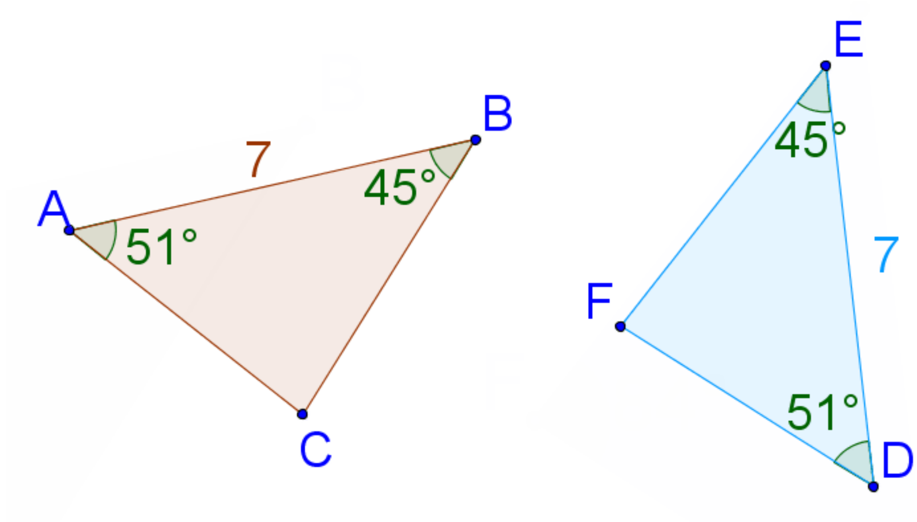
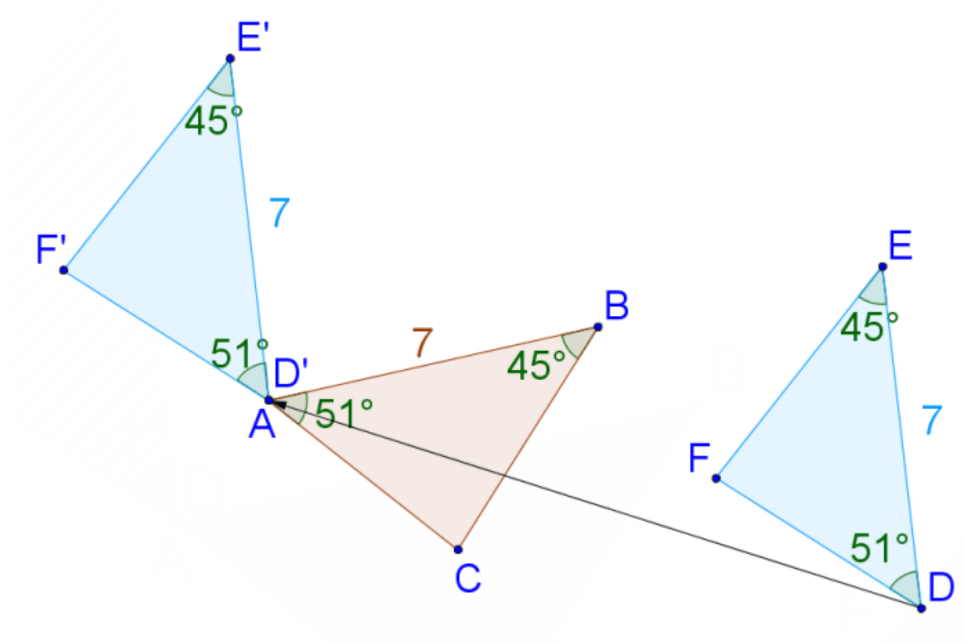
Performing a Rigid Transformation

Perform a rigid transformation to bring point D to point A.



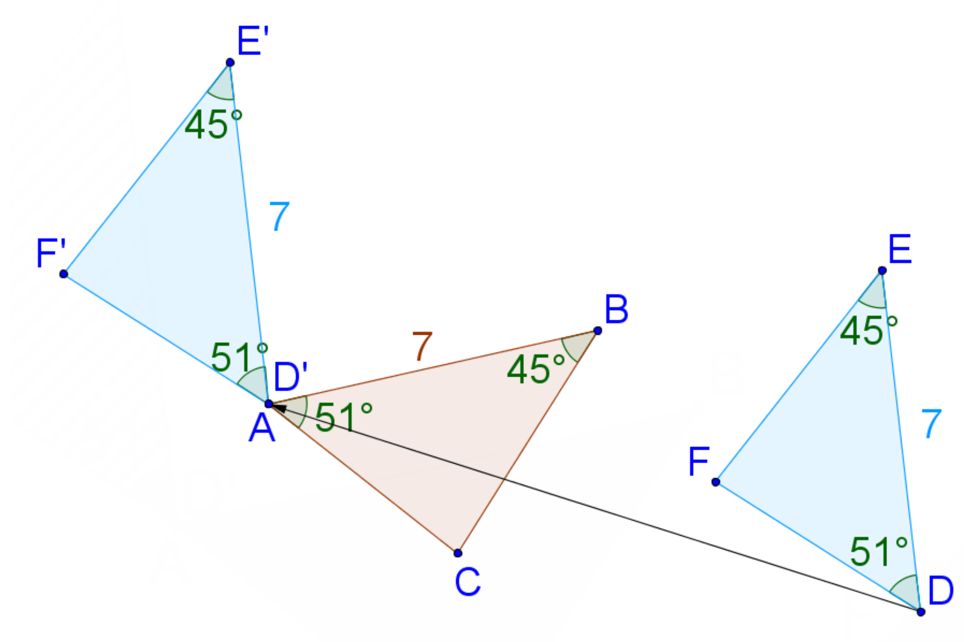
Drawing a Vector

Draw a vector from point D to point A. Translate ΔDEF along the vector to create $\Delta D'E'F'$.

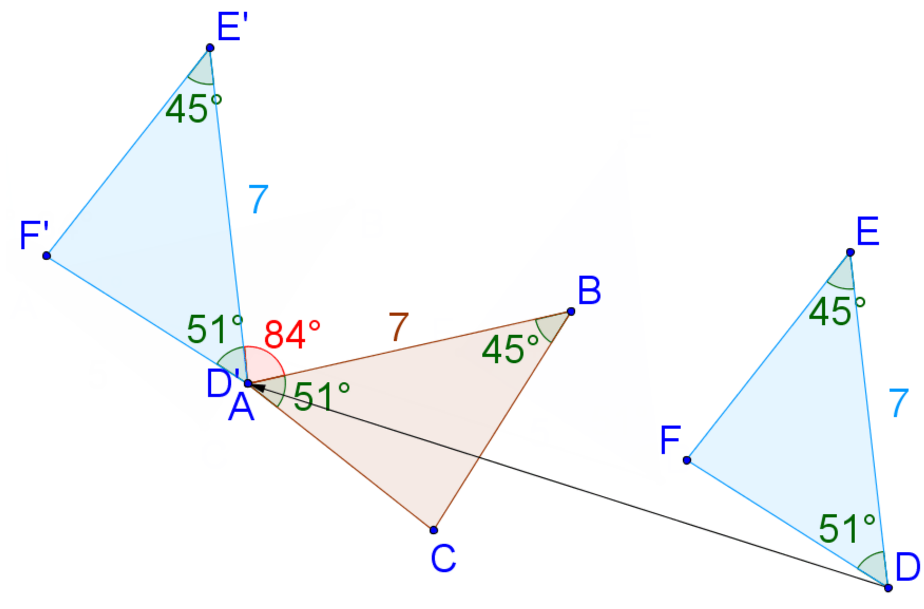


Rotating Triangles

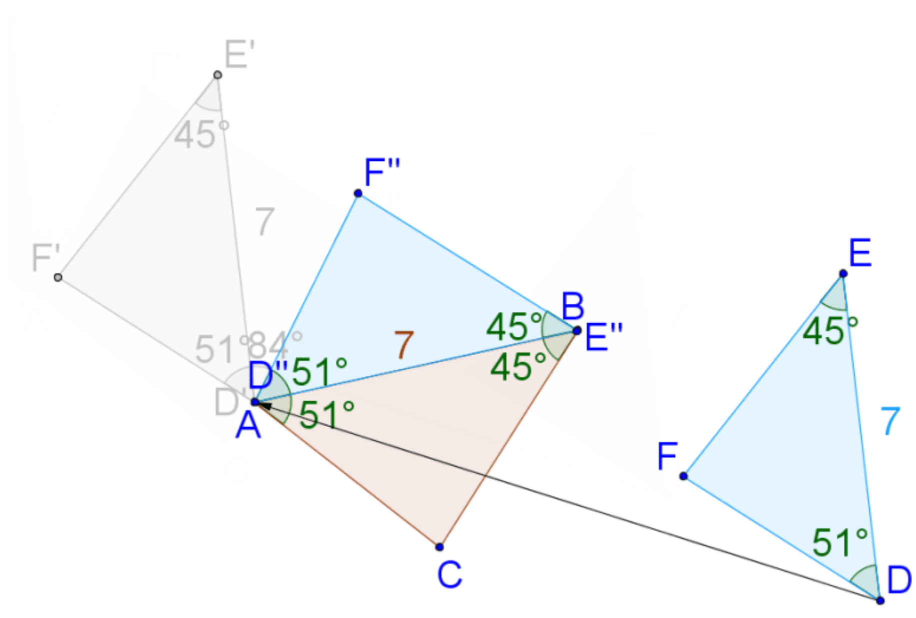
1. Rotate $\Delta D'E'F'$ to map $\overline{D'E'}$ to $\overline{AB}$.



2. Measure $\angle BD'E'$. In this case, $m\angle BD'E' = 84°$.

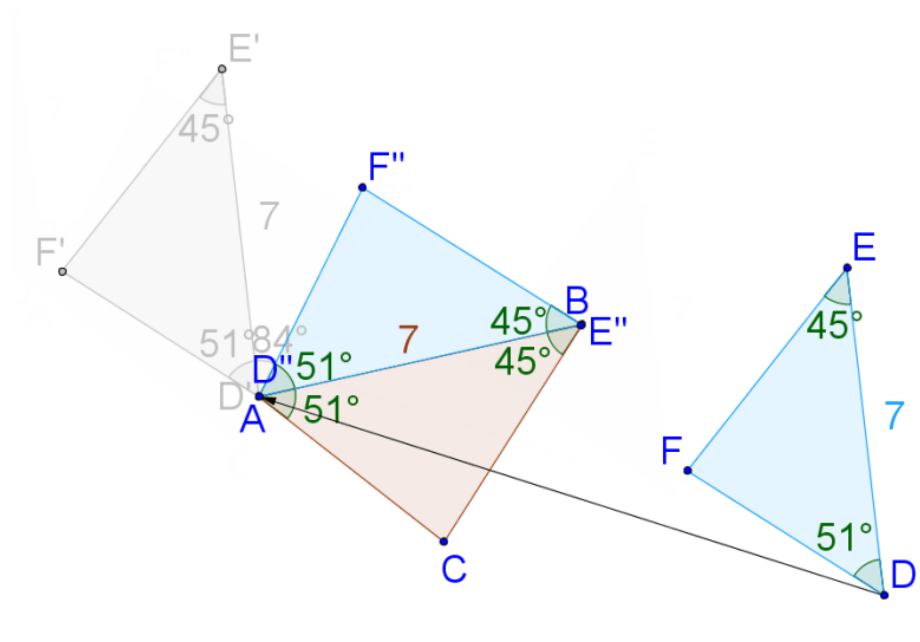


3. Rotate $\Delta D'E'F'$ clockwise that number of degrees about point D' to create $\Delta D''E''F''$. Note that because $\overline{DE} \cong \overline{AB}$ and rigid transformations preserve distance, $\overline{D''E''}$ matches up perfectly with $\overline{AB}$.

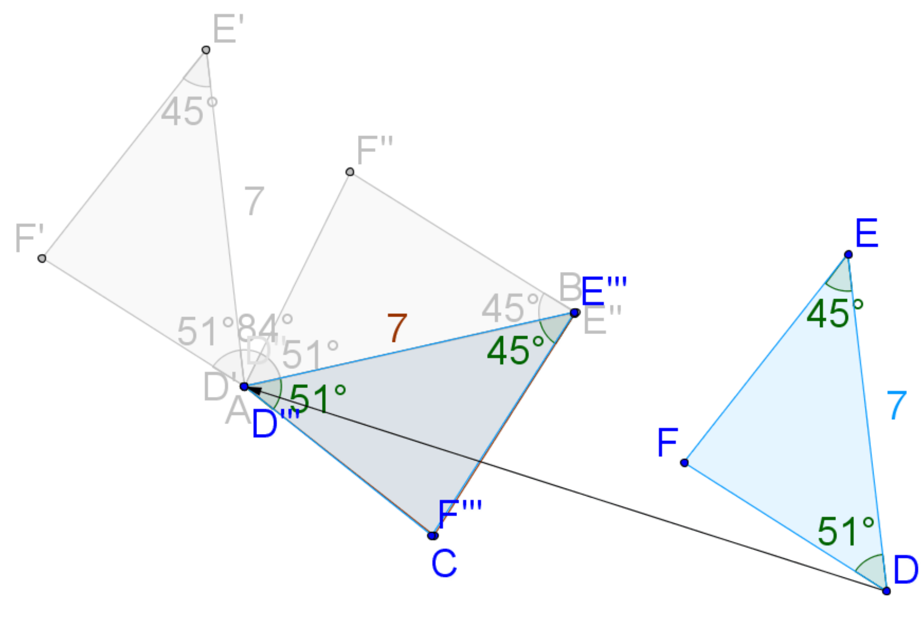


Performing Reflections

1. Reflect $\Delta D''E''F''$ to map it to ΔABC. Can you be confident that the triangles are congruent?



2. Reflect $\Delta D''E''F''$ across $\overline{D''E''}$ (which is the same as $\overline{AB}$).



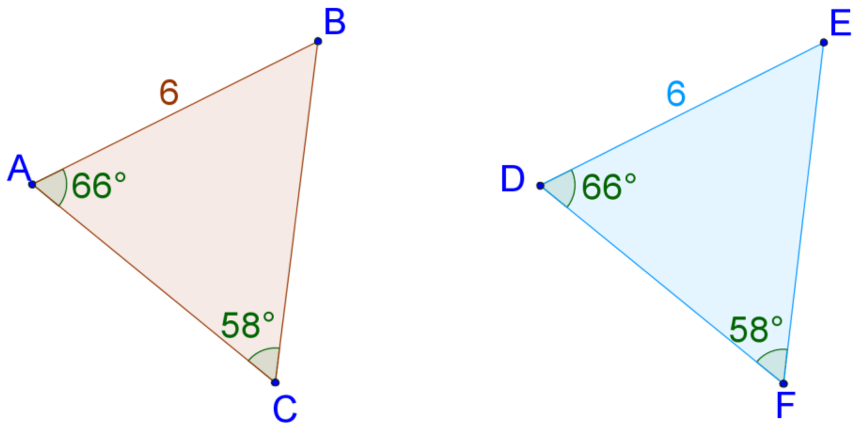
Because $\angle F''D''E'' \cong \angle CAB$ and $\angle F''E''D'' \cong \angle CBA$, the triangles must match up exactly (in particular, F''' must map to C), and the triangles are congruent.

This means that even though you didn't know all the side lengths and angle measures, because you knew two pairs of angles and the included sides were congruent, the triangles had to be congruent overall. At this point you can use the ASA criterion for showing triangles are congruent without having to go through all of these transformations each time (but make sure you can explain why ASA works in terms of the rigid transformations!).

Examples

Example 1

Earlier, you were asked how could you use "ASA" to verify that the triangles are congruent.

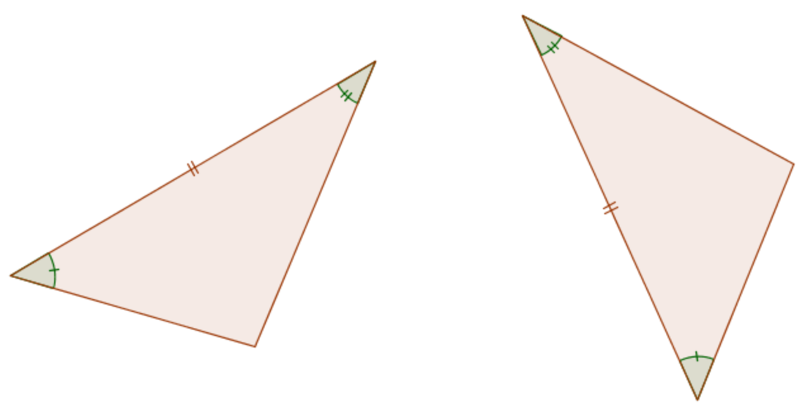


Because the three angles of a triangle always have a sum of 180°, $m\angle B = 56^\circ$ and $m\angle E = 56^\circ$. Therefore, the triangles are congruent by ASA due to the fact that $\angle A \cong \angle D, \overline{AB} \cong \overline{DE}, \angle B \cong \angle E$.

This example shows how if ASA is a criterion for triangle congruence, then AAS must also be a criterion for triangle congruence.

Example 2

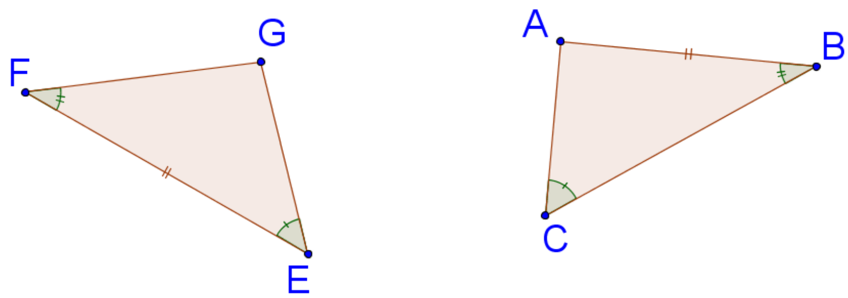
Are the following triangles congruent? Explain.



The triangles are congruent by ASA.

Example 3

Are the following triangles congruent? Explain.

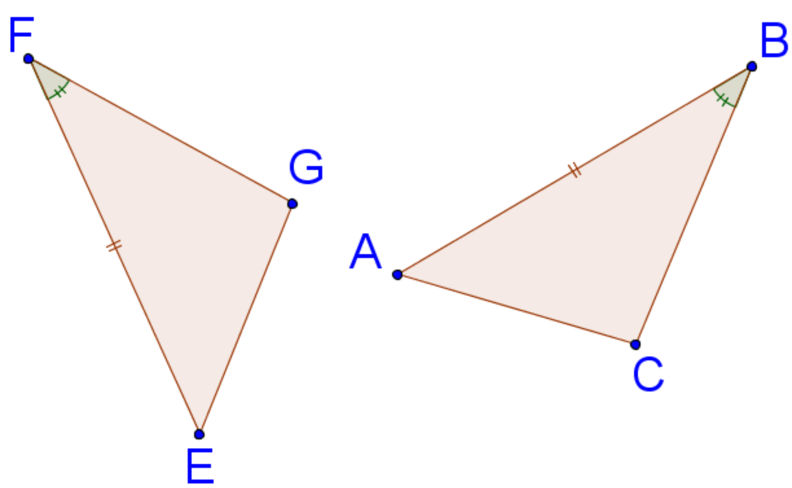


The triangles are not necessarily congruent. The information for ΔABC is AAS while the information for ΔEFG is ASA. There is not enough information about corresponding sides that are congruent.

Example 4

Are the following triangles congruent? Explain.

What additional information would you need in order to be able to state that the triangles below are congruent by AAS?



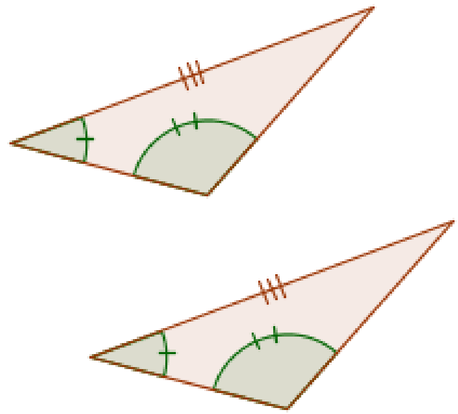
You would need to know that $\angle G \cong \angle C$.

Review

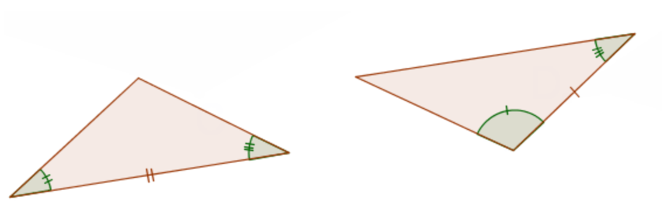
1. What does ASA stand for? How is it used?
2. What does AAS stand for? How is it used?
3. Draw an example of two triangles that must be congruent due to ASA.
4. Draw an example of two triangles that must be congruent due to AAS.

For each pair of triangles below, state if they are congruent by ASA, congruent by AAS, or if there is not enough information to determine whether or not they are congruent.

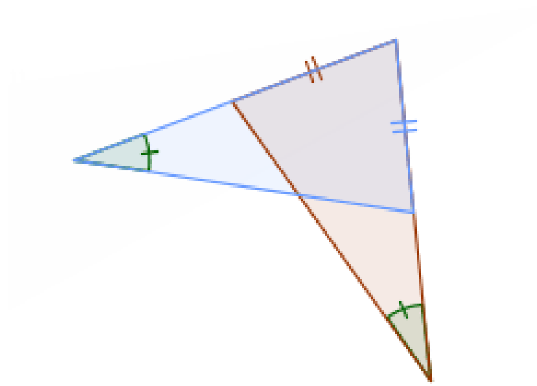
5.



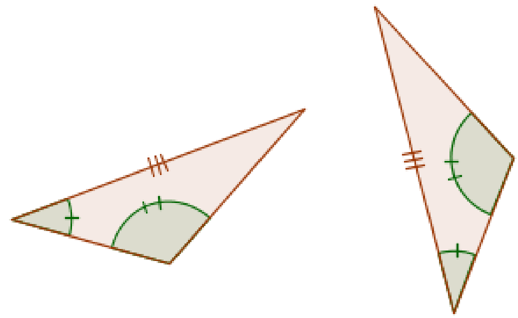
6.



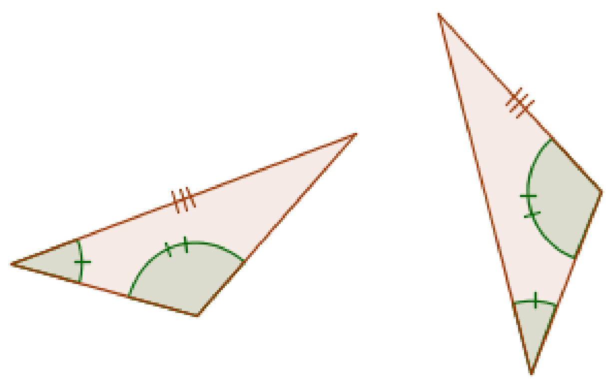
7.



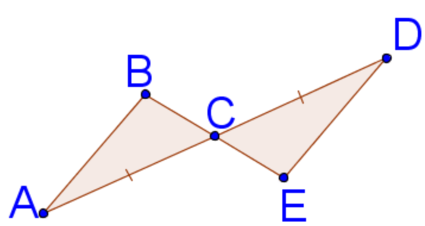
8.



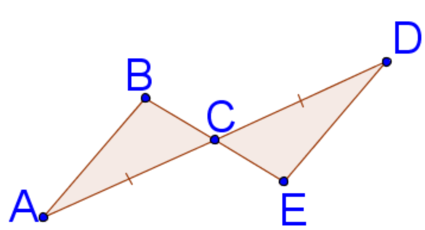
9.



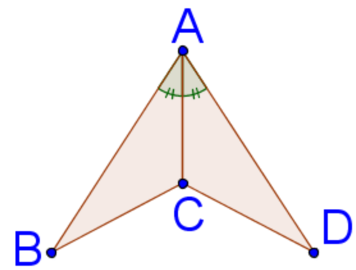
10. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by AAS? Assume that points B, C, and E are collinear.



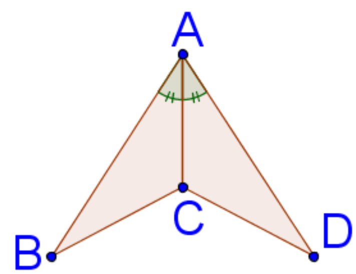
11. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by ASA? Assume that points B, C, and E are collinear.



12. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by AAS?

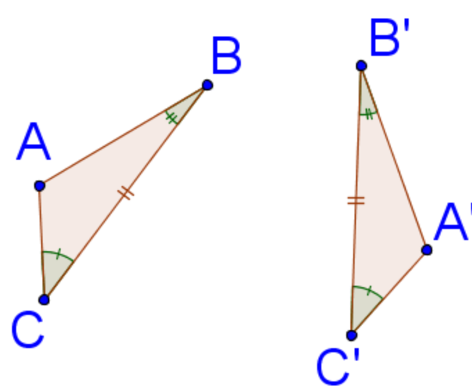


13. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by ASA?



14. If you can show that two triangles are congruent with AAS, can you also show that they are congruent with ASA?

15. Show how the ASA criterion for triangle congruence works: use rigid transformations to help explain why the triangles below are congruent.



16. Given the triangles in the image, if possible, describe a sequence of rigid motion transformations that maps $\overline{AB}$ to $\overline{A'B'}$. Now, describe a sequence of rigid motion transformations that maps $\overline{AC}$ to $\overline{A'C'}$. Was the sequence the same in both cases? In order for an original and an image to be congruent, what must be true about the transformations? Is it possible to describe a series of rigid motion transformations that map $\overline{BC}$ to $\overline{B'C'}$? Why or why not?

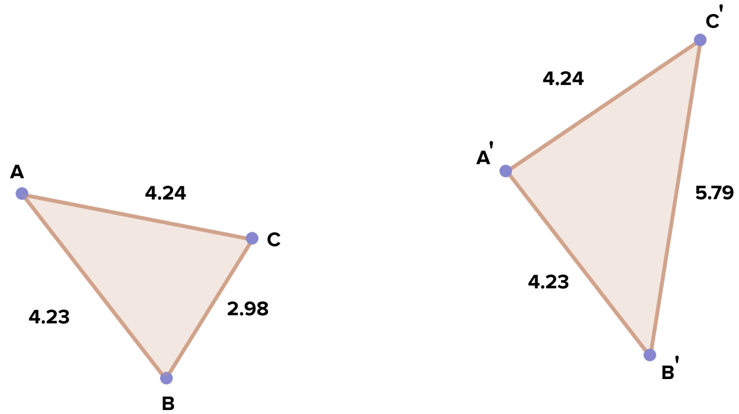


FIGURE 4.9

17. Given the triangles in the image, if possible, describe a sequence of rigid motion transformations that maps $\overline{AB}$ to $\overline{A'B'}$ and $\angle BAC$ to $\angle B'A'C'$. Will C and C' be collinear after this transformation? Why or why not? Will C map to C' as a result of this transformation? Why or why not? What does that tell you about $\overline{BC}$ and $\overline{B'C'}$?

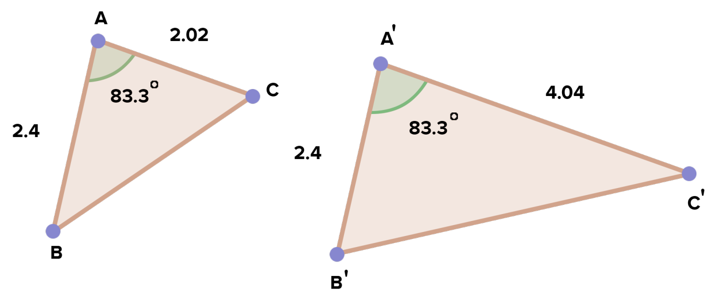


FIGURE 4.10

18. If possible, describe a series of rigid motion transformations that maps ΔABC to $\Delta A'B'C'$.

19. With a partner or on your own, select a side length and two different angles for a triangle such that the third angle will also be different. Without looking at each other's paper (or your prior work), use a ruler and a protractor to sketch the triangle by arranging the sides and angles in two different ways. One person (or your first sketch) should show the selected side length as a side of **both** selected angles. The second person (or your second sketch) should show the selected side length as a side of **only one** of the selected angles. Are your triangles congruent or not? Why or why not? What does this tell you about the ASA Triangle Congruence Theorem?

20. If two angles of a triangle are congruent to two angles of another triangle, what must be true about the remaining corresponding angles? Why? How does this relate to the fact that AAS follows directly from ASA?

21. Given the diagram below, explain why ΔABC and ΔDBF are or are not congruent.

22. Given the diagram below, explain why ΔABC and ΔDBF are or are not congruent.

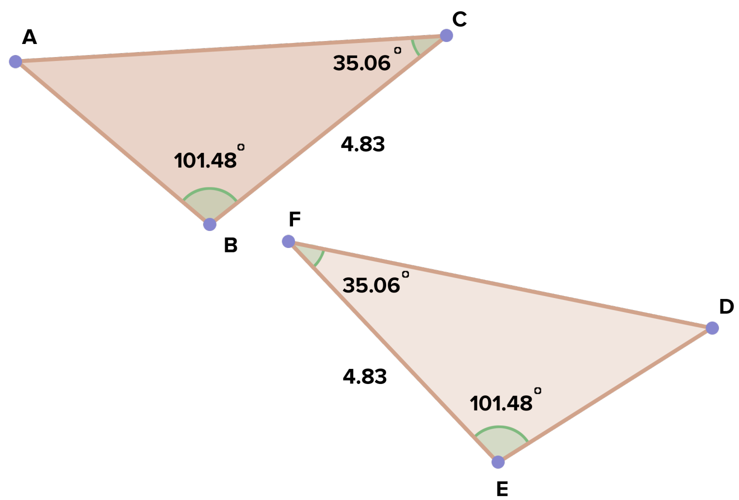


FIGURE 4.11

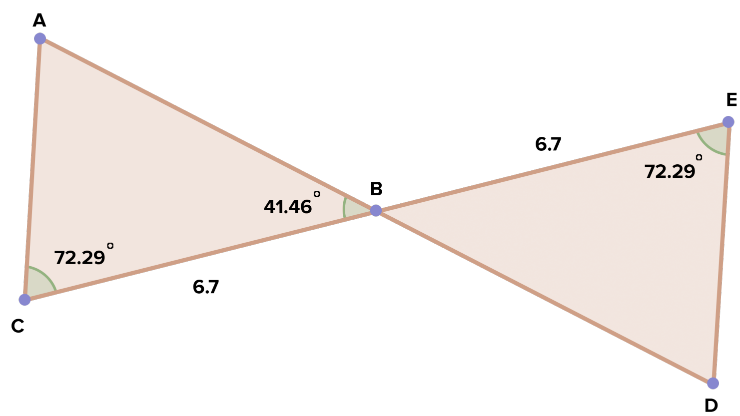


FIGURE 4.12

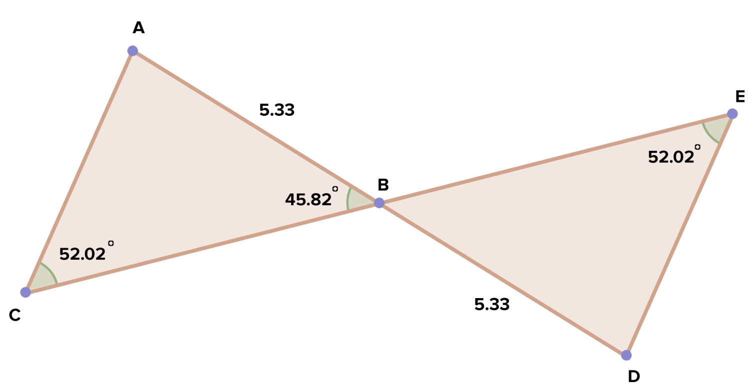


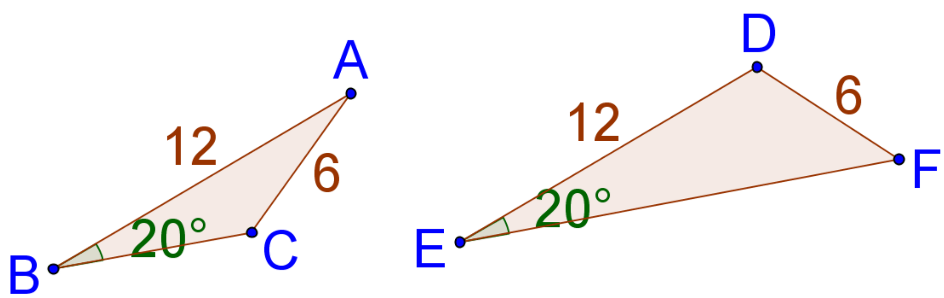
FIGURE 4.13

4.4 SAS Triangle Congruence

Learning Objectives

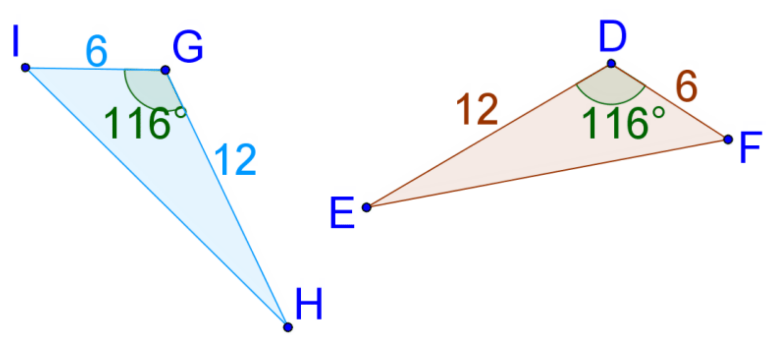
Here you will explore the SAS criterion for triangle congruence.

When two triangles have two pairs of sides and their included angles congruent, the triangles are congruent. What if the angles aren't included angles?



SAS Triangle Congruence

If two triangles are **congruent** it means that all corresponding angle pairs and all corresponding sides are congruent. However, in order to be sure that two triangles are congruent, you do not necessarily need to know that all angle pairs and side pairs are congruent. Consider the triangles below.



In these triangles, you can see that $\angle G \cong \angle D$, $\overline{IG} \cong \overline{FD}$, and $\overline{GH} \cong \overline{DE}$. The information you know about the congruent corresponding parts of these triangles is a **s**ide, an **a**ngle, and then another **s**ide. This is commonly referred to as "side-angle-side" or "SAS".

The SAS criterion for triangle congruence states that if two triangles have two pairs of congruent sides and the included angle in one triangle is congruent to the included angle in the other triangle, then the triangles are congruent.

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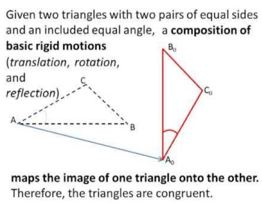

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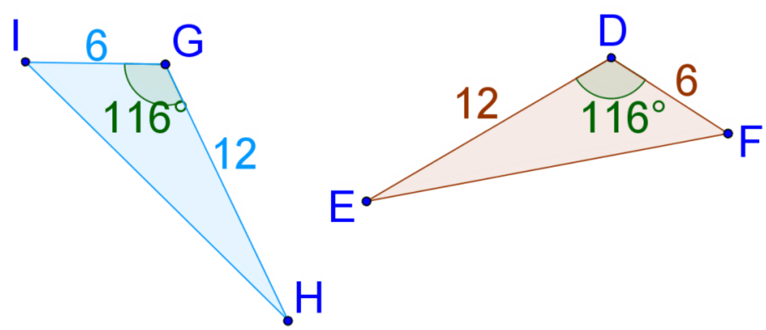
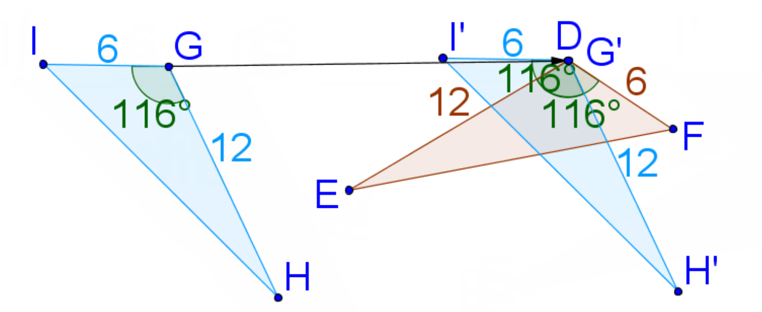
Let's take a look at some example problems.

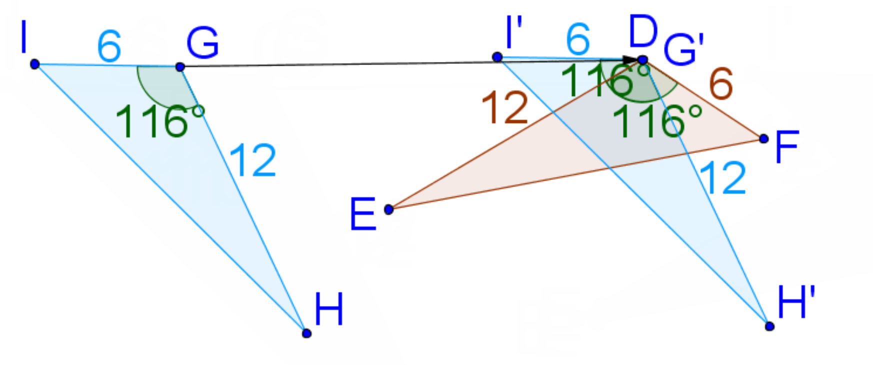
1. Perform a rigid transformation to bring point G to point D.



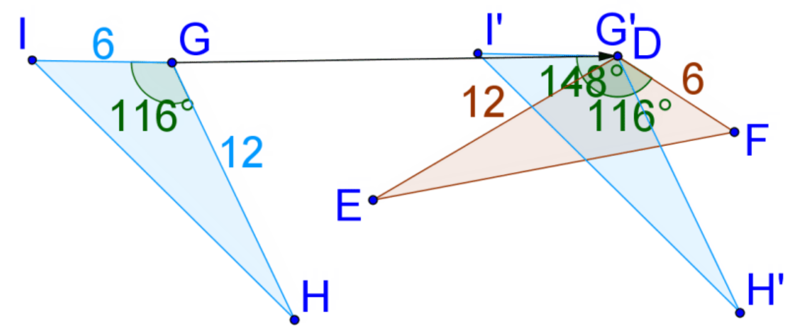
2. Draw a vector from point G to point D. Translate $\triangle GHI$ along the vector to create $\triangle G'H'I'$.



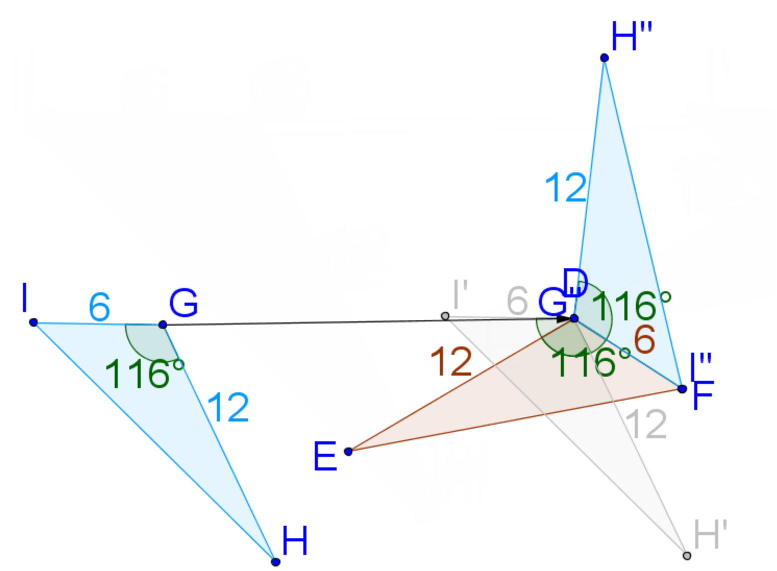
3. Rotate $\triangle G'H'I'$ to map to $\overline{G'I'}$ to $\overline{DF}$.



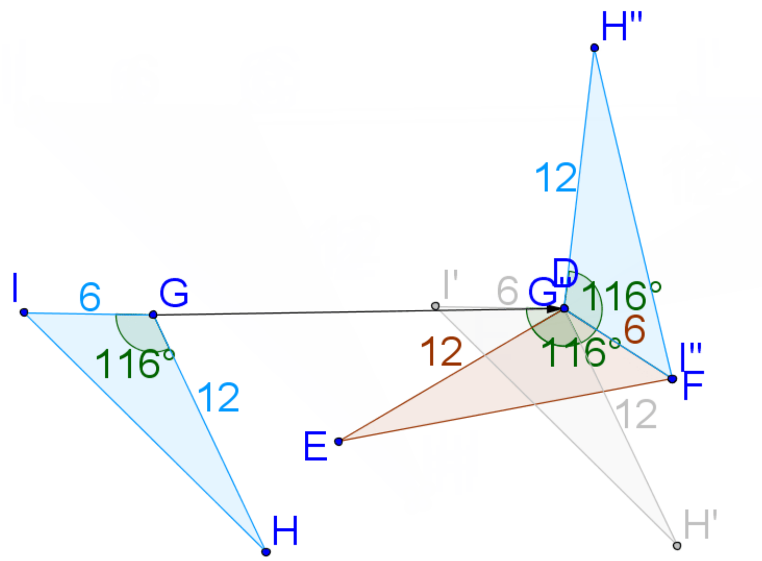
4. Measure $\angle I'DF$. In this case, $m\angle I'DF = 148°$.



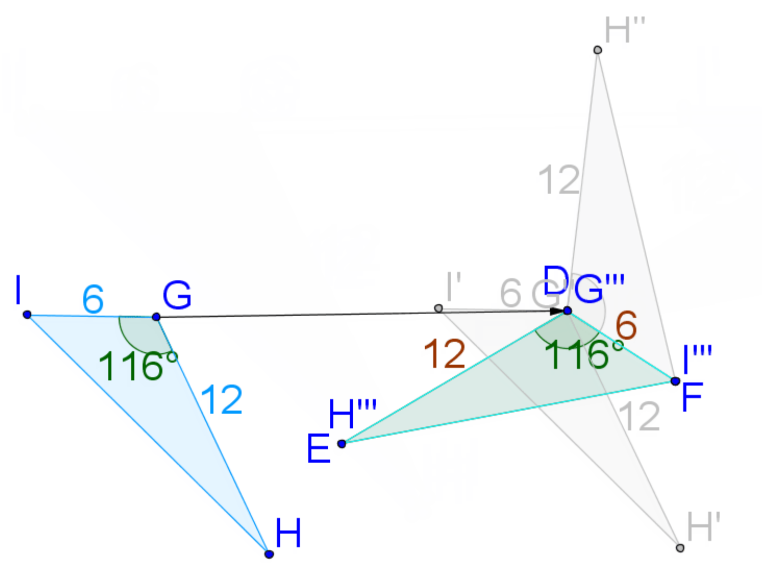
5. Rotate $\triangle G'H'I'$ counterclockwise that number of degrees about point G' to create $\triangle G''H''I''$. Note that because $\overline{GI} \cong \overline{DF}$ and rigid transformations preserve distance, $\overline{G''I''}$ matches up perfectly with $\overline{DF}$.



6. Reflect $\triangle G''H''I''$ to map it to $\triangle DEF$. Can you be confident that the triangles are congruent?



7. Reflect $\triangle G''H''I''$ across $\overline{G''I''}$ (which is the same as $\overline{DF}$).



Because $\angle EDF \cong \angle H''G''I''$ and $\overline{G''H''} \cong \overline{DE}$, the triangles must match up exactly (in particular, H''' must map to E), and the triangles are congruent.

This means that even though you didn't know all the side and angle measures, because you knew two pairs of sides and the included angles were congruent, the triangles had to be congruent overall. At this point you can use the SAS criterion for showing triangles are congruent without having to go through all of these transformations each time (but make sure you can explain why SAS works in terms of the rigid transformations!).

Determining if Two Triangles are Congruent

Is the pair of triangles congruent? If so, write the congruence statement and why.

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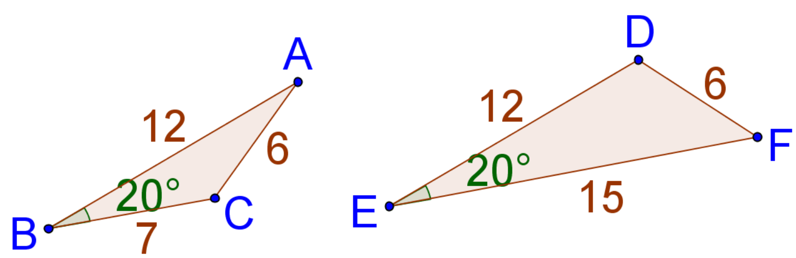
While the triangles have two pairs of sides and one pair of angles that are congruent, the angle is not in the same place in both triangles. The first triangle fits with SAS, but the second triangle is SSA. There is not enough information for us to know whether or not these triangles are congruent.

Examples

Example 1

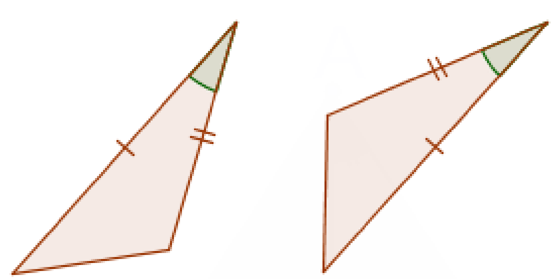
Earlier, you were asked what if the angles are not included angles.



Even though these triangles have two pairs of sides and one pair of angles that are congruent, the triangles are clearly not congruent. SSA is NOT a criterion for triangle congruence. In order to use two pairs of sides and one pair of angles to show that triangles are congruent, the pair of angles must be included between the pairs of congruent sides.

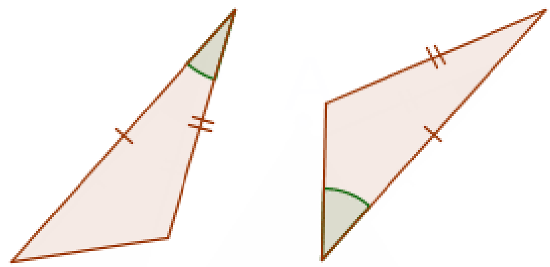
Example 2

Are the following triangles congruent? Explain.



The triangles are congruent by SAS.

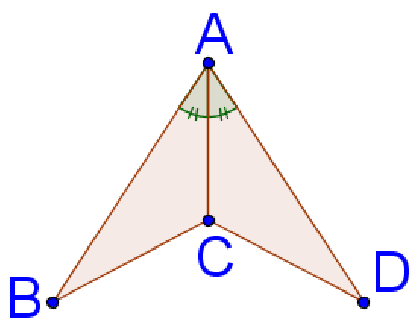
Example 3



The triangles are not necessarily congruent. The given angle is not the included angle in both triangles.

Example 4

What additional information would you need in order to be able to state that the triangles below are congruent by SAS?



You would need to know that $\overline{AB} \cong \overline{AD}$.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

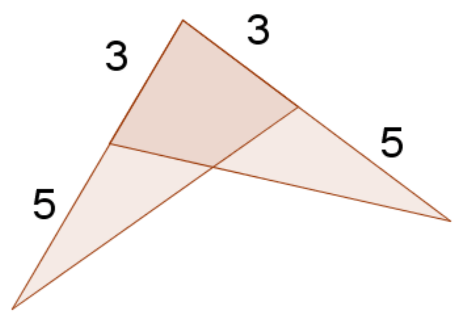
URL: http://www.ck12.org/geometry/sas-triangle-congruence/plix/Side-Angle-Side-Triangle-Congruence-5410dc215aa4132edb4e970c

Review

1. What does SAS stand for? What does it have to do with congruent triangles?
2. What does SSA stand for? What does it have to do with congruent triangles?
3. Draw an example of two triangles that must be congruent due to SAS.
4. Draw an example of two triangles that **are not congruent** because all you know is SSA.

For each pair of triangles below, state if they are congruent by SAS or if there is not enough information to determine whether or not they are congruent.

5.



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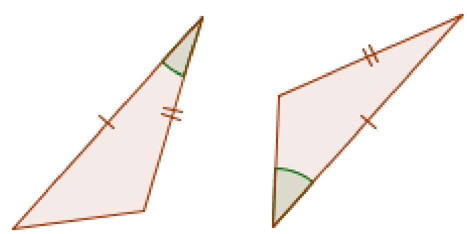

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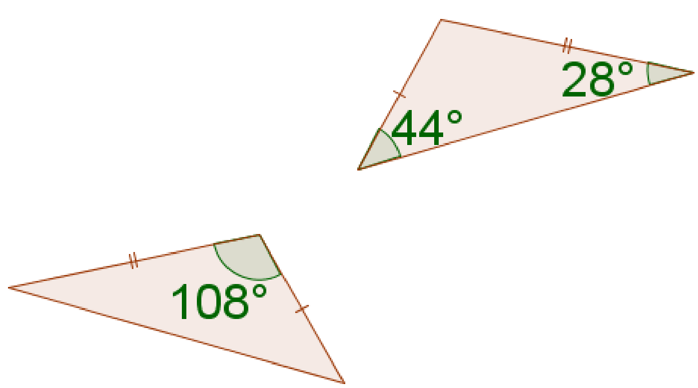

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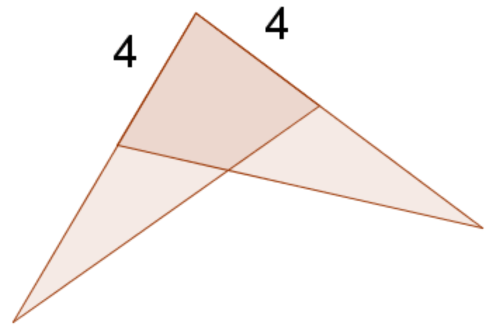

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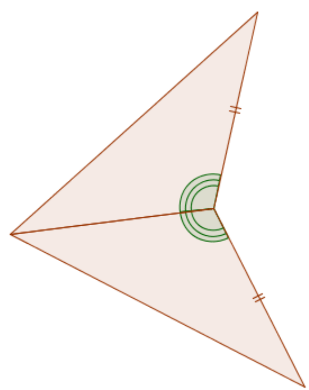
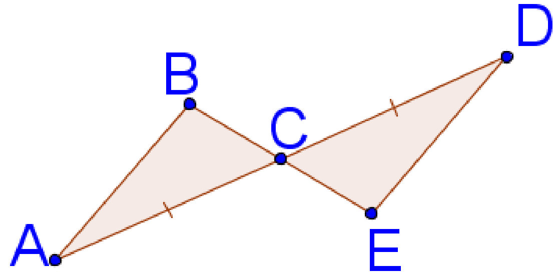
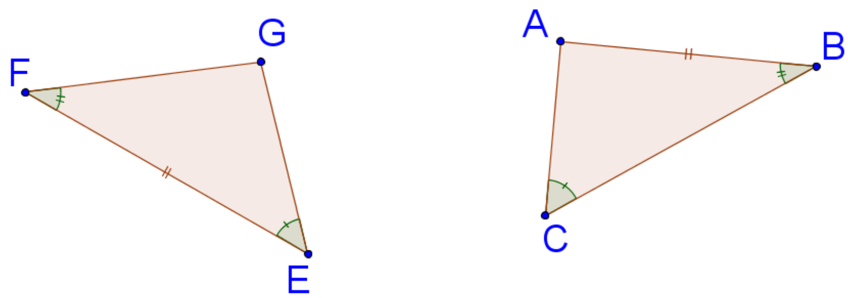


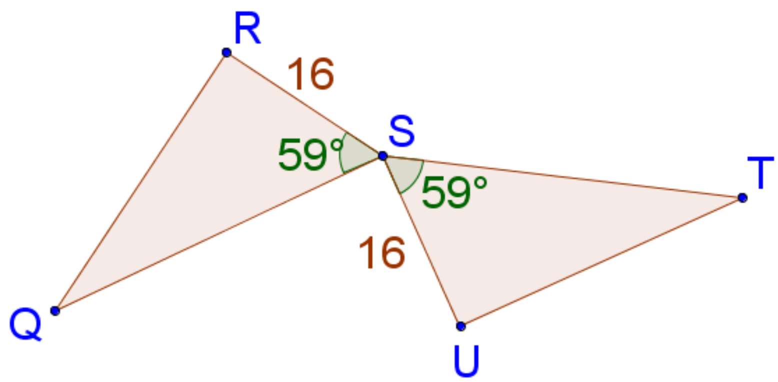
10. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by SAS? Assume that points B, C, and E are collinear.



11. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by SAS?



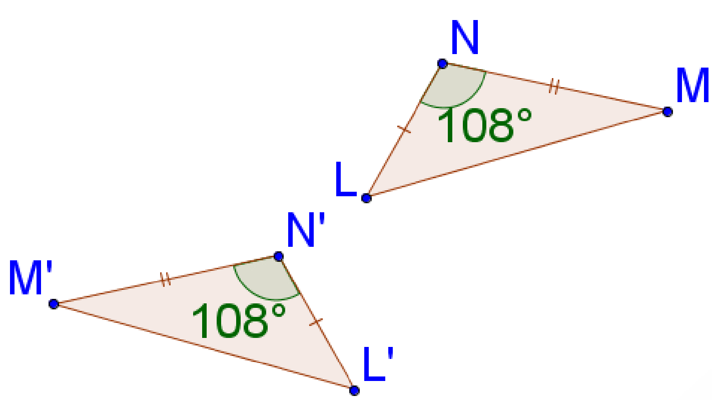
12. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by SAS?



13. Do you think you always need at least three pairs of congruent sides/angles to show that two triangles are congruent? Explain.

14. If the two pairs of legs are congruent on two right triangles, are the triangles congruent? Explain. Draw a picture to support your reasoning.

15. Show how the SAS criterion for triangle congruence works: use rigid transformations to help explain why the triangles below are congruent.



16. Given the diagram below, is there a rigid motion transformation that maps $\overline{AB}$ to $\overline{DE}$? If so, describe. Is there such a transformation that maps $\angle ABC$ to $\angle DEF$? If so, is it the **same** transformation that mapped $\overline{AB}$ to $\overline{DE}$?

Explain. Finally, is there a rigid motion transformation that maps $\overline{AC}$ to $\overline{DF}$? If so, is it the **same** transformation as those used for the previous objects? Explain any difference. Can it be concluded that the triangles are congruent? Explain.

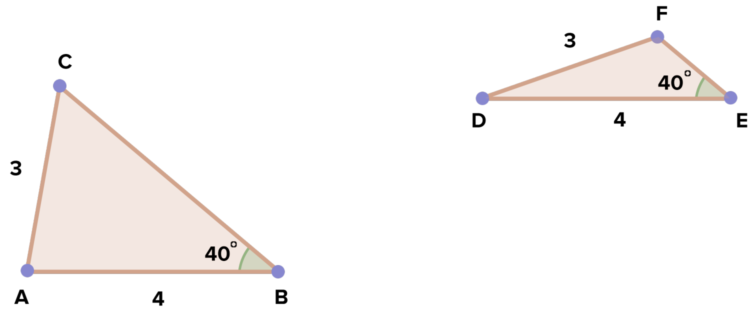


FIGURE 4.14

17. Explain how the diagram below shows that there is not a single rigid motion transformation that maps all the parts of the triangle on the left to the triangle on the right.

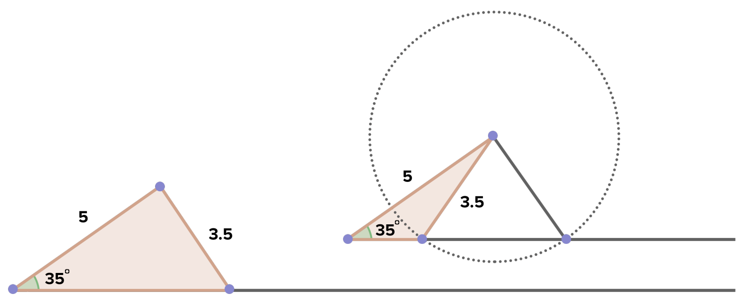


FIGURE 4.15

18. Referring to the side of length 3.5 in the above diagram, is there a length to which this side could be changed that would make it so that the two triangles are congruent? What would be the relationship between this segment and the horizontal ray? Explain.

Review (Answers)

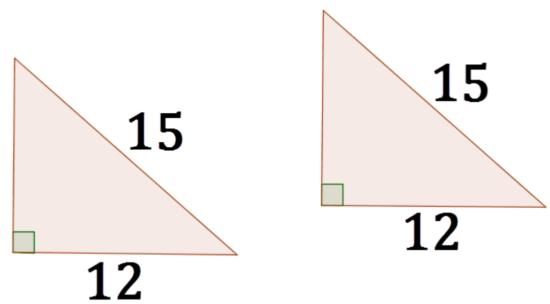
To see the Review answers, open this PDF file , and look for section 3.3.

4.5 SSS Triangle Congruence

Learning Objectives

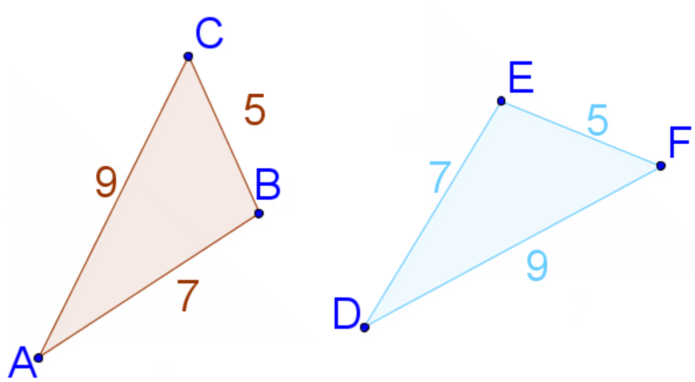
Here you will explore the SSS criterion for triangle congruence and the HL criterion for right triangle congruence.

How can you use the SSS criterion for triangle congruence to show that the triangles below are congruent?



SSS Triangles

If two triangles are **congruent** it means that all corresponding angle pairs and all corresponding sides are congruent. However, in order to be sure that two triangles are congruent, you do not necessarily need to know that all angle pairs and side pairs are congruent. Consider the triangles below.



In these triangles, you can see that all three pairs of sides are congruent. This is commonly referred to as "side-side-side" or "SSS".

The SSS criterion for triangle congruence states that if two triangles have three pairs of congruent sides, then the triangles are congruent.

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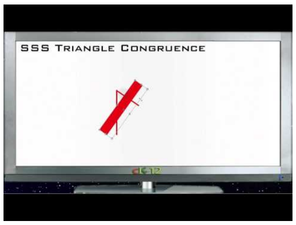
URL: http://www.ck12.org/flx/render/embeddedobject/223463



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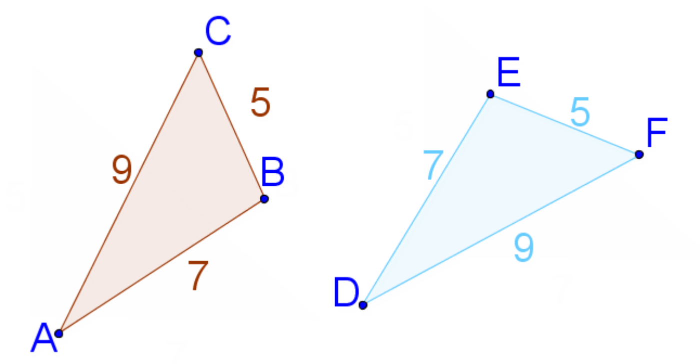
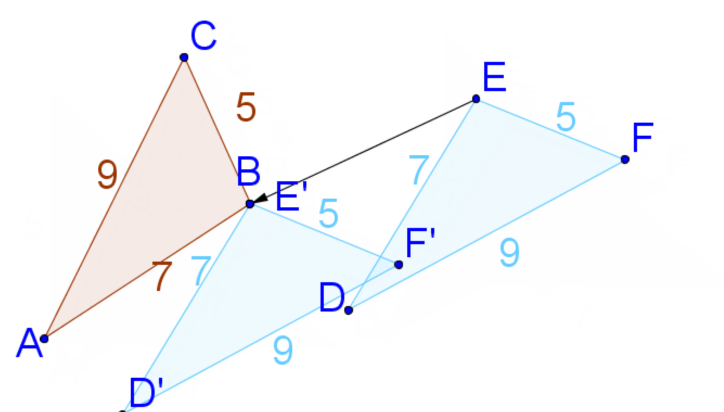
In the examples, you will use rigid transformations to show why the above SSS triangles must be congruent overall, even though you don't know the measures of any of the angles.

Let's take a look at some example problems.

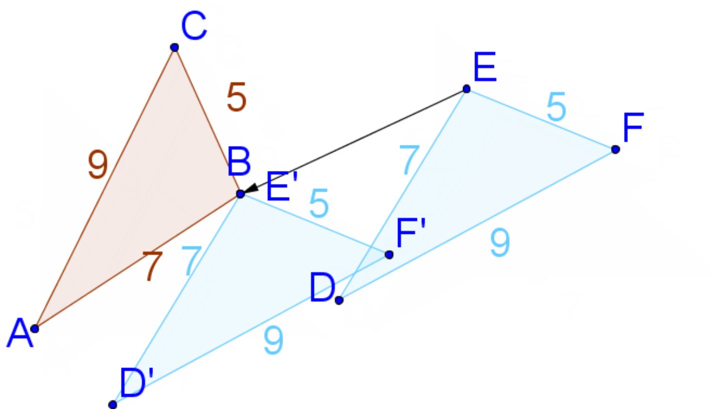
1. Perform a rigid transformation to bring point E to point B.



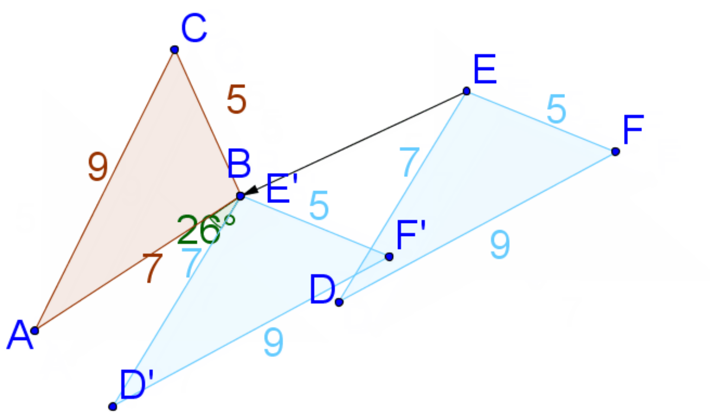
2. Draw a vector from point E to point B. Translate $\triangle DEF$ along the vector to create $\triangle D'E'F'$.



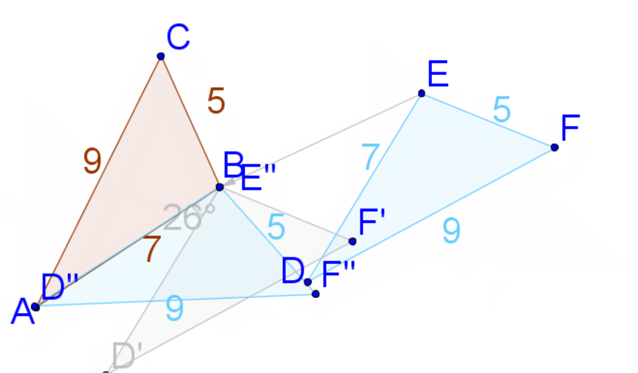
3. Rotate $\triangle D'E'F'$ to map $\overline{D'E'}$ to $\overline{AB}$.



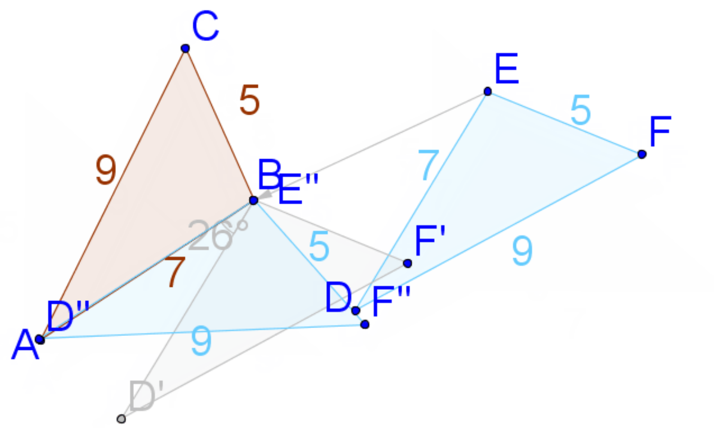
4. Measure $\angle ABD'$. In this case, $m\angle ABD' = 26^\circ$.



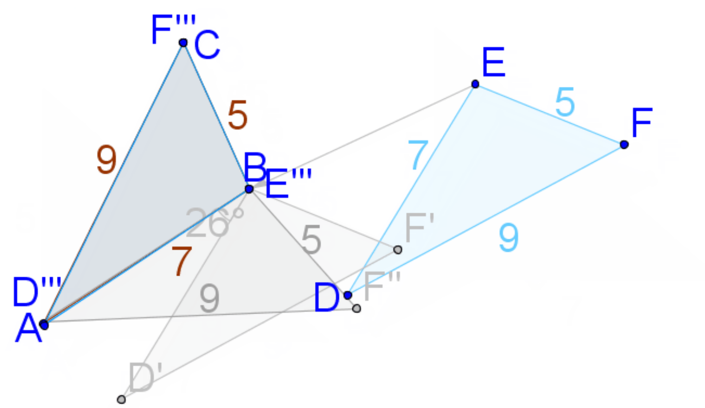
5. Rotate $\triangle D'E'F'$ clockwise that number of degrees to create $\triangle D''E''F''$. Note that because $\overline{DE} \cong \overline{AB}$ and rigid transformations preserve distance, $\overline{D''E''}$ matches up perfectly with $\overline{AB}$.



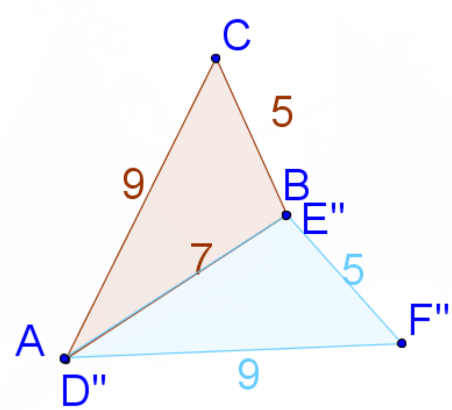
6. Reflect $\triangle D''E''F''$ to map it to $\triangle ABC$. Can you be confident that the triangles are congruent?



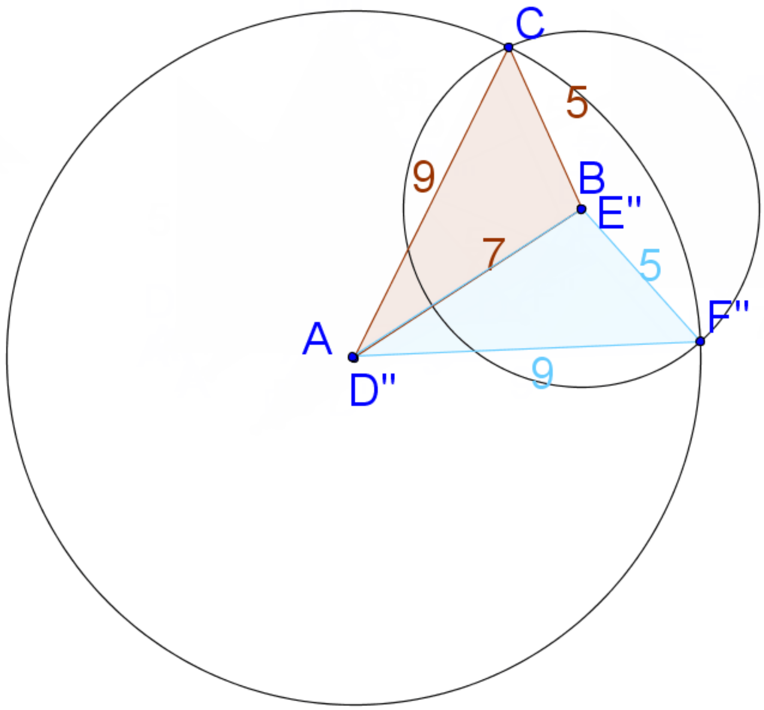
7. Reflect $\triangle D''E''F''$ across $\overline{D''E''}$ (which is the same as $\overline{AB}$).



In this case, it looks like the triangles match up exactly and are therefore congruent, but how can you always be confident that F'' will map to C? Consider the previous step, with the two triangles below:



You know that wherever F'' ends up after it is reflected, it has to stay 5 units away from E'' and 9 units away from D''. Create a circle centered at E'' with radius 5 units to find all the points besides F'' that are 5 units away from E''. Also create a circle centered at D'' with radius 9 units to find all the points besides F'' that are 9 units away from D''. Notice that there are only two points in the whole plane that are both 5 units away from E'' and 9 units away from D'': point F'' and point C. Since reflections preserve distance, when F'' is reflected, it must end up at point C. Therefore, a reflection will always map $\triangle D''E''F''$ to $\triangle ABC$ at this step.

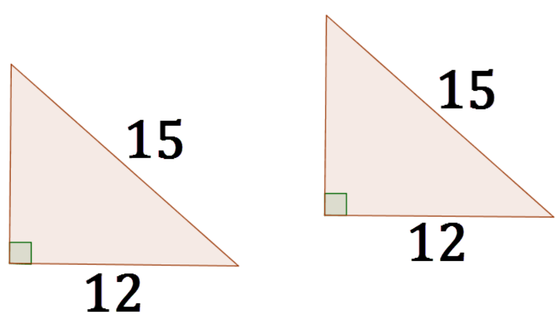


This means that even though you didn't know the angle measures, because you knew three pairs of sides were congruent, the triangles had to be congruent overall. At this point you can use the SSS criterion for showing triangles are congruent without having to go through all of these transformations each time (but make sure you can explain why SSS works in terms of the rigid transformations!).

Examples

Example 1

Earlier, you were asked how can you use the SSS criterion for triangle congruence to show that the triangles below are congruent.

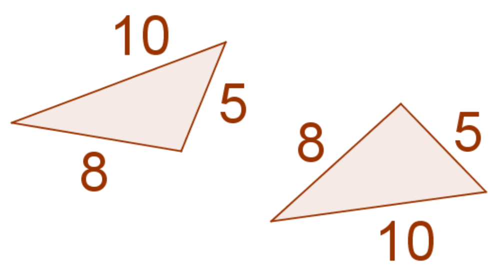


Because these are right triangles, you can use the Pythagorean Theorem to find the third side of each triangle. The third side of each triangle will be $\sqrt{15^2 - 12^2} = 9$. Now you know that all three pairs of sides are congruent, so the triangles are congruent by SSS.

In general, anytime you have the hypotenuses congruent and one pair of legs congruent for two right triangles, the triangles are congruent. This is often referred to as "HL" for "hypotenuse-leg". Remember, it only works for right triangles because you can only use the Pythagorean Theorem for right triangles.

Example 2

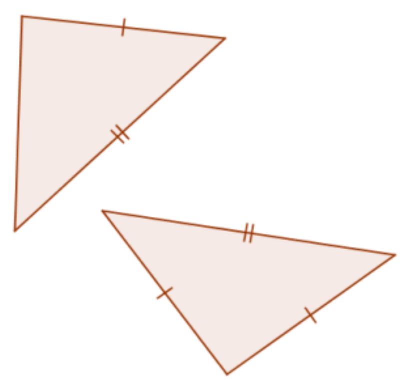
Are the following triangles congruent? Explain.



Yes, the triangles are congruent by SSS.

Example 3

Are the following triangles congruent? Explain.

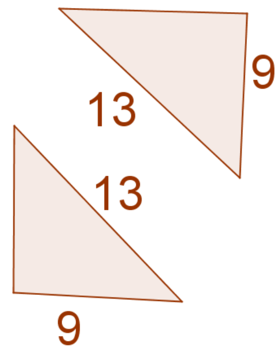


There is not enough information to determine if the triangles are congruent. You need to know how the unmarked side compares to the other sides, or if there are right angles.

Example 4

Are the following triangles congruent. Explain.

What additional information would you need in order to be able to state that the triangles below are congruent by HL?



You would need to know that the triangles are right triangles in order to use HL.

MEDIA

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CK-12 PLIX Interactive

PLIX

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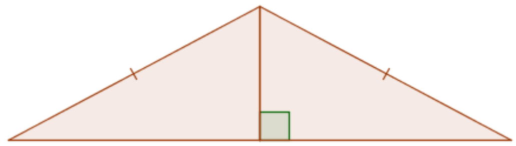
URL: http://www.ck12.org/geometry/sss-triangle-congruence/plix/SSS-Triangle-Congruence-Congruence-Conundrum-540a3ee65aa4136b3235cc5f

1. What does SSS stand for? How is it used?
2. What does HL stand for? How is it used?
3. Draw an example of two triangles that must be congruent due to SSS.
4. Draw an example of two triangles that must be congruent due to HL.

For each pair of triangles below, state if they are congruent by SSS, congruent by HL, or if there is not enough information to determine whether or not they are congruent.

5.



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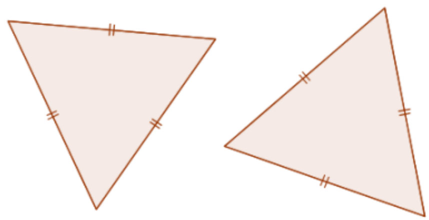

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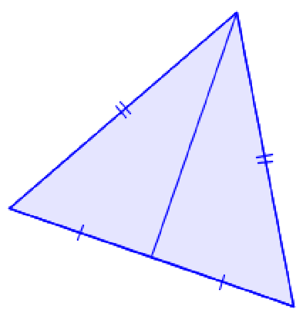

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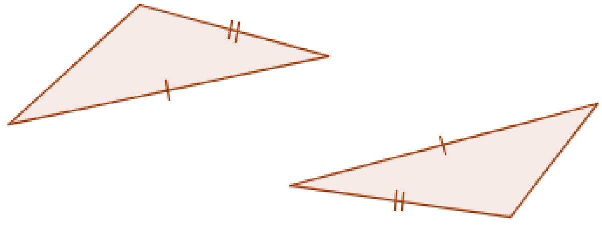

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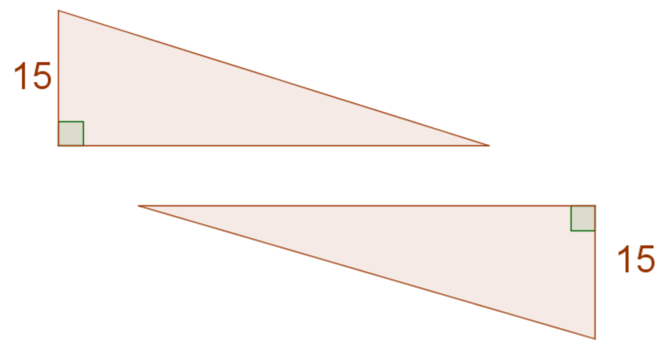
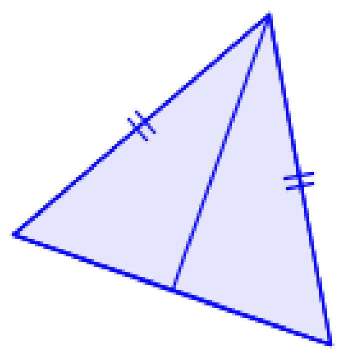
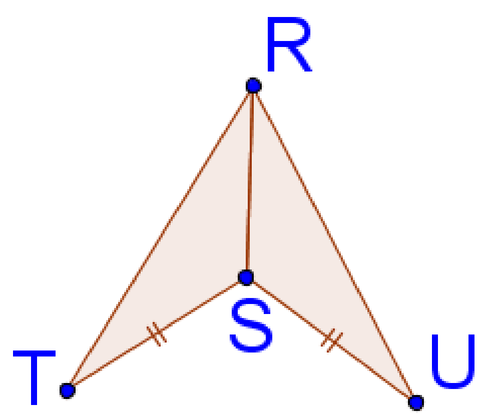


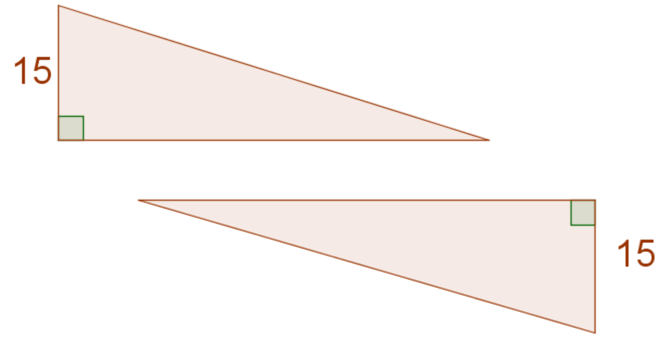
10. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by HL?



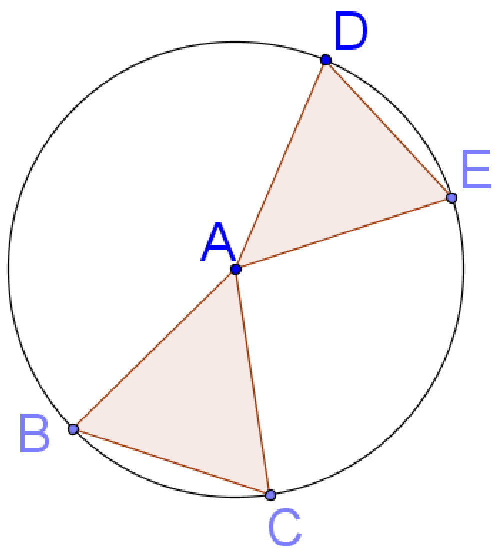
11. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by SSS?



12. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by HL?

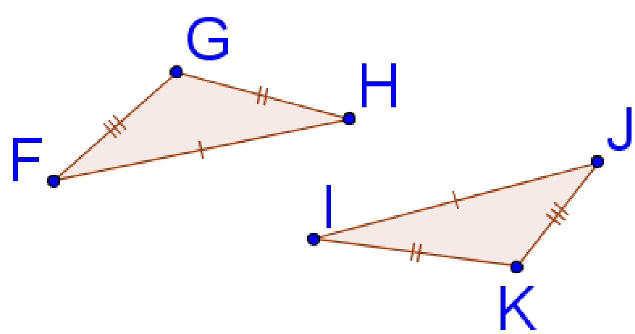


13. Point A is the center of the circle below. What is the minimum additional information you would need in order to be able to state that the triangles below are congruent by SSS?



14. If you can show that two triangles are congruent by HL, can you also show that they are congruent by SAS?

15. Show how the SSS criterion for triangle congruence works: use rigid transformations to help explain why the triangles below are congruent.



16. Below is another example which demonstrates why SSS can be used to establish triangle congruence. Three sides of ΔABC are given as congruent to three sides of ΔDEF as shown. Rigid motion transformations were performed to

map $\overline{AB}$ to $\overline{DE}$. But how can one be sure that a reflection maps C to F? One can try a different approach from the one used in the notes above. Remember, SAS is a triangle congruence theorem. What kind of triangle is ΔACF? So what is known about $\angle 1$ and $\angle 2$? What about $\angle 3$ and $\angle 4$? Why? So what about $\angle ACE$ and $\angle DFE$? Now, is there enough information to conclude the triangles are congruent? Why? Explain the process just used in words to a peer.

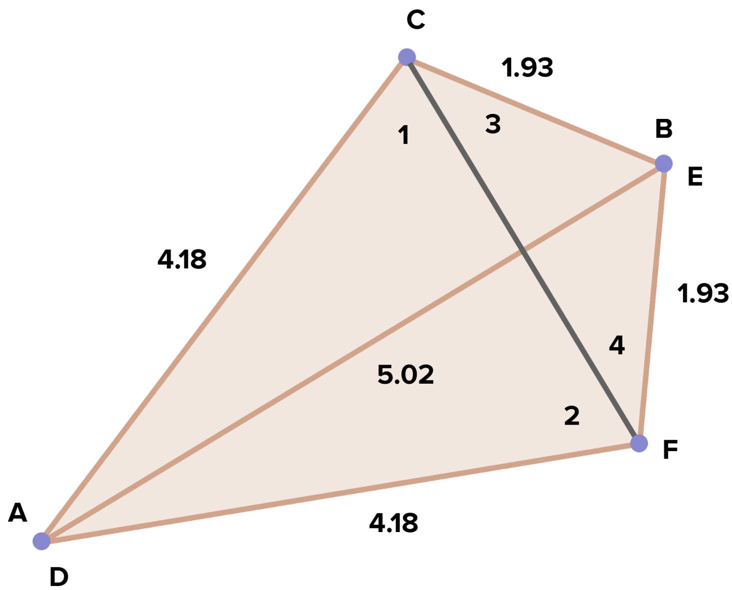


FIGURE 4.16

17. The two triangles below have two pairs of congruent sides and a pair of non-included congruent angles. Are the triangles congruent? Why or why not? Under what conditions would the triangles be congruent?

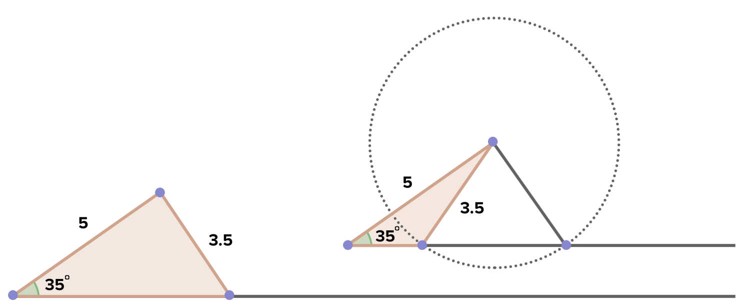


FIGURE 4.17

18. The two triangles below have one pair of congruent legs and congruent hypotenuses. Remember, SSA is **not** a congruence theorem. But a similar method used in number 16 can be used to establish that these two triangles are congruent. Explain the process. Based on this, can one conclude that HL is a congruence theorem that can be used for any pair of right triangles that have a pair of congruent legs and a pair of congruent hypotenuses? Why or why not?

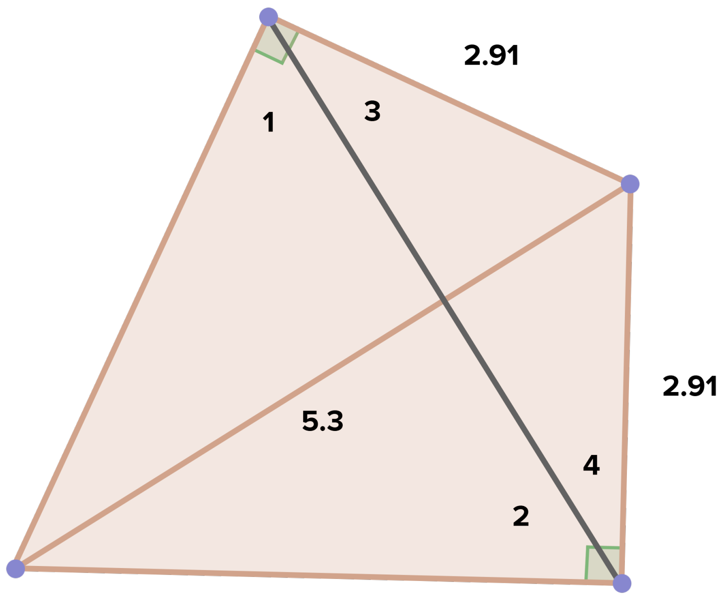


FIGURE 4.18

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.4.

4.6 Applications of Congruent Triangles

Learning Objectives

Here you will use the criteria for triangle congruence to solve problems.

Max constructs a triangle using an online tool. He tells Alicia that his triangle has a 42° angle, a side of length 12, and a side of length 8. With only this information, will Alicia be able to construct a triangle that must be congruent to Max's triangle?

Applications for Congruent Triangles

Two triangles are **congruent** if and only if corresponding pairs of sides and corresponding pairs are congruent. While one way to show that two triangles are congruent is to verify that all side and angle pairs are congruent, there are five "shortcuts". The following list summarizes the different criteria that can be used to show triangle congruence.

- **AAS (Angle-Angle-Side):** If two triangles have two pairs of congruent angles and a non-common side of the angles in one triangle is congruent to the corresponding side in the other triangle, then the triangles are congruent.
- **ASA (Angle-Side-Angle):** If two triangles have two pairs of congruent angles and the common side of the angles in one triangle is congruent to the corresponding side in the other triangle, then the triangles are congruent.
- **SAS (Side-Angle-Side):** If two triangles have two pairs of congruent sides and the included angle in one triangle is congruent to the included angle in the other triangle, then the triangles are congruent.
- **SSS (Side-Side-Side):** If two triangles have three pairs of congruent sides, then the triangles are congruent.
- ***[FOR RIGHT TRIANGLES]*** **HL (Hypotenuse-Leg):** If two ***right*** triangles have one pair of legs congruent and hypotenuses congruent, then the triangles are congruent.

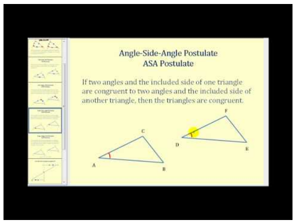
If two triangles don't satisfy at least one of the criteria above, you cannot be confident that they are congruent.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/hl-triangle-congruence/plix/HL-Triangle-Congruence-5589babb8e0e08346175796e



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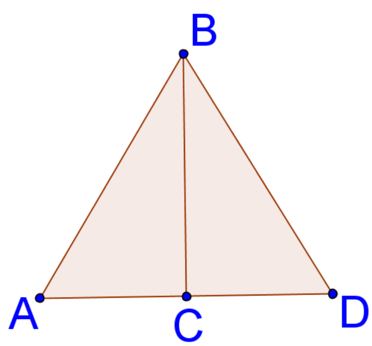
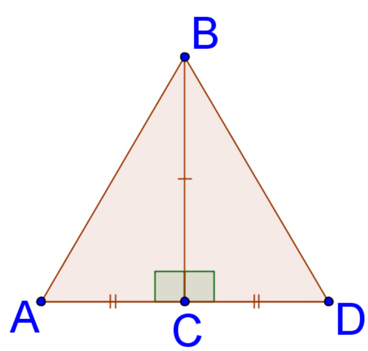
Recognizing Perpendicular Bisectors

$\overline{BC}$ is the perpendicular bisector of $\overline{AD}$. Is $\triangle ABC \cong \triangle ADC$?



If $\overline{BC}$ is the perpendicular bisector of $\overline{AD}$, then $\overline{AC} \cong \overline{CD}$. Also, $m\angle ACB = 90^\circ$ and $m\angle DCB = 90^\circ$, so $\angle ACB \cong \angle DCB$. You also know that $\overline{BC}$ is a side of both triangles, and is clearly congruent to itself (this is called the **reflexive property**).



The triangles are congruent by SAS. *Note that even though these are right triangles, you would not use HL to show triangle congruence in this case since you are not given that the hypotenuses are congruent.*

Measuring Angles

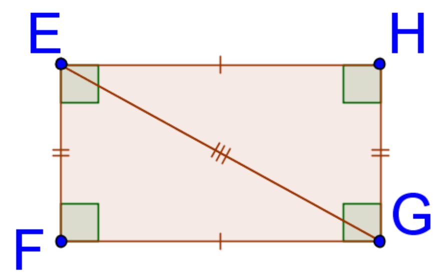
Using the information from the previous problem, if $m\angle A = 50^\circ$, what is $m\angle D$?

$m\angle D = 50^\circ$. Since the triangles are congruent, all of their corresponding angles and sides must be congruent. $\angle A$ and $\angle D$ are corresponding angles, so $\angle A \cong \angle D$.

Congruent Triangles

Does one diagonal of a rectangle divide the rectangle into congruent triangles?

Recall that a rectangle is a quadrilateral with four right angles. The opposite sides of a rectangle are congruent.



There is more than enough information to show that $\triangle EFG \cong \triangle GHE$.

- **Method #1:** The triangles have three pairs of congruent sides, so they are congruent by SSS.
- **Method #2:** The triangles have two pairs of congruent sides and congruent included angles, so they are congruent by SAS.
- **Method #3:** The triangles are right triangles with congruent hypotenuses and a pair of congruent legs, so they are congruent by HL.

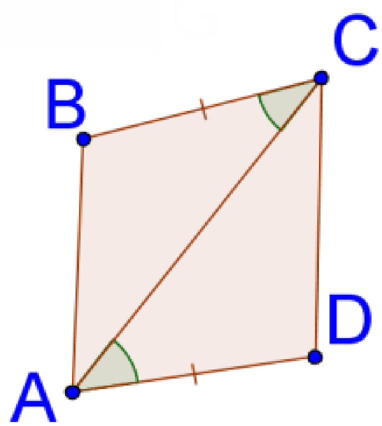
Examples

Example 1

Earlier, you were asked will Alicia be able to construct a triangle that must be congruent to Max's triangle.

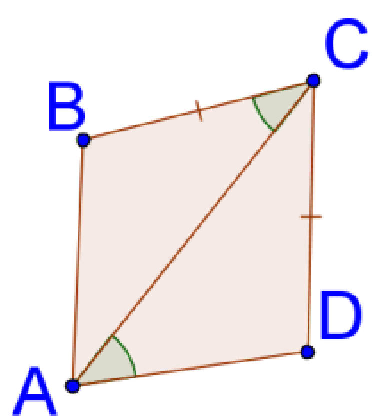
Max constructs a triangle using interactive geometry software. He tells Alicia that his triangle has a 42° angle, a side of length 12, and a side of length 8. If Max also told Alicia that the angle was in between the two sides, then she would be able to construct a triangle that must be congruent due to SAS. If the angle is not between the two sides, she cannot be confident that her triangle is congruent because SSA is not a criterion for triangle congruence. Because Max did not state where the angle was in relation to the sides, Alicia cannot create a triangle that must be congruent to Max's triangle.

For each pair of triangles, tell whether the given information is enough to show that the triangles are congruent. If the triangles are congruent, state the criterion that you used to determine the congruence and write a congruency statement. *Note that the figures are not necessarily drawn to scale!*



Notice that besides the one pair of congruent sides and the one pair of congruent angles, $\overline{AC} \cong \overline{CA}$. $\triangle ACB \cong \triangle CAD$ by SAS.

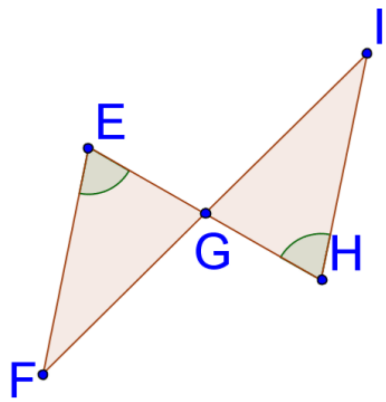
Example 3



The congruent sides are not corresponding in the same way that the congruent angles are corresponding. The given information for $\triangle ACB$ is SAS while the given information for $\triangle CAD$ is SSA. The triangles are not necessarily congruent.

Example 4

G is the midpoint of $\overline{EH}$.



Because G is the midpoint of $\overline{EH}$, $\overline{EG} \cong \overline{GH}$. You also know that $\angle EGF \cong \angle HGI$ because they are vertical angles. $\triangle EGF \cong \triangle HGI$ by ASA.

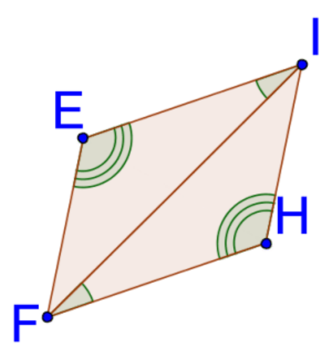
Review

1. List the five criteria for triangle congruence and draw a picture that demonstrates each.

2. Given two triangles, do you always need at least three pieces of information about each triangle in order to be able to state that the triangles are congruent?

For each pair of triangles, tell whether the given information is enough to show that the triangles are congruent. If the triangles are congruent, state the criterion that you used to determine the congruence and write a congruency statement.

3.



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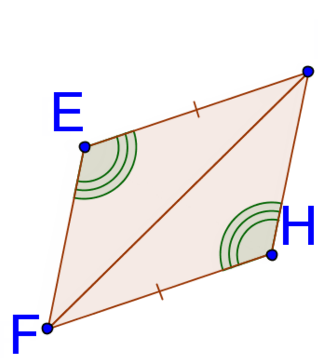

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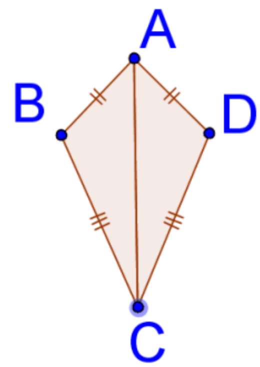

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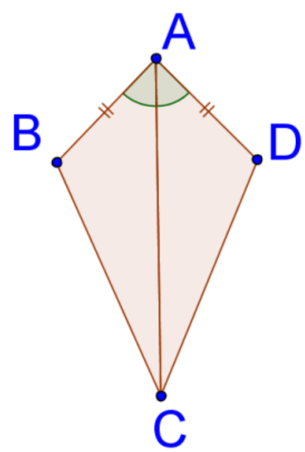

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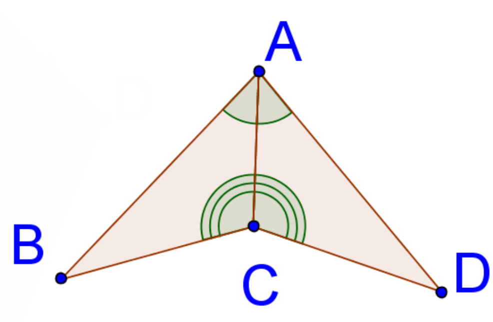

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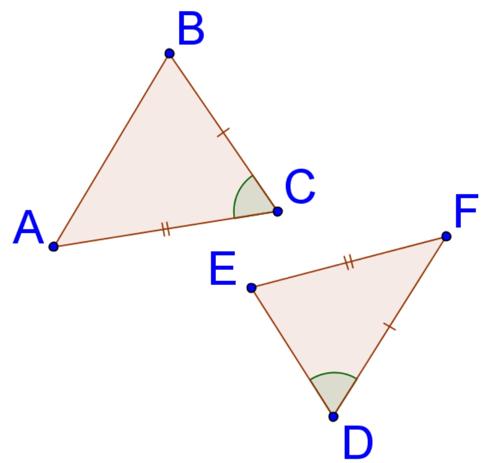


For 9-11, state whether the given information about a hidden triangle would be enough for you to construct a triangle that must be congruent to the hidden triangle. Explain your answer.

9. $\triangle ABC$ with $m\angle A = 72^\circ$, $AB = 6\ cm$, $BC = 8\ cm$.

10. $\triangle ABC$ with $m\angle A = 90^\circ$, $AB = 4\ cm$, $BC = 5\ cm$.

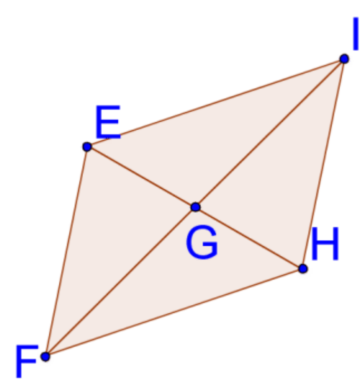
11. $\triangle ABC$ with $m\angle A = 72^\circ$, $AB = 6\ cm$, $AC = 8\ cm$.

12. Recall that a square is a quadrilateral with four right angles and four congruent sides. Show and explain why a diagonal of a square divides the square into two congruent triangles.

13. Show and explain using a different criterion for triangle congruence why a diagonal of a square divides the square into two congruent triangles.

14. Recall that a kite is a quadrilateral with two pairs of adjacent, congruent sides. Will one of the diagonals of a kite divide the kite into two congruent triangles? Show and explain your answer.

15. In the picture below, G is the midpoint of both $\overline{EH}$ and $\overline{FI}$. Explain why $\overline{FH} \cong \overline{IE}$ and $\overline{FE} \cong \overline{HI}$.



16. Explain why AAA is not a criterion for triangle congruence.

17. We have five different triangle congruence theorems we can use to establish that two triangles are congruent. Define and describe them. What does it mean to say that two triangles are congruent? If two triangles are congruent, are all of their corresponding sides and angles of the same measure? Explain.

18. We can use our triangle congruence shortcuts to establish previously discussed properties of quadrilaterals. Below is a kite. Define kite. One of the properties of a kite is that one pair of opposite angles are congruent. We now have the tools to prove this. A diagonal has been drawn to create two triangles. Which pairs of segments are congruent by the definition of a kite? Which segment is congruent to itself by the reflexive property? Can you prove the triangles are congruent? Why? What can be concluded about $\angle ACB$ and $\angle ADB$? Why?

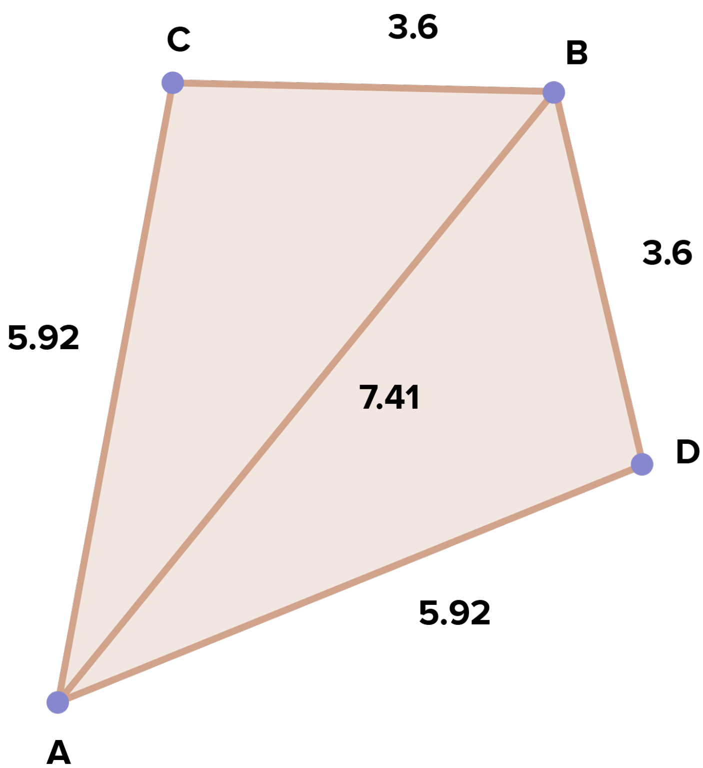


FIGURE 4.19

19. Below is given $\overline{AD}$, with perpendicular bisector $\overleftrightarrow{BC}$. What is a perpendicular bisector? Prove that any point on the perpendicular bisector (in this case,

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.5.

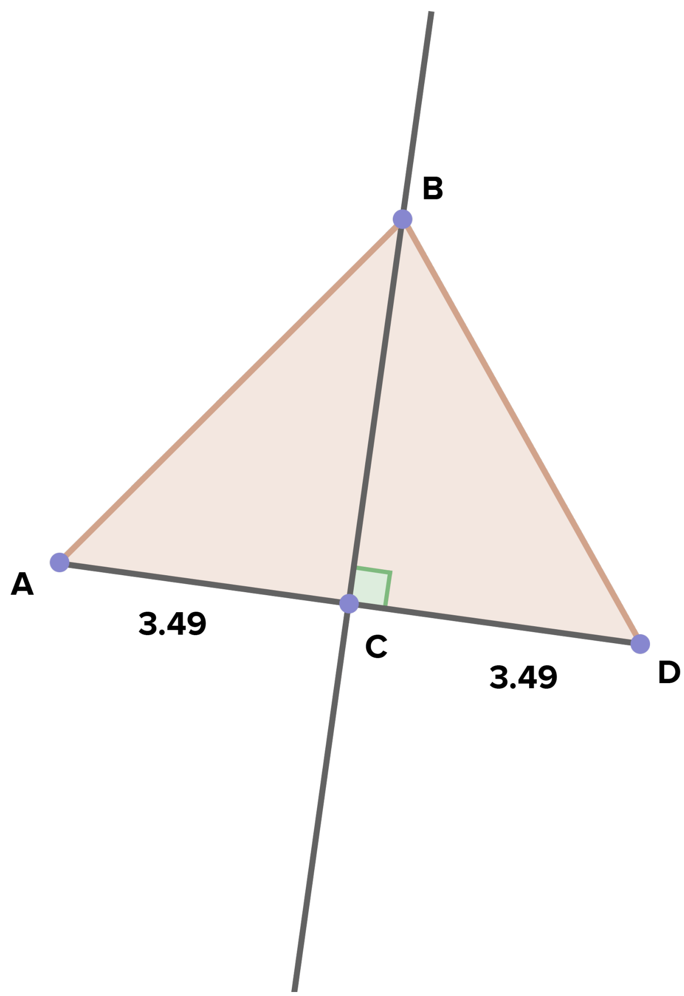


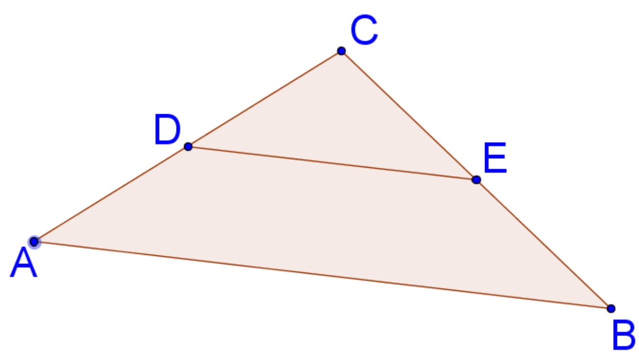
FIGURE 4.20

4.7 Theorems about Triangles

Learning Objectives

Here you will learn theorems about triangles and how to prove them.

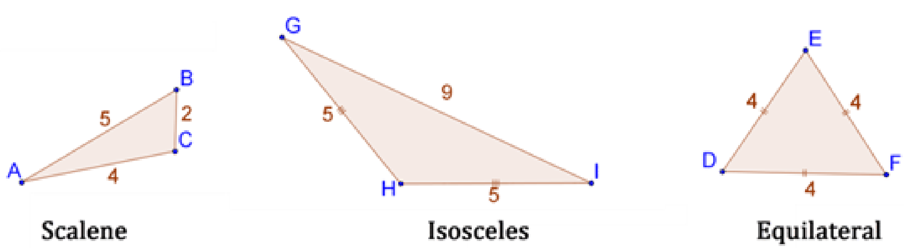
In the triangle below, point D is the midpoint of $\overline{AC}$ and point E is the midpoint of $\overline{BC}$. Make a conjecture about how $\overline{DE}$ relates to $\overline{AB}$.



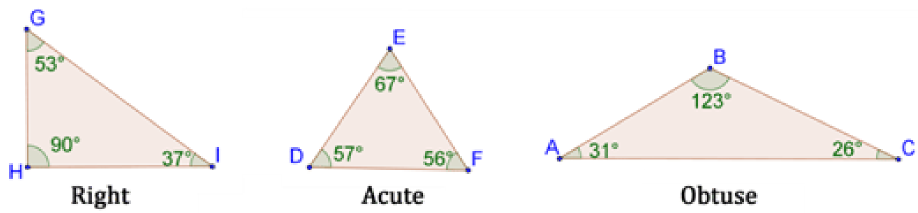
Theorems about Triangles

Recall that a **triangle** is a shape with exactly three sides. Triangles can be classified by their sides and by their angles.

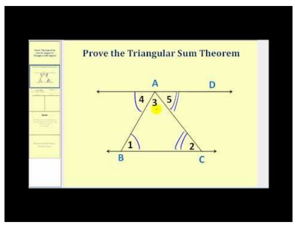
When classifying a triangle by its sides, you should look to see if any of the sides are the same length. If no sides are the same length, then it is a **scalene triangle**. If two sides are the same length, then it is an **isosceles triangle**. If all three sides are the same length, then it is an **equilateral triangle.**



When classifying a triangle by its angles, you should look at the size of the angles. If there is a right angle, then it is a **right triangle**. If the measures of all angles are less than 90°, then it is an **acute triangle**. If the measure of one angle is greater than 90°, then it is an **obtuse triangle**. Additionally, if all angles of a triangle are the same, the triangle is **equiangular.**



In the examples and practice, you will learn how to prove many different properties of triangles.



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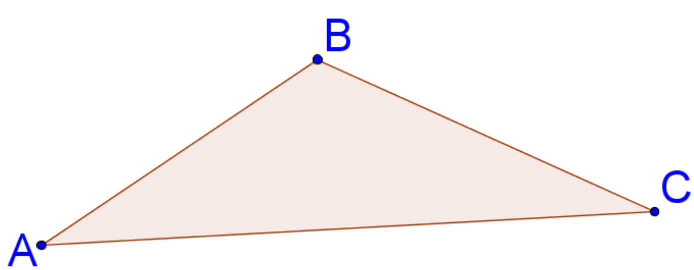
Let's take a look at some example problems.

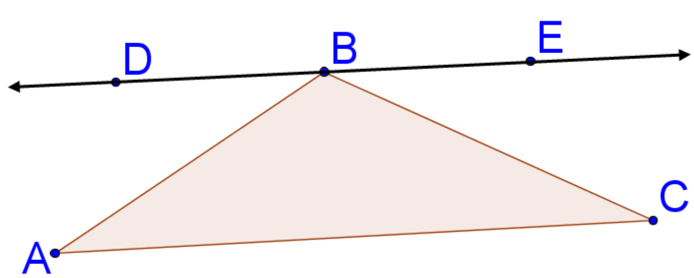
1. Prove that the sum of the interior angles of a triangle is 180°.

This is a property of triangles that you have heard and used before, but you may not have ever seen a proof for why it is true. Here is a proof in the paragraph format, that relies on parallel lines and alternate interior angles.

Consider the generic triangle below.



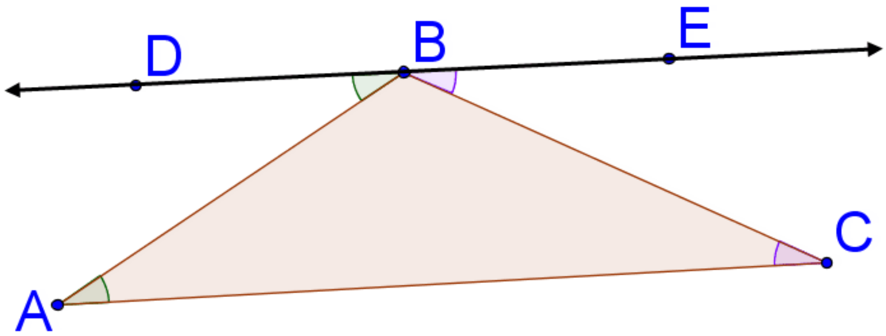
By the parallel postulate, there exists exactly one line parallel to $\overline{AC}$ through B. Draw this line.



$\angle DBA \cong \angle A$ because they are alternate interior angles and alternate interior angles are congruent when lines are parallel. Therefore, $m\angle DBA = m\angle A$. Similarly, $\angle EBC \cong \angle C$ because they are also alternate interior angles, and so

$m\angle EBC = m\angle C$. $m\angle DBA + m\angle ABC + m\angle EBC = 180^\circ$ because these three angles form a straight line. By substitution, $m\angle A + m\angle ABC + m\angle C = 180^\circ$.

The picture below uses color coding to show the angles that are congruent, referenced in the above proof.



The statement "*the sum of the measures of the interior angles of a triangle is* 180°" is a **theorem**. Now that it has been proven, you can use it in future proofs without proving it again.

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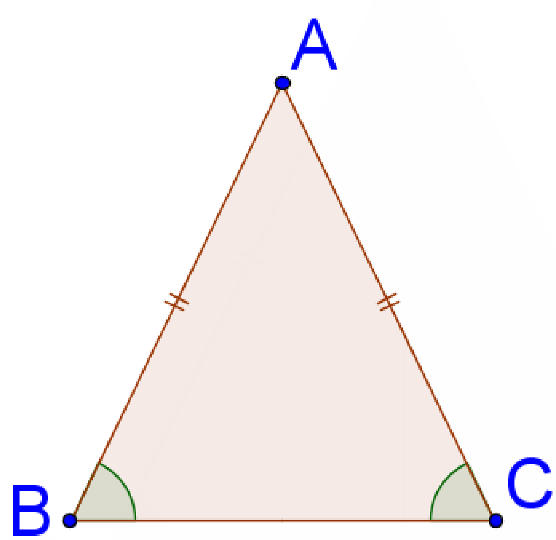
2. Prove that the base angles of an isosceles triangle are congruent.

The base angles of an isosceles triangle are the angles opposite the congruent sides. Below, the base angles are marked for isosceles ΔABC.



Your job is to prove that $\angle B \cong \angle C$ given that $\overline{AB} \cong \overline{AC}$. Here is a proof in the two-column format, that relies on angle bisectors and congruent triangles. The proof will reference the picture below.

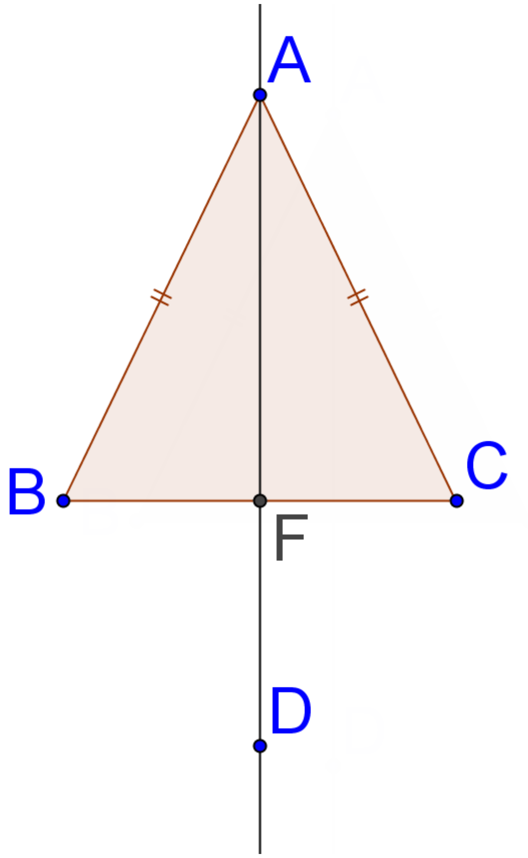


TABLE 4.1:

Statements	**Reasons**
Isosceles ΔABC	Given
$\overline{AB} \cong \overline{AC}$	Definition of isosceles triangle
Construct $\overleftrightarrow{AD}$, the angle bisector of $\angle A$, with F the intersection of $\overline{BC}$ and $\overleftrightarrow{AD}$	An angle has only one angle bisector
$\overline{AF} \cong \overline{AF}$	Reflexive Property
$\angle BAF \cong \angle CAF$	Definition of angle bisector
$\Delta ABF \cong \Delta ACF$	$SAS \cong$
$\angle B \cong \angle C$	CPCTC

The statement "*the base angles of an isosceles triangle are congruent*" is a **theorem**. Now that it has been proven, you can use it in future proofs without proving it again.

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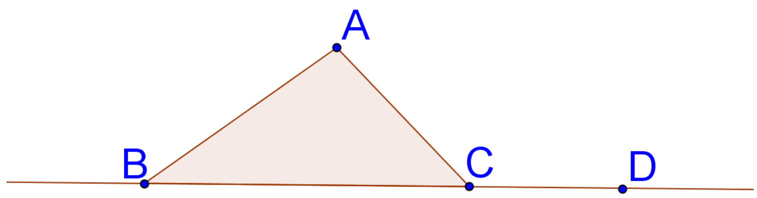
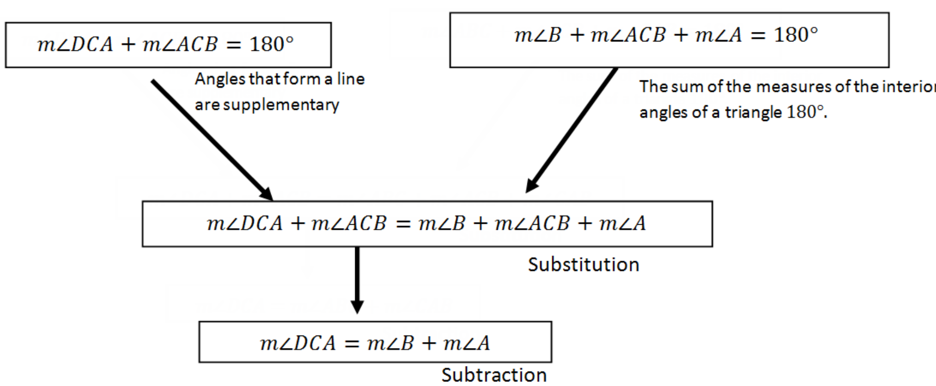
3. Prove that the measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles.

An exterior angle of a triangle is an angle outside of a triangle created by extending one of the sides of the triangles. Below, $\angle ACD$ is an exterior angle. For exterior angle $\angle ACD$, the angles $\angle A$ and $\angle B$ are the **remote interior angles**, because they are the interior angles that are not adjacent to the exterior angle.



Here is a flow diagram proof of this theorem.



The statement *"the measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles*" is a **theorem**. Now that it has been proven, you can use it in future proofs without proving it again.

In the interactive below, move the red points to change the shape of the triangle. Move the blue points to compare angles $\angle 1$, $\angle 2$, and $\angle 3$.

Notice that because they are vertical angles, the angle pairs $m\angle 1 = m\angle 4$, $m\angle 2 = m\angle 5$, and $m\angle 3 = m\angle 6$.

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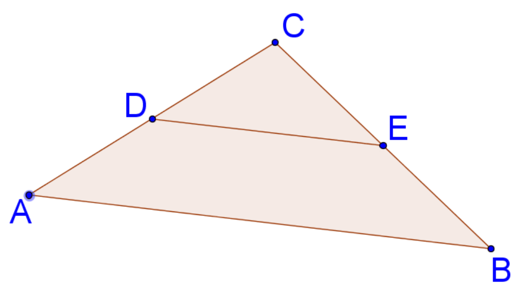
$$m\angle 1 + m\angle 2 + m\angle 3 = 360^\circ$$
$$m\angle 4 + m\angle 5 + m\angle 6 = 360^\circ$$

Example 1

Earlier, you were asked to make a conjecture how $\overline{DE}$ relates to $\overline{AB}$.

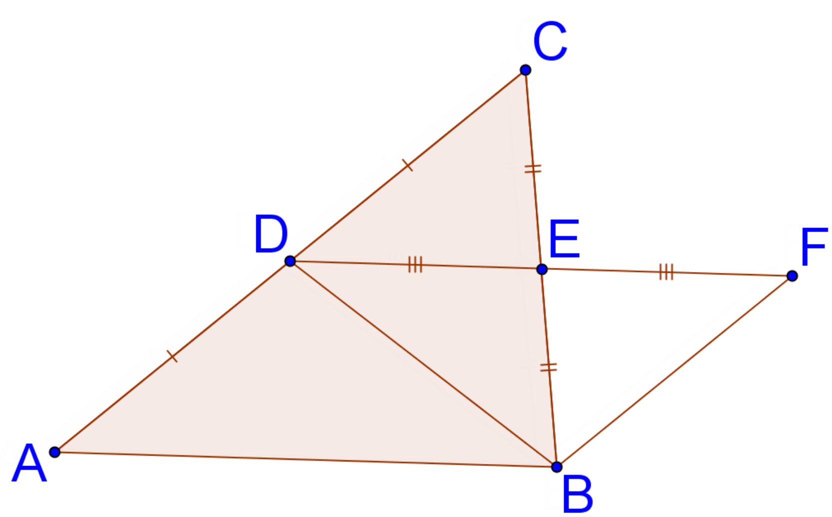


A **conjecture** is a guess about something that might be true. After making a conjecture, usually you will try to prove it. Two possible conjectures are:

1. $\overline{DE} \parallel \overline{AB}$
2. The length of $\overline{DE}$ is half the length of $\overline{AB}$

Both of these conjectures will be proved in the examples and Review questions.

Consider the picture below. The following questions will guide you through proving that $\overline{DE} \parallel \overline{AB}$.



Example 2

Prove that $\Delta FEB \cong \Delta DEC$.

Example 3

Continue your proof from #1 to prove that $\overline{BF} \parallel \overline{AC}$.

Example 4

Continue your proof from #2 to prove that $\Delta ADB \cong \Delta FBD$.

Example 5

Continue your proof from #3 to prove that $\overline{DE} \parallel \overline{AB}$.

TABLE 4.2:

	Statements	**Reasons**
2.	$\overline{DC} \cong \overline{AD}$, $\overline{CE} \cong \overline{EB}$, $\overline{DE} \cong \overline{EF}$	Given
	$\angle CED \cong \angle FEB$	Vertical angles are congruent
	$\Delta FEB \cong \Delta DEC$	$SAS \cong$
3.	$\angle FBE \cong \angle ECD$	CPCTC
	$\overline{BF} \parallel \overline{AC}$	If alternate interior angles are congruent then lines are parallel.
4.	$\angle ADB \cong \angle DBF$	If lines are parallel then alternate interior angles are congruent.
	$\overline{DB} \cong \overline{DB}$	Reflexive property
	$\overline{BF} \cong \overline{DC}$	CPCTC
	$\overline{BF} \cong \overline{AD}$	Substitution
	$\Delta ADB \cong \Delta FBD$	$SAS \cong$
5.	$\angle ABD \cong \angle FDB$	CPCTC
	$\overline{DE} \parallel \overline{AB}$	If alternate interior angles are congruent then lines are parallel.

Note that there are other ways to prove that the two segments are parallel. One method relies on similar triangles, which will be explored in another concept.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/isosceles-triangles/plix/Isosceles-Triangles-5286a89f5aa4137e691c5134

Review

1. In Example A you proved that the sum of the interior angles of a triangle is 180° using a paragraph proof. Now, rewrite this proof in the two-column format.

2. Rewrite the proof from #1 again in the flow diagram format.

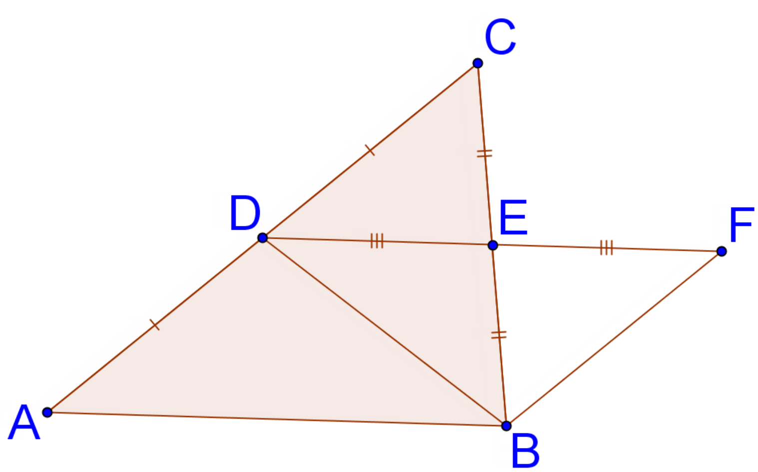
3. In Example B you proved that the base angles of an isosceles triangle are congruent using a two-column proof. Now, rewrite this proof in the paragraph format.

4. Rewrite the proof from #3 again in the flow diagram format.

5. In Example C you proved that the measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles using the flow diagram format. Now, rewrite this proof in the paragraph format.

6. Rewrite the proof from #5 again in the two-column format.

7. In the Guided Practice questions you proved that the segment joining midpoints of two sides of a triangle is parallel to the third side of the triangle. Given the diagram below and that $\Delta ADB \cong \Delta FBD$ as proved in the Guided Practice questions, prove that $DE = \frac{1}{2}AB$.



8. In Example B you proved that "if a triangle is isosceles, the base angles are congruent". What is the converse of this statement? Do you think the converse is also true?

9. Prove that if two angles of a triangle are congruent, then the triangle is isosceles. Use the diagram and two-column proof below and fill in the blanks to complete the proof.

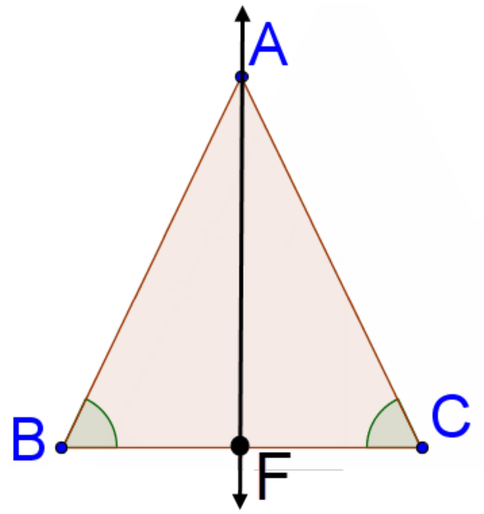
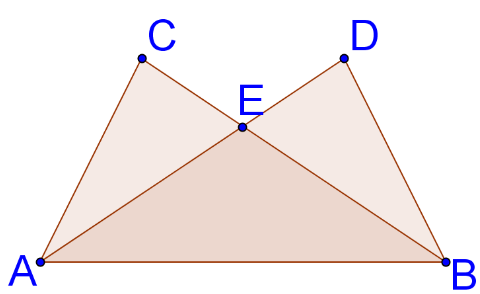


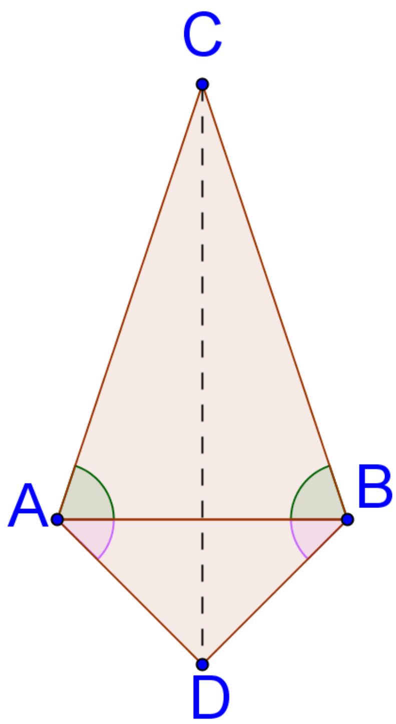
TABLE 4.3:

Statements	Reasons
$\angle B \cong \angle C$	________
Construct $\overleftrightarrow{AF}$, the angle bisector of $\angle A$, with F the intersection of $\overline{BC}$ *and* $\overleftrightarrow{AF}$	An angle has only one angle bisector
________	Definition of angle bisector
________	Reflexive Property
$\Delta ABF \cong \Delta ACF$	________
________	CPCTC

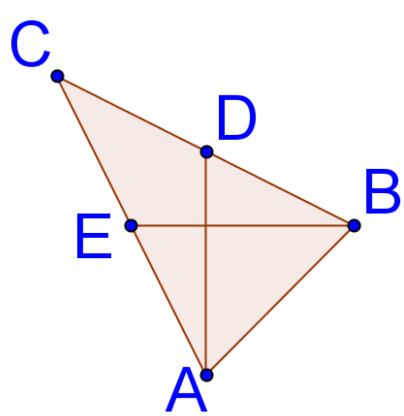
10. Rewrite the proof from #9 in the flow diagram format.
11. Rewrite the proof from #9 in the paragraph format.
12. Given that $\Delta ABC \cong \Delta BAD$, prove that ΔAEB is isosceles.



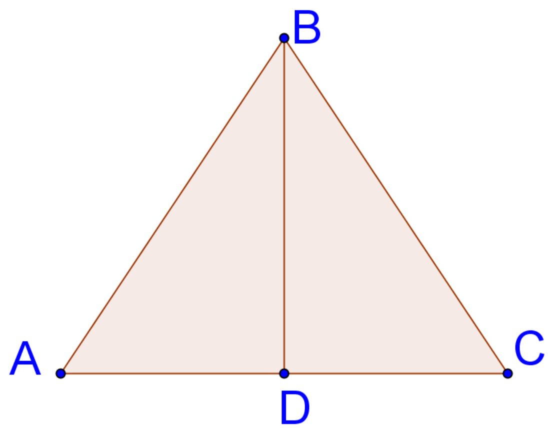
13. Given the markings in the picture below, explain why $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$.



14. In the picture below, ΔABC is isosceles with $\overline{AC} \cong \overline{CB}$. E is the midpoint of $\overline{AC}$ and D is the midpoint of $\overline{CB}$. Prove that $\Delta EAB \cong \Delta DBA$.



15. Explain why knowing that ΔABC is isosceles is not enough information to prove that $\Delta ABD \cong \Delta CBD$.



16 Given: $\angle CBD \cong \angle EFD$; $\overline{CB} \cong \overline{EF}$

Prove: $\angle DBF \cong \angle DFB$

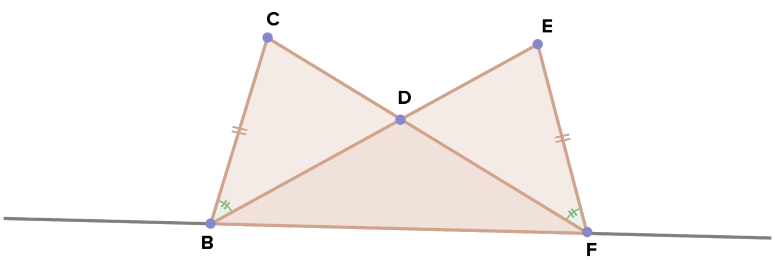


FIGURE 4.21

17. Given: $\overline{BC} \cong \overline{EF}$; $\overline{CF} \cong \overline{EB}$

Prove: ΔBDF is isosceles

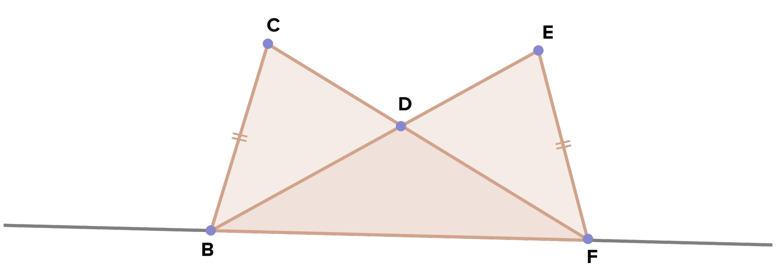


FIGURE 4.22

18. Given: $\overline{DE}$ midsegment of ΔABC ; $\overline{GF} \cong \overline{JI}$; $\overline{FH} \cong \overline{IK}$

Prove: $\Delta GFH \cong \Delta JIK$

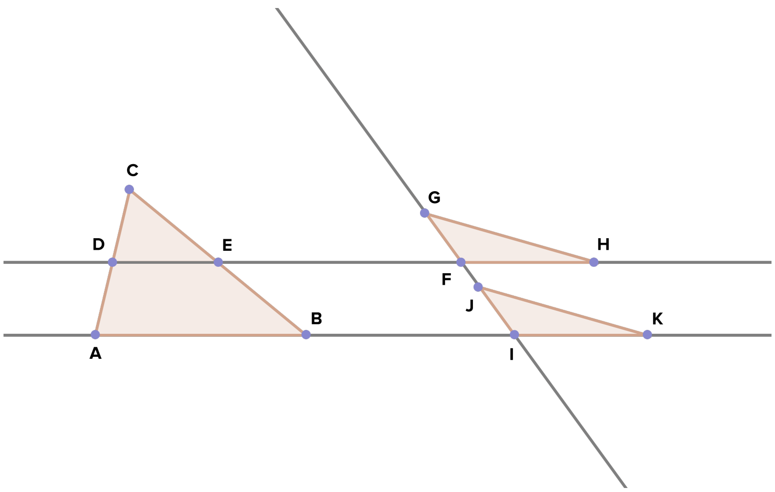


FIGURE 4.23

4.8 Applications of Triangle Theorems

Learning Objectives

Here you will use triangle theorems to solve problems.

Find a piece of cardstock or thick paper. Use a ruler and pencil to draw a fairly large random triangle on the paper. Use your ruler to help you to construct the centroid of the triangle. Carefully cut out the triangle and try to balance it on the tip of your pencil. Where is the balancing point?

Applications of Triangle Theorems

There are nine theorems related to triangles that are helpful to know.

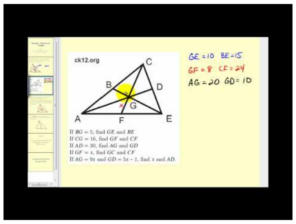
1. The sum of the measures of the interior angles of a triangle is 180°.
2. The base angles of an isosceles triangle are congruent.
3. If a triangle has two congruent angles then it is isosceles. (Note that this is the converse of #2)
4. The measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles.
5. The segment connecting two midpoints of a triangle is both parallel to and one half the length of the third side of the triangle.
6. The three medians of a triangle meet at a point called the centroid. The centroid divides each median in a 2:1 ratio with the larger segment being the one from the vertex to the centroid.
7. The three angle bisectors of a triangle meet at a point called the incenter.
8. The three altitudes of a triangle meet at a point called the orthocenter.
9. The three perpendicular bisectors of a triangle meet at a point called the circumcenter.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/medians/plix/Triangle-Relationships-560182a9da2cfe744f7b7dcf



MEDIA

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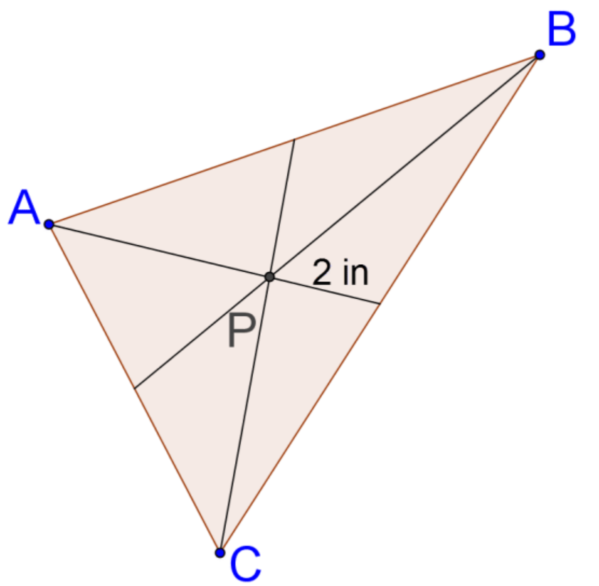
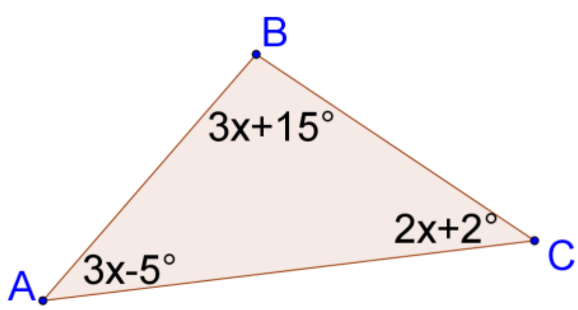
Let's look at some problems that involve triangle theorems.

1. Point P is the centroid of ΔABC. Find the length of $\overline{AP}$.



Because P is the centroid, it divides each median in a 2:1 ratio, where the length of the segment from the vertex to the centroid is the longer segment. This means that $\overline{AP}$ is twice the length of the segment that is marked as 2 *in*. Therefore, $AP = 4$.

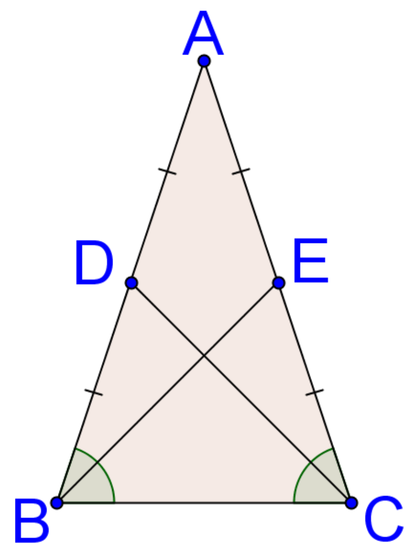
2. Solve for x.



The sum of the measures of the angles is 180°. Set up an equation and solve for x.

3. Prove that two of the medians of an isosceles triangle are congruent.

You can complete this proof using the triangle below. Your goal is to prove that $\Delta DBC \cong \Delta ECB$ and then show that the medians are congruent because they are corresponding parts of the triangles.

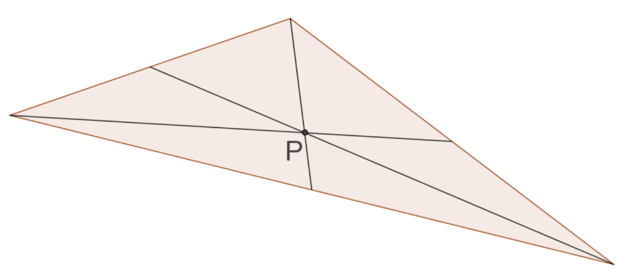


Consider isosceles ΔABC with $\overline{AB} \cong \overline{AC}$. Midpoints D and E divide $\overline{AB}$ and $\overline{AC}$ respectively, so $\overline{AD} \cong \overline{DB} \cong \overline{AE} \cong \overline{EC}$. Because it is an isosceles triangle, $\angle ABC \cong \angle ACB$. $\overline{BC} \cong \overline{BC}$ because any segment is congruent to itself. Therefore, $\Delta DBC \cong \Delta ECB$ by $SAS \cong$. $\overline{DC} \cong \overline{EB}$ because corresponding parts of congruent triangles are congruent.

Examples

Example 1

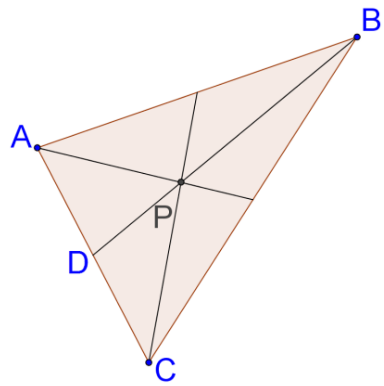
Earlier, you were asked where is the balancing point.



You should find that the centroid is the balancing point of the triangle. This means that the centroid is the center of gravity for the triangle when constructed in real life.

Example 2

Point P is the centroid of ΔABC and $BD = 18$. Find the length of $\overline{BP}$.



The centroid divides the median in a 2:1 ratio where $\overline{BP}$ is the longer segment.

$$\begin{aligned} 2x + 1x &= 18 \\ x &= 6 \\ 2x &= 12 \end{aligned}$$

The two segments have lengths of 6 and 12, so $BP = 12$.

Example 3

Where is the orthocenter of a triangle?

The three altitudes of a triangle meet at a point called the orthocenter.

The altitudes of a triangle are line segments intersecting each vertex and the point on the opposite side where the segment meets the side at a 90-degree angle. In other words, an altitude is a line segment perpendicular to a base and intersecting the opposite vertex.

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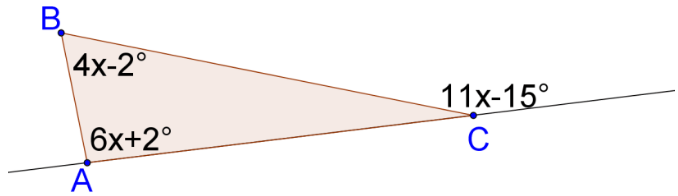
Example 4

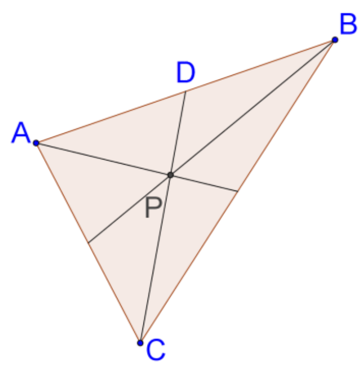
Solve for x.



The measure of the exterior angle is equal to the sum of the measures of the remote interior angles. Set up an equation to solve for x.

Review

1. Point P is the centroid of ΔABC and $CD = 12$. Find the lengths of $\overline{CP}$ and $\overline{DP}$.



2. What are the four points of concurrency for a triangle?

3. The three medians of a triangle meet at a point called the _____________.

4. The three angle bisectors of a triangle meet at a point called the ______________.

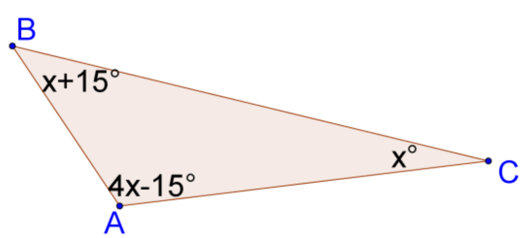
5. The three perpendicular bisectors of a triangle meet at a point called the ______________.

6. Investigate which points of concurrency are always **inside** a triangle and which points of concurrency are sometimes **outside** a triangle (use geometry software to help). What did you find out?

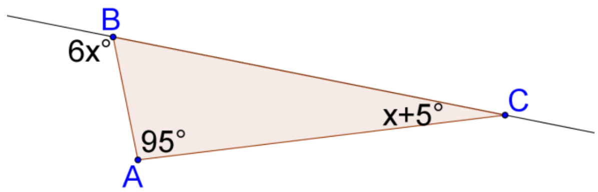
7. Explore the points of concurrency for an equilateral triangle (use geometry software to help). What do you notice?

Solve for x.

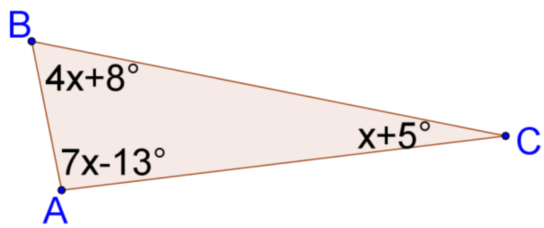
8.



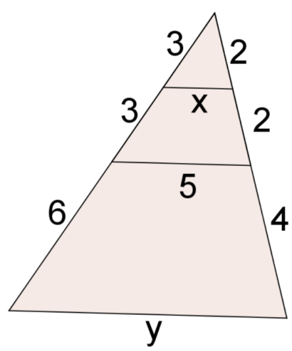
9.



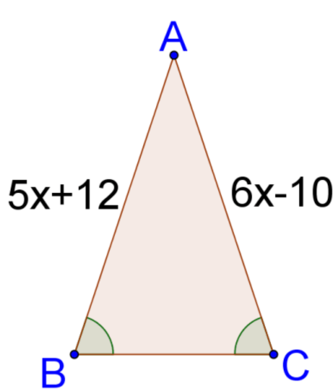
10.



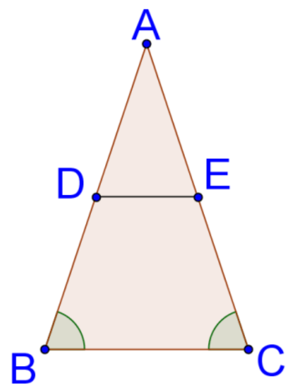
11. Solve for x and y.



12. Solve for x.



In the triangle below, $BC = x$, $AE = x - 2$, D is the midpoint of $\overline{AB}$, and E is the midpoint of $\overline{AC}$, and the perimeter of the triangle is 42.



13. Solve for x.

14. Find DE, BC, AE, and AD.

15. A kite has diagonals with lengths 4 and 6. An inner quadrilateral is formed by joining the midpoints of each of the four sides of the kite. What is the perimeter of this inner quadrilateral?

16. Given: E, F, G, and H are midpoints of their respective sides

Prove: EFGH is a parallelogram

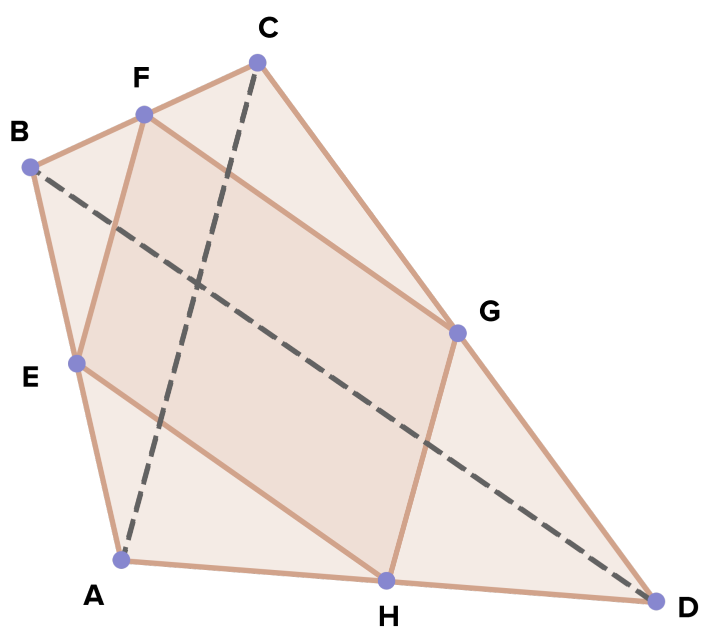


FIGURE 4.24

17. The centroid is the center of mass of the triangle, its balance point. To balance a metal rod horizontally on one's finger, where is the finger placed? How about the center of mass for a rectangle? How would you locate it? What

assumptions are being made about the density of the material of the rectangle? Make some sketches of segments and rectangles, and discuss the calculation of the center of mass for these figures.

18. Now, construct a triangle in the coordinate plane such that the coordinates of the vertices are integers. This can be constructed by hand or with interactive geometry software. Construct the centroid. Estimate the coordinates of the centroid. Now, calculate the average of x-values of the vertices of the triangle, and the average of the y-values. What do you observe? Why is this the case?

19. Think about how to find the center of mass for any quadrilateral. Experiment and make some conjectures.

20. Construct the angle bisectors for any quadrilateral. Are they concurrent? Connect their intersection points to create a smaller quadrilateral called a **cyclic quadrilateral**. Now construct the perpendicular bisectors of the sides of the smaller quadrilateral. Are they concurrent? Construct a circle using this point as the center and one of the points on the cyclic quadrilateral. What do you observe? Can any quadrilateral be circumscribed? Explain why or why not.

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.6.

4.9 Theorems about Concurrence in Triangles

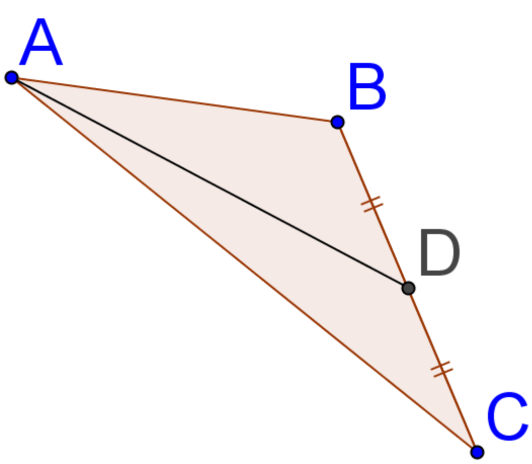
Learning Objectives

Here you will learn about points of concurrency in triangles.

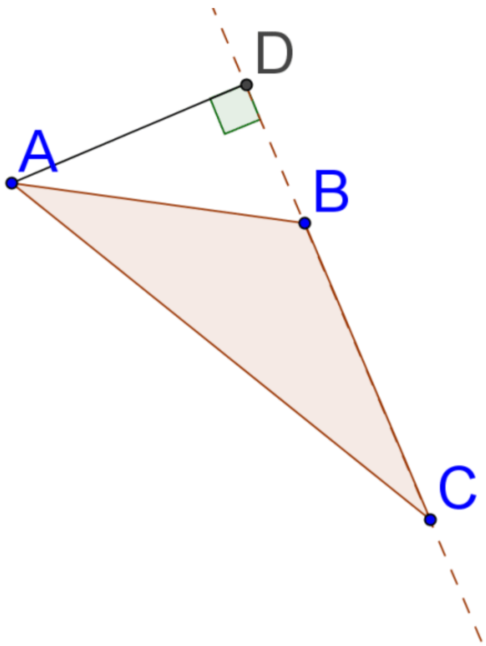
For any triangle, the perpendicular bisectors of the three sides of the triangle meet at one point called the **circumcenter**. Why is this point called the circumcenter and what does it have to do with circles?

Theorems about Concurrence in Triangles

The **median** of a triangle is a line segment that connects the **midpoint** of one side of the triangle with the opposite **vertex**. In the triangle below, $\overline{AD}$ is a median. Something interesting happens when you consider the three medians of a triangle. (See Guided Practice #1-#3)



The **altitude** of a triangle is a line that is **perpendicular** to one side of the triangle and passes through the opposite **vertex**. The altitude is always a height of the triangle. Sometimes the altitude is outside the triangle. Below, $\overline{AD}$ is an altitude. Something interesting happens when you consider the three altitudes of a triangle. (See Practice #6-#9)



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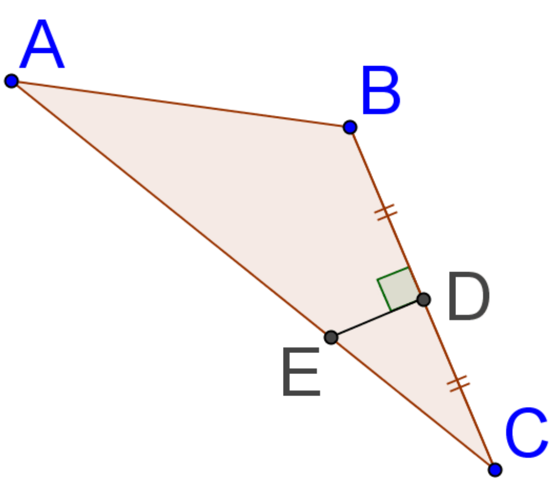
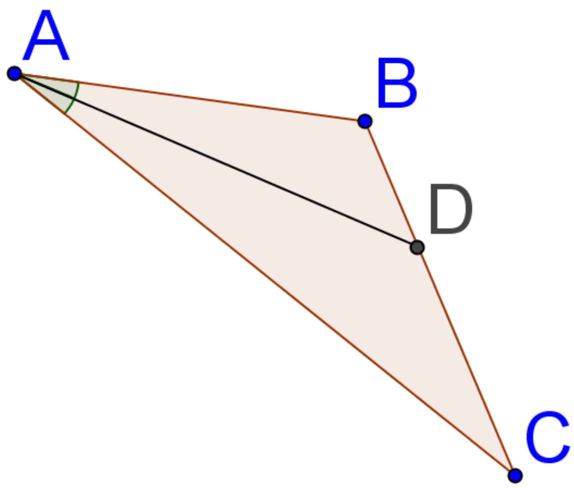
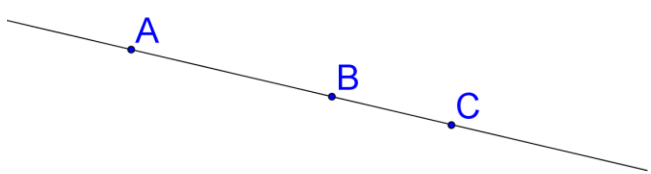
Recall that a **perpendicular bisector** of a segment is a line that bisects the segment and meets the segment at a right angle. Below, $\overline{ED}$ is a perpendicular bisector of $\overline{BC}$. Something interesting happens when you consider the perpendicular bisectors of each segment of a triangle. (See Examples A, B, C)



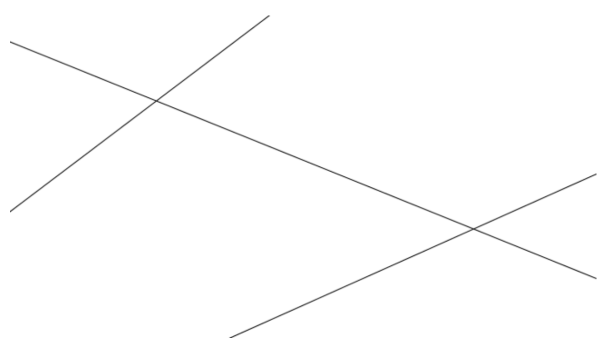
Also recall that an **angle bisector** of an angle is a line that bisects the angle. This means that the line divides the angle into two congruent angles. Something interesting happens when you consider the angle bisectors of each angle of a triangle. (See Practice #1-#5)



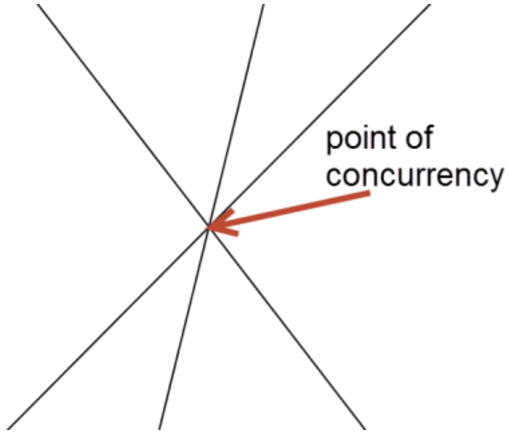
Given three points, it is unlikely that the same line will pass through all three points. ***If*** three points do lie on the same line, they are called collinear.

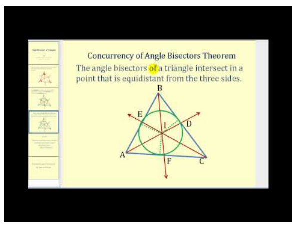


Similarly, given three random lines, it is unlikely that all three lines will intersect at the same point.



When three lines ***do*** intersect at a point, the point is called a **point of concurrency**. In the examples and practice, you will investigate theorems about different points of concurrency for triangles.





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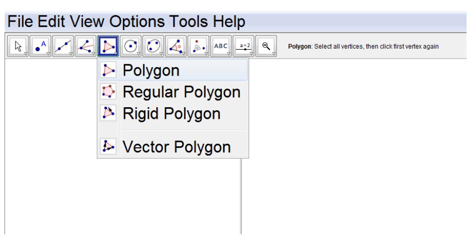
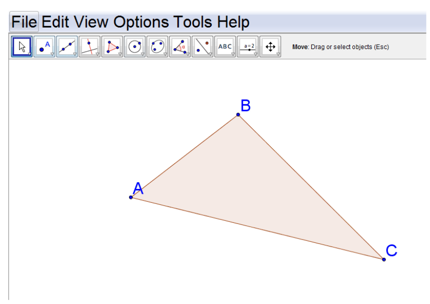
Using Software

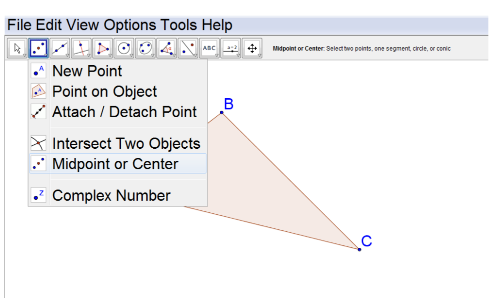
Use interactive geometry software to construct a triangle and its three perpendicular bisectors. Make a conjecture about the perpendicular bisectors of a triangle.

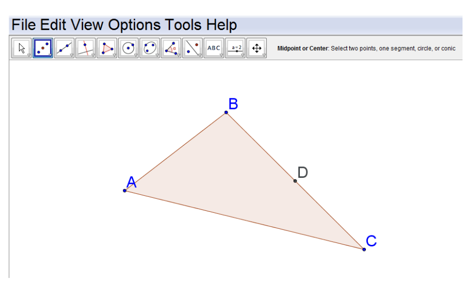
Start by constructing a triangle:



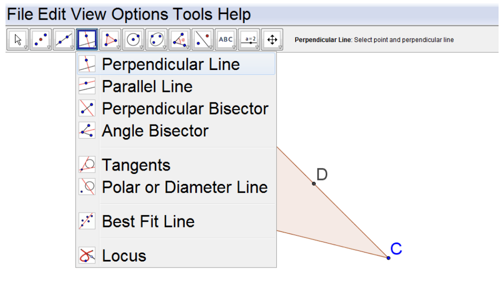


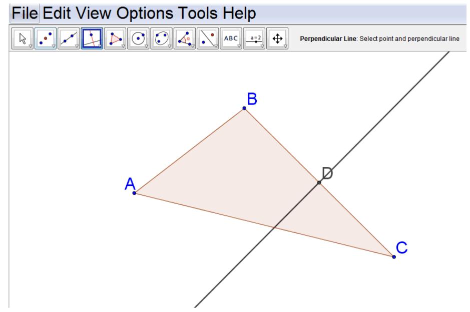
Next construct the midpoint of one side of the triangle:



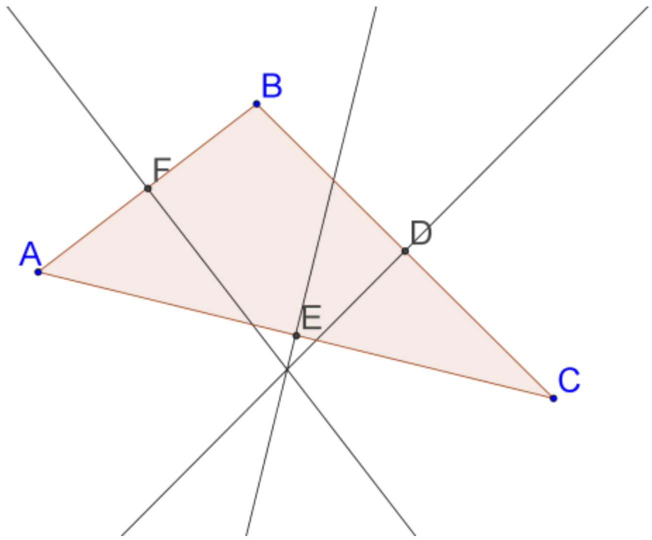


Next construct a line perpendicular to that side of the triangle that passes through the midpoint:





Repeat for the other two sides of the triangle:



You should notice that the three perpendicular bisectors meet at a point. Also notice that if you move the points that define the triangle, the perpendicular bisectors remain perpendicular bisectors, and they continue to meet at a point. This point of concurrency is called the **circumcenter** of a triangle.

In the interactive below, you can attempt the construction yourself.

MEDIA

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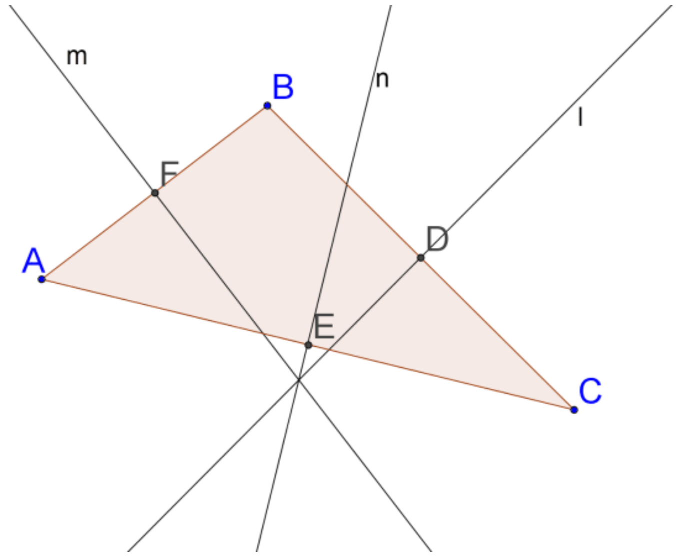
URL: http://www.ck12.org/flx/render/embeddedobject/237091

Prove that the three perpendicular bisectors of any triangle meet at a point.

Consider ΔABC with perpendicular bisectors m, n and l.



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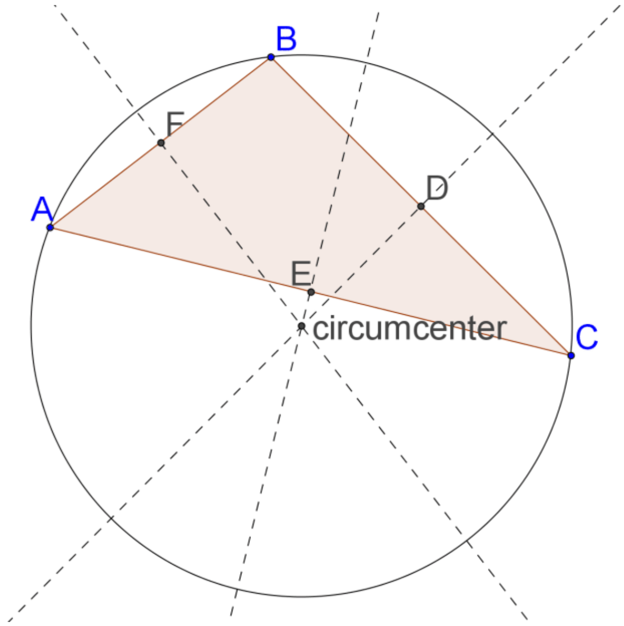
Lines m and n intersect at a point. This point is equidistant from points A and B because it is on line m, the perpendicular bisector of $\overline{AB}$. This point is also equidistant from points A and C because it is on line n, the perpendicular bisector of $\overline{AC}$. Therefore, the point of intersection is equidistant from B and C, and so must lie on line l, the perpendicular bisector of $\overline{BC}$. Line l intersects lines m and n at the same point, so the three perpendicular bisectors meet at one point.

The statement "*the three perpendicular bisectors of any triangle meet at a point*" is a **theorem**. Now that it has been proven, you can use it in future proofs without proving it again.

Investigating the Circumcenter of a Triangle

Use interactive geometry software to investigate the circumcenter of a triangle. In particular, construct a circle with the circumcenter as its center. Can you find any interesting circles?

If you construct a circle with the circumcenter as its center that passes through vertex A, you should notice that the circle **also** passes through vertices B and C. Does this seem surprising?



For any triangle, there exists only one circle that passes through all three of its vertices. This circle is said to ***circumscribe*** the triangle, and the circumcenter is always the center of this circle.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/perpendicular-bisectors/plix/Circumcenter-561bd9778e0e0865e7c90b84

Example 1

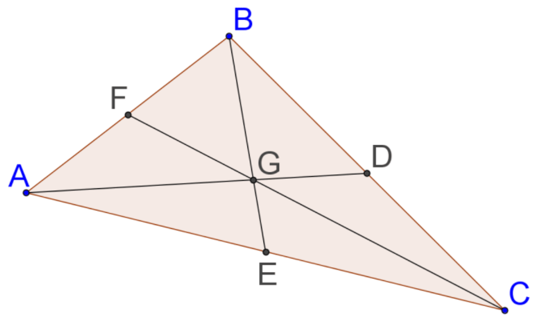
Earlier, you were asked why is a point on a triangle called a circumcenter, and what does it have to do with circles.

As shown in the examples, the circumcenter is not only the intersection of the three perpendicular bisectors, but it is also the ***center*** of the circle that ***circum***scribes the triangle. This is why it is called the ***circumcenter***.

Example 2

Use interactive geometry software to construct a triangle and its three medians. Move the triangle around and see what happens. Make a conjecture about the medians of a triangle.

Start by constructing a triangle and the midpoints of the three sides (see Example A for help). Then, construct line segments connecting each midpoint to the opposite vertex.



You should notice that the three medians meet at a point, even if you move vertices A, B, or C. As the centroid, the point of intersection divides each median in a 2:1 ratio.

$AG = \frac{2}{3}AD$, $BG = \frac{2}{3}BE$, and $CG = \frac{2}{3}CF$.

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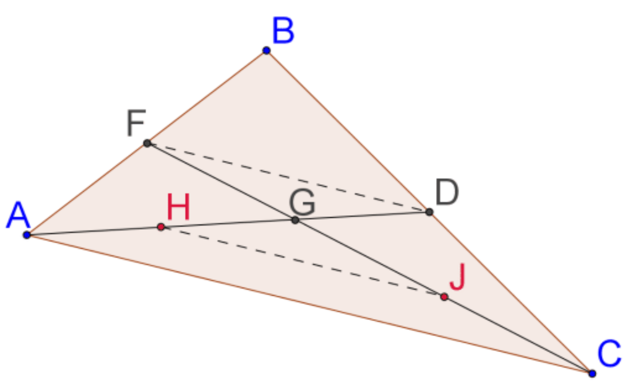
Prove that the point of intersection for two medians of a triangle divides each median into two segments in a 2:1 ratio.

Consider two medians ($\overline{FC}$ and $\overline{AD}$) and their point of intersection (G), as shown below. Also shown below is point H, the midpoint of $\overline{AG}$ and point J, the midpoint of $\overline{GC}$.



$\overline{FD}$ is a segment that connects the midpoints of a triangle. This means it is half the length of $\overline{AC}$ and parallel to $\overline{AC}$. Therefore, $FD = \frac{1}{2}AC$ and $\overline{FD} || \overline{AC}$.

Similarly, $\overline{HJ}$ is a segment that connects the midpoints of a triangle (ΔAGC), so it is also half the length of $\overline{AC}$ and parallel to $\overline{AC}$. Therefore, $HJ = \frac{1}{2}AC$ and $\overline{HJ} || \overline{AC}$.

By substitution, $HJ = FD$ and thus $\overline{HJ} \cong \overline{FD}$. Also, $\overline{HJ} || \overline{FD}$ because both lines are parallel to $\overline{AC}$. This means that $\angle GHJ \cong \angle GDF$ and $\angle GJH \cong \angle GFD$ because they are alternate interior angles, which are congruent when lines are parallel.

Because two pairs of angles and a pair of included sides is congruent, $\Delta GHJ \cong \Delta GDF$ by $ASA \cong$. Therefore, $\overline{HG} \cong \overline{GD}$ and $\overline{GJ} \cong \overline{GF}$ because they are corresponding parts of congruent triangles.

H is the midpoint of $\overline{AG}$, so $\overline{AH} \cong \overline{HG}$. This means the ratio of $AG : GD$ must be 2:1. Similarly, J is the midpoint of $\overline{CF}$, so $\overline{CJ} \cong \overline{JG}$. The ratio of $CG : GF$ must also be 2:1.

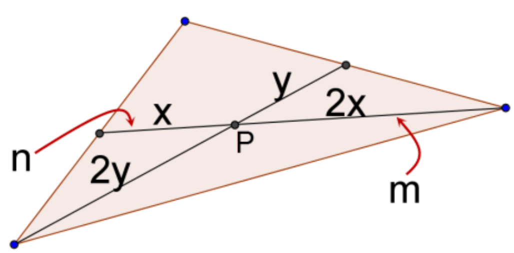
Note that the statement *"the medians of a triangle intersect at a point that divides each median in a 2:1 ratio"* is a **theorem.**

Example 4

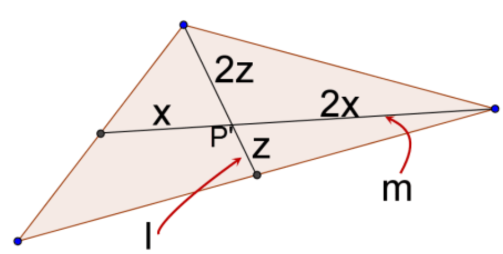
Use #3 to help explain why the three medians of any triangle meet at a point.

By investigation, you can see that the point of intersection of any two medians is in a 2:1 ratio such that the **segment from the vertex to the point of intersection** is the **longer** segment.

Now, consider a triangle with two medians, m and n, constructed. Those medians meet at a point P that divides each median in a 2:1 ratio.



Consider the same triangle with medians m and l constructed. Those medians must also meet at a point P' that divides each median in a 2:1 ratio.

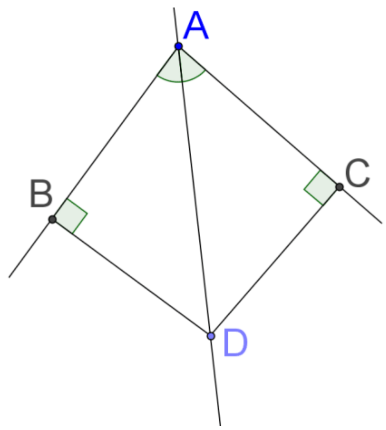


P must be the same as P', because both points divide median m in the same 2:1 ratio. Therefore, all three medians must meet at the same point P.

Note that the statement "*the three medians of a triangle intersect at a point*" is a **theorem**.

Review

1. Prove the **theorem** "*any point on the angle bisector of an angle is equidistant from the two rays that define the angle*", by proving that $\overline{BD} \cong \overline{DC}$ in the diagram below, where $\overline{AD}$ is the angle bisector of $\angle A$.

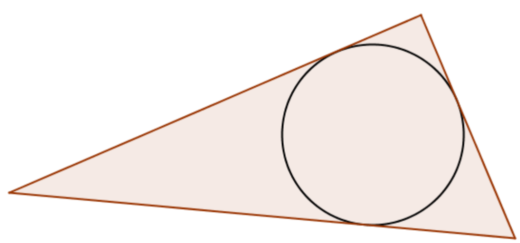


2. Use interactive geometry software to construct a triangle and its three angle bisectors. Move the triangle around and see what happens. Make a conjecture about the angle bisectors of a triangle.

3. Prove that the three angle bisectors of any triangle meet at a point. *Hint: Use Example B to help.*

4. The three angle bisectors of a triangle meet at a point called the **incenter**. Use geometry software to investigate the **incenter**. In particular, construct a circle with the incenter as its center. Can you find any interesting circles?

5. The circle below is inscribed in the triangle. The center of the circle is the incenter of the triangle. Why do you think the **incenter** is called the **incenter**?



6. Use interactive geometry software to construct a triangle and its three altitudes. Move the triangle around and see what happens. Make a conjecture about the altitudes of a triangle.

7. The three altitudes of a triangle always meet at a point called the **orthocenter**. Where is the orthocenter located for right triangles?

8. Where is the **orthocenter** of a triangle located for acute triangles?

9. Where is the **orthocenter** of a triangle located for obtuse triangles?

10. Use interactive geometry software to construct a triangle and its circumcenter, centroid, and orthocenter. *Hint: It can help to hide lines when you do not need them anymore and/or label the points as you construct them.* Move the triangle around and see what happens. Make a conjecture about the relationship between these three points.

11. The circumcenter, centroid, and orthocenter are always collinear, creating a line segment called the **Euler segment**. Are the three points always in the same order on the Euler segment? Make a conjecture.

Continue with your triangle from #10 and #11.

12. Construct the midpoints of the three sides of a triangle.

13. Construct a segment from each vertex to the orthocenter.

14. Construct the midpoints of the three previous segments (the segments connecting the vertices to the orthocenter).

15. Construct the midpoint of the Euler segment.

16. Construct a circle centered at the midpoint of the Euler segment that passes through one of the midpoints of the triangle. What do you notice about this circle?

17. Under what conditions do the various points of concurrency lie within, on, or outside the triangle? Explain.

18. Construct a triangle and its Euler line as previously completed. Now construct a medial triangle, that is, a triangle connecting the midpoints of the sides of the initial triangle. Then construct the Euler line for the medial triangle. Make observations about the relationship between the new Euler line and the original, as well as comparing the circumcenter, centroid, and orthocenter to those of the original.

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.5.

4.10 References

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CHAPTER 5 Triangle Similarity

Chapter Outline

5.1 Definition of Similarity

Learning Objectives

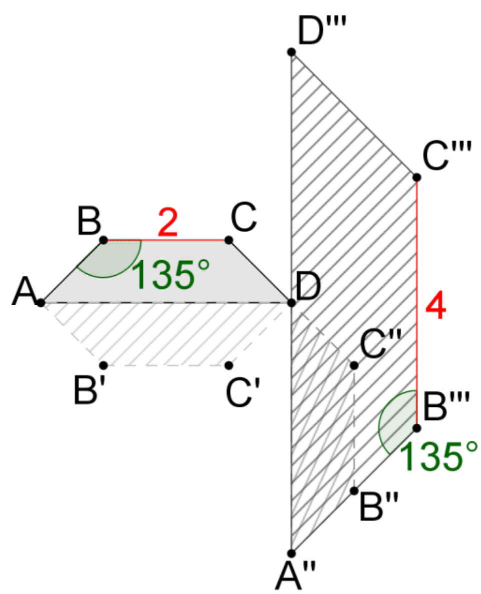
Here you will learn what it means for two figures to be similar.

What does similarity have to do with transformations?

Similarity

A **similarity transformation** is one or more rigid transformations (reflection, rotation, translation) followed by a dilation. When a figure is transformed by a **similarity transformation**, an image is created that is **similar** to the original figure. In other words, two figures are **similar** if a similarity transformation will carry the first figure to the second figure.

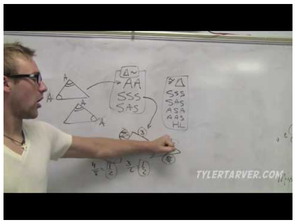
In the picture below, trapezoid $ABCD$ has been reflected, then rotated, and then dilated with a scale factor of 2. The first three trapezoids are all congruent. The final trapezoid is similar to each of the first three trapezoids.



In the trapezoids above, notice that $\angle B \cong \angle B'''$. Also notice that $B'''C''' = 2BC$. In general, similarity transformations preserve angles. Side lengths are enlarged or reduced according to the scale factor of the dilation. This means that **similar figures** will have *corresponding angles that are the same measure and corresponding sides that are proportional.*

In order to determine if two shapes are similar, you can:

1. Carefully describe the sequence of similarity transformations necessary to carry the first figure to the second. AND/OR
2. Verify that all corresponding pairs of angles are congruent and all corresponding pairs of sides are proportional.



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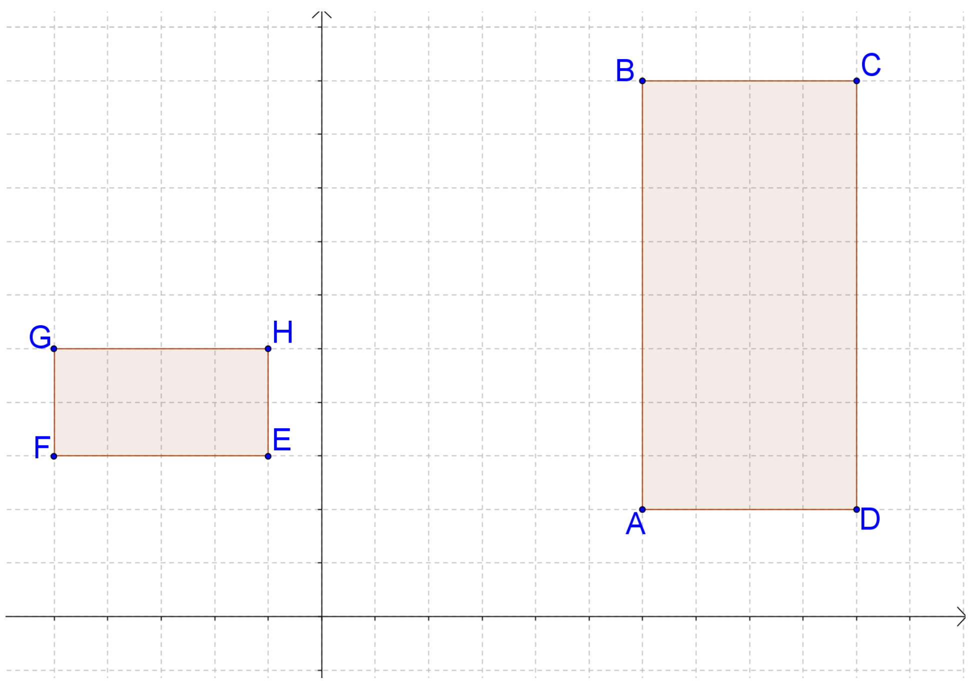
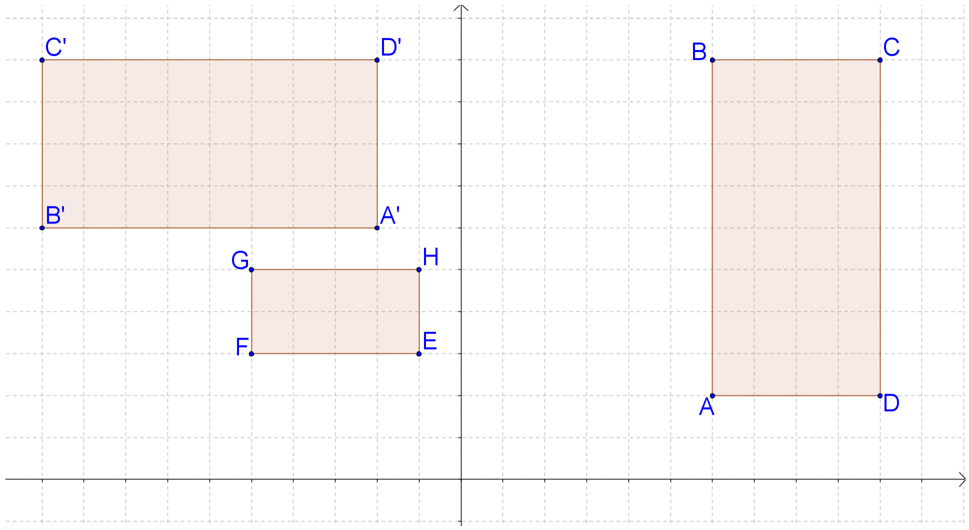
URL: http://www.ck12.org/flx/render/embeddedobject/73901

1. Are the two rectangles similar? Explain.



One way to determine whether the two rectangles are similar is to see if a similarity transformation would carry rectangle $ABCD$ to rectangle $EFGH$. First rotate rectangle $ABCD$ 90° counterclockwise about the origin to create rectangle $A'B'C'D'$. Then dilate rectangle $A'B'C'D'$ about the origin with a scale factor of $\frac{1}{2}$ to create rectangle $EFGH$.



A rotation followed by a dilation is a similarity transformation. Therefore, the two rectangles are similar.

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2. Give another explanation for why the two rectangles from #1 are similar.

Another way to check if two shapes are similar is to verify that all corresponding angles are congruent and all corresponding sides are proportional. Because both shapes are rectangles, all angle measures are 90°. Therefore, all pairs of corresponding angles are congruent. For the sides:

- $\frac{AD}{EH} = \frac{4}{2} = 2$
- $\frac{DC}{HG} = \frac{8}{4} = 2$
- $\frac{CB}{GF} = \frac{4}{2} = 2$
- $\frac{BA}{FE} = \frac{8}{4} = 2$

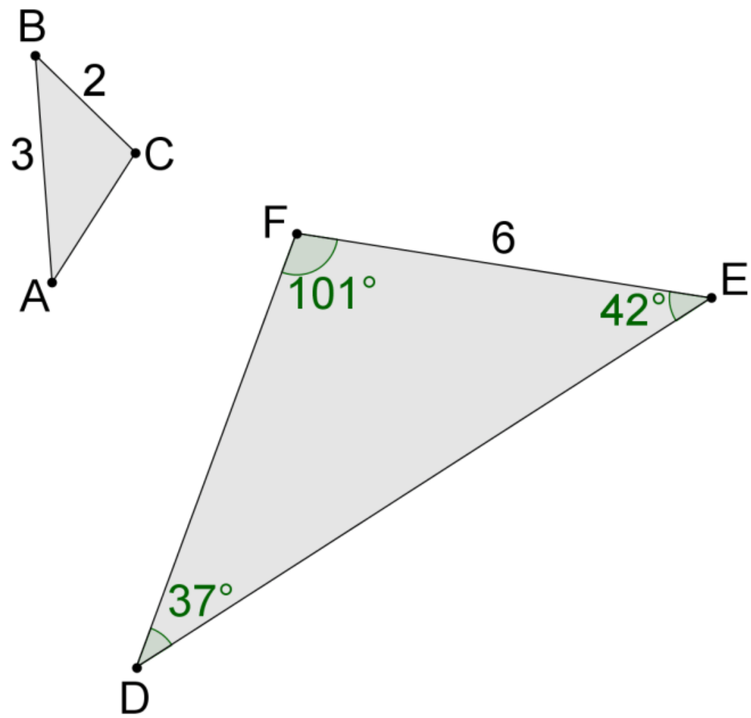
Because all corresponding side lengths are in the same ratio, they are proportional.

All corresponding angles are congruent and all corresponding sides are proportional, so the rectangles are similar.

The symbol for similarity is $\sim$. $\Delta ABC \sim \Delta DEF$ means "triangle ABC is similar to triangle DEF". Just like with congruence statements, the order of the letters matters. A corresponds to D, B corresponds to E and C corresponds to F.

Finding the Length of Sides

The two triangles below are similar with $\Delta ABC \sim \Delta DEF$. What is $m\angle A$? What is the length of $\overline{DE}$?



Because the triangles are similar, corresponding angles are congruent and corresponding sides are proportional.

- $\angle A$ corresponds to $\angle D$. Since $m\angle D = 37°$, $m\angle A = 37°$.
- $\overline{BC}$ corresponds to $\overline{EF}$. $\frac{EF}{BC} = \frac{6}{2} = 3$, so the scale factor is 3. $\overline{DE}$ corresponds to $\overline{AB}$. This means that $\frac{DE}{AB} = 3$. You know that $AB = 3$, so $\frac{DE}{3} = 3$. This means $DE = 9$.

CK-12 PLIX Interactive

PLIX

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URL: http://www.ck12.org/geometry/translation-applications-in-circle-similarity/plix/Are-All-Circles-Similar-56ba27ed9616aa67eb68fd34

Examples

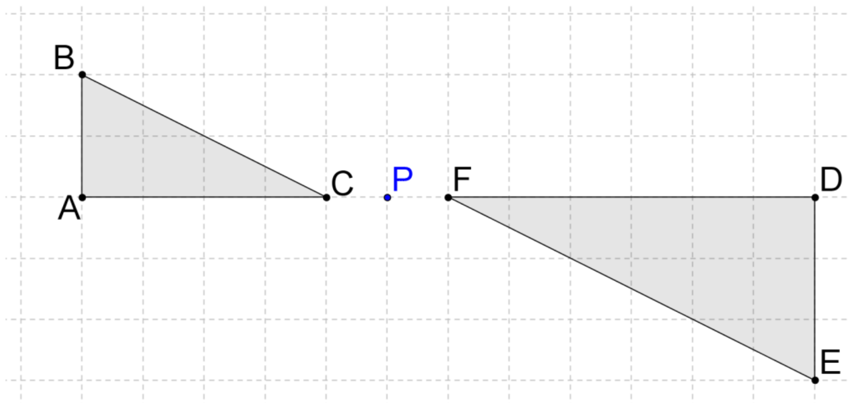
Example 1

Earlier, you were asked what does similarity have to do with transformations?

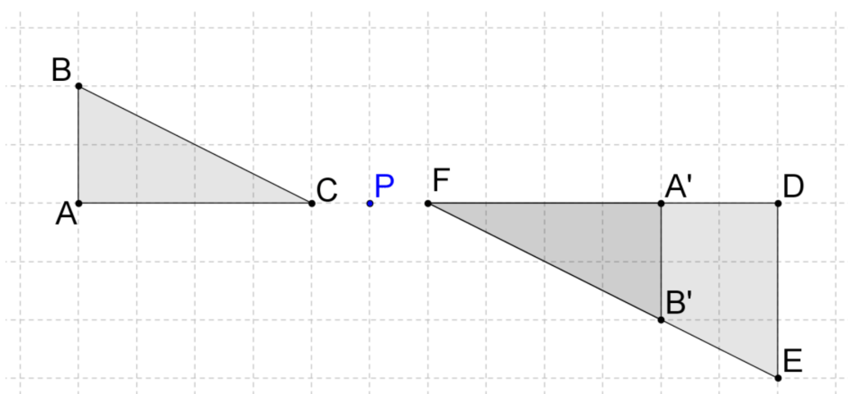
Similarity transformations produce similar figures. You might think of similar figures as "shapes that are the same shape but different sizes", but similar figures can always be linked to rigid motions and dilations as well. If two figures are similar, you will always be able to perform a sequence of rigid motions followed by a dilation on one to create the other.

Example 2

Are the two triangles similar? Explain.



Yes. Rotate ΔABC 180° about point P. Then, dilate about point F with a scale factor of 1.5 to create ΔDEF.



A rotation followed by a dilation is a similarity transformation. Therefore, the two triangles are similar.

Example 3

Use the triangles from #2 to fill in the blanks.

- $\Delta ABC \sim$ _____

- $\angle C \cong$ _____
- $\frac{FD}{CA} = \frac{D}{\_}$

Notice that A corresponds to D, B corresponds to E, and C corresponds to F. This means that $\Delta ABC \sim \Delta DEF$.

$\angle C \cong \angle F$.

$\frac{FD}{CA} = \frac{ED}{BA}$ because $\overline{ED}$ and $\overline{BA}$ are corresponding sides.

Example 4

Suppose that $\Delta DOG \sim \Delta CAT$ with $\frac{AT}{OG} = \frac{1}{3}$, State what you know about the sides and angles of the two triangles.

$\angle D \cong \angle C$, $\angle O \cong \angle A$, $\angle G \cong \angle T$. ΔDOG is the bigger triangle because $\frac{AT}{OG} = \frac{1}{3}$. This means that $DO = 3CA$, $OG = 3AT$, and $DG = 3CT$.

Review

1. If two shapes are similar, must they be congruent? Explain.

2. If two shapes are congruent, must they be similar? Explain.

$\Delta ABC \sim \Delta DEF$. Decide if each statement is true or false and explain your answer.

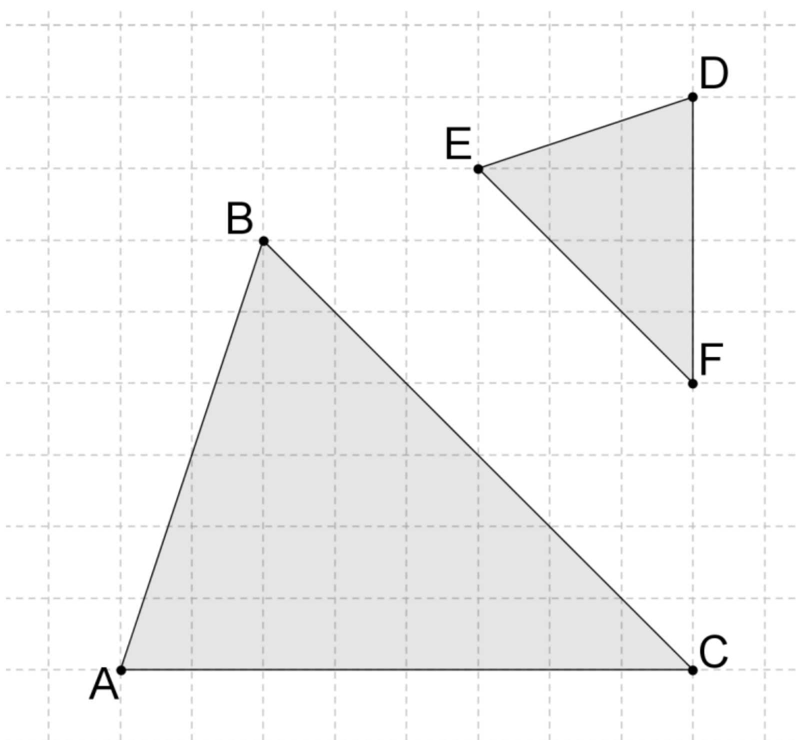
3. $\frac{AB}{DE} = \frac{BC}{EF}$

4. $AC \cdot BC = DF \cdot EF$

5. $\angle B \cong \angle E$

For #6-#9, are the two triangles similar? If so, give a similarity statement and explain how you know. If not, explain.

6.



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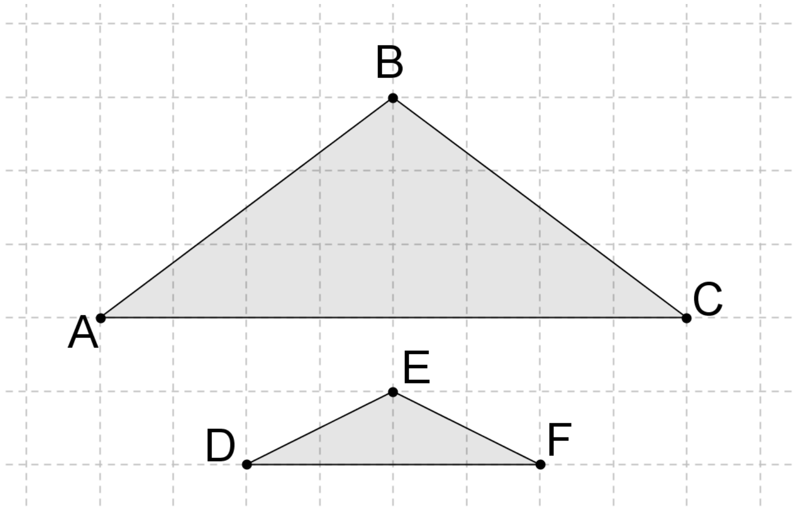

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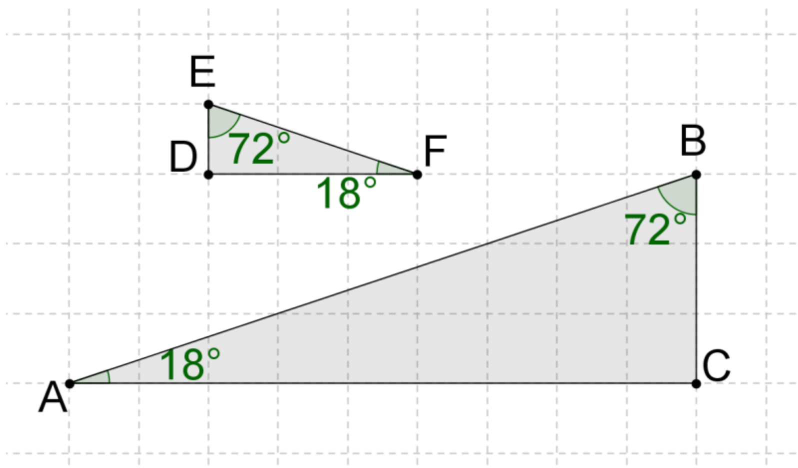

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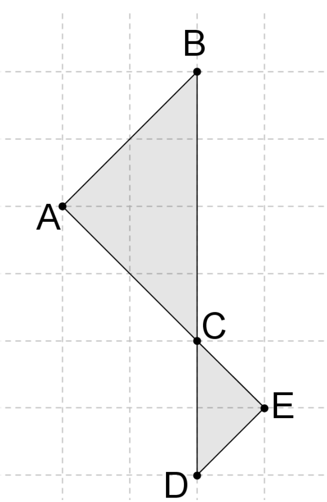
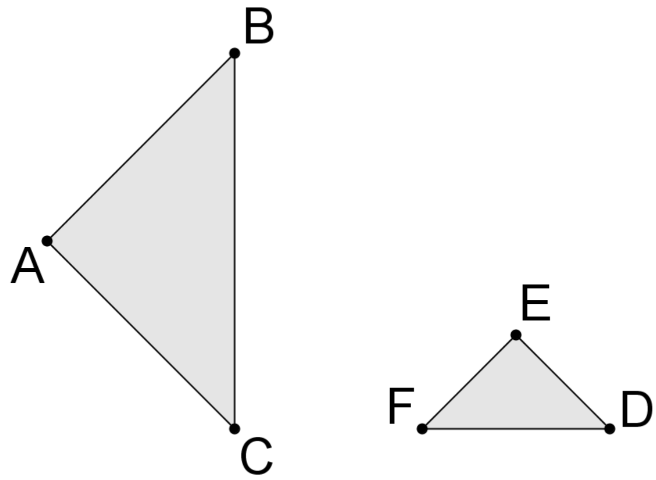


FIGURE 5.1

For #10-#12, $\Delta BAC \sim \Delta DEF$. Note that the triangles below are not drawn to scale.

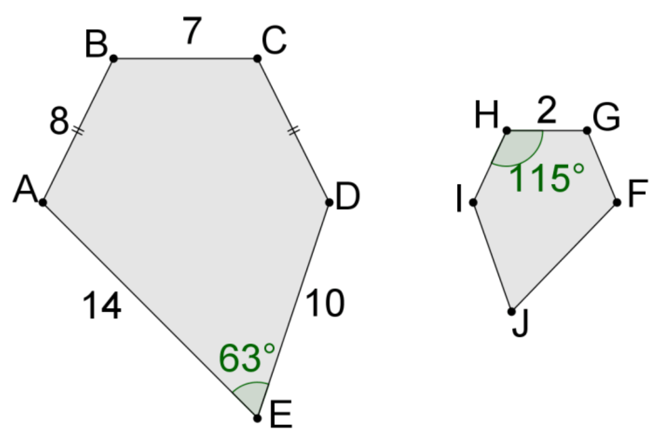


10. If $m\angle A = 85^\circ$ and $m\angle D = 50^\circ$, what is $m\angle F$?

11. If $AB = 5$, $ED = 2$, and $FD = 3$, what is CB?

12. If $AC = 3$, $DE = 1$, and $AC \cong AB$, what is EF?

For #13-#16, $ABCDE \sim FGHIJ$. Note that the pentagons below are not drawn to scale.



13. $GF =$?

14. $JF =$?

15. $ED =$?

16. $m\angle J =$?

17. Explain two ways to determine whether or not two triangles are similar.

18. Assuming the two triangles below are drawn to scale, are they similar? Why or why not? Explain.

19. Assuming the two triangles below are drawn to scale, are they similar? Why or why not? Explain.

20. One triangle has side lengths of measures 3, 4, and 5. Another has side lengths of 6, 8, and 10. The corresponding angles in each triangle are congruent. Sketch the scenario. Are the triangles similar? Why or why not? Now sketch two triangles with side lengths of 4,4,5 and 6,8,10. Describe the triangles, compare their angles and the ratios of their sides. Are these two triangles similar? Why or why not?

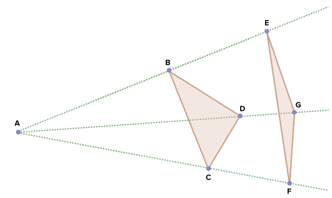


FIGURE 5.2

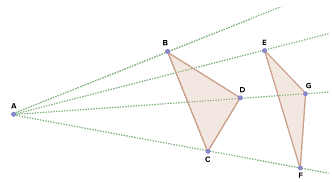


FIGURE 5.3

21. Using interactive geometry software, construct a polygon. Perform the following sequence of transformations in order: a translation, a reflection, a dilation, a rotation, then a second dilation. Is the final image similar to the original? Why or why not? Is there a single dilation that will transform the original into the final image? Why or why not?

22. The triangle below has been dilated by a negative scale factor. What scale factor? What happened? Explain. What sequence of two transformations would also map the original to the image? Are the figures similar? Explain.

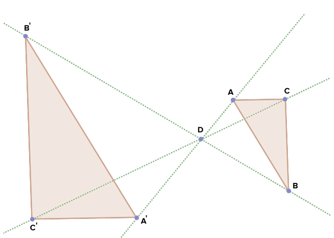


FIGURE 5.4

23. The triangle below has been dilated by a negative scale factor. What scale factor? What happened? Explain. What single rigid motion transformation would also map the original to the image? Are the figures similar? Explain.

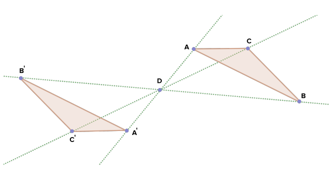


FIGURE 5.5

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.2.

5.2 AA Triangle Similarity

Learning Objectives

Here you will explore the AA criterion for triangle similarity.

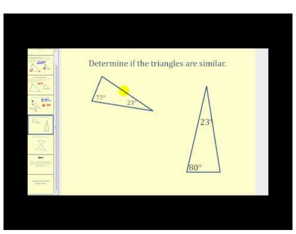
Why don't you have to verify that all three pairs of corresponding angles are congruent in order to show that two triangles are similar?

AA Triangle Similarity

If two triangles are **similar** it means that all corresponding angle pairs are congruent and all corresponding sides are proportional. However, in order to be sure that two triangles are similar, you do not necessarily need to have information about all sides and all angles.

The AA criterion for triangle similarity states that if two triangles have two pairs of congruent angles, then the triangles are similar.

In the examples, you will use similarity transformations and criteria for triangle congruence to show why AA is a criterion for triangle similarity.



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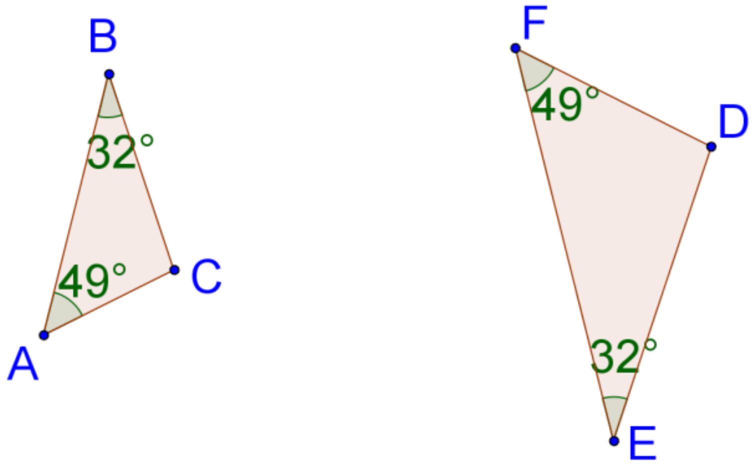
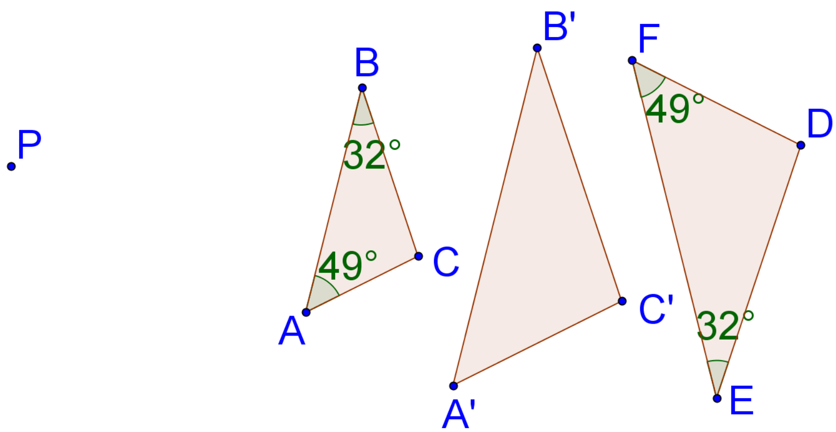
Let's take a look at some problems regarding AA triangle similarity.

1. Consider the triangles below. Dilate ΔABC with a scale factor of $\frac{FE}{AB}$ to create $\Delta A'B'C'$. What do you know about the sides and angles of $\Delta A'B'C'$?



Below, ΔABC is dilated about point P with a scale factor of $\frac{FE}{AB}$ to create $\Delta A'B'C'$.



Corresponding angles are congruent after a dilation is performed, so $\angle B \cong \angle B'$. Therefore, $\angle B' \cong \angle E$ as well. Similarly, $\angle A \cong \angle A'$, and therefore $\angle A' \cong \angle F$. Because the scale factor was $\frac{FE}{AB}$, $A'B' = \frac{FE}{AB} \cdot AB = FE$. So, $\overline{A'B'} \cong \overline{FE}$.

2. Use your work from #1 to prove that $\Delta ABC \sim \Delta FED$.

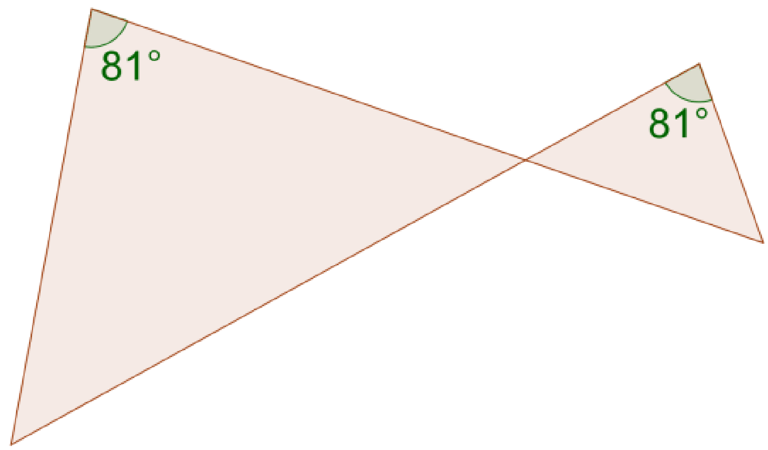
From #1, you know that $\angle B' \cong \angle E$, $\angle A' \cong \angle F$ and $\overline{A'B'} \cong \overline{FE}$. This means that $\Delta A'B'C' \cong \Delta FED$ by $ASA \cong$. Therefore, there must exist a sequence of rigid transformations that will carry ΔFED to $\Delta A'B'C'$.

$\Delta ABC \sim \Delta FED$ because a series of rigid transformations will carry ΔFED to $\Delta A'B'C'$, and then a dilation will carry $\Delta A'B'C'$ to ΔABC.

All that was known about the original two triangles in #1 was two pairs of congruent angles. Therefore, you have proved that AA is a criterion for triangle similarity.

Now, let's take a look at a problem about determining whether two triangles are similar.

Are the triangles below similar? Explain.



One pair of angles is marked as being congruent. You also have another pair of congruent angles due to the vertical angles in the center of the picture. Therefore, the triangles are similar by $AA \sim$.

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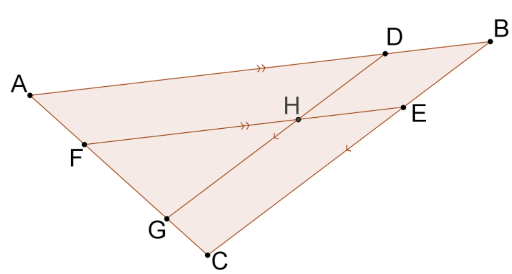
Examples

Example 1

Earlier, you were asked why don't you have to verify that all three pairs of corresponding angles are congruent in order to show that two triangles are similar.

Why don't you have to verify that all three pairs of corresponding angles are congruent in order to show that two triangles are similar?

If two pairs of angles are congruent, then three pairs of angles must be congruent due to the fact that the sum of the measures of the interior angles of a triangle is 180°. Therefore, three pairs of congruent angles is more information than you need.



Example 2

Prove that $\Delta CBA \sim \Delta GDA$.

Note that $\overline{CB} \parallel \overline{GD}$. This means that $\angle AGD \cong \angle ACB$ because they are corresponding angles. $\angle A$ is shared by both triangles. Two pairs of angles are congruent, so $\Delta CBA \sim \Delta GDA$ by $AA \sim$.

Example 3

Prove that $\Delta CBA \sim \Delta CEF$.

Note that $\overline{AB} \parallel \overline{FE}$. This means that $\angle BAC \cong \angle EFC$ because they are corresponding angles. $\angle C$ is shared by both triangles. Two pairs of angles are congruent, so $\Delta CBA \sim \Delta CEF$ by $AA \sim$.

Example 4

Prove that $\Delta CBA \sim \Delta GHF$.

In #1 you found that $\angle AGD \cong \angle ACB$. In #2 you found that $\angle BAC \cong \angle EFC$. Two pairs of angles are congruent, so $\Delta CBA \sim \Delta GHF$ by $AA \sim$.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

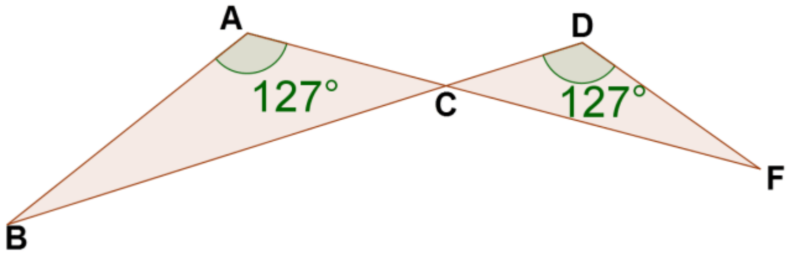
URL: http://www.ck12.org/geometry/aa-similarity/plix/Similar-Triangle-5727aa819616aa625dbe92dc

Review

1. What does AA stand for? How is it used?
2. Draw an example of two triangles that must be similar due to AA.

For each pair of triangles below, state if they are congruent, similar or not enough information. If they are similar or congruent, write a similarity or congruence statement. Explain your answer.

3.



4.

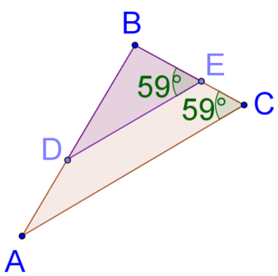
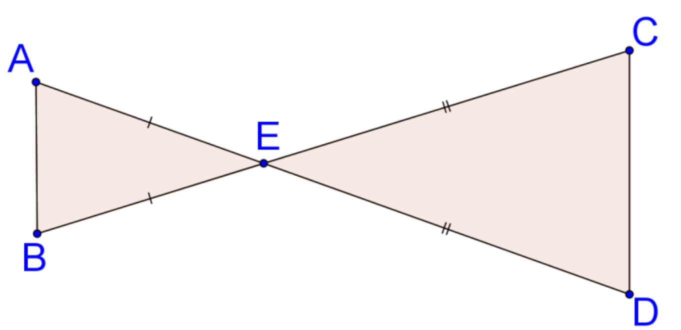


FIGURE 5.6

5.



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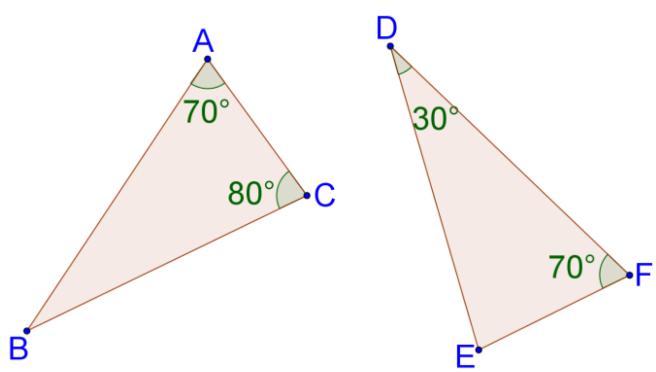

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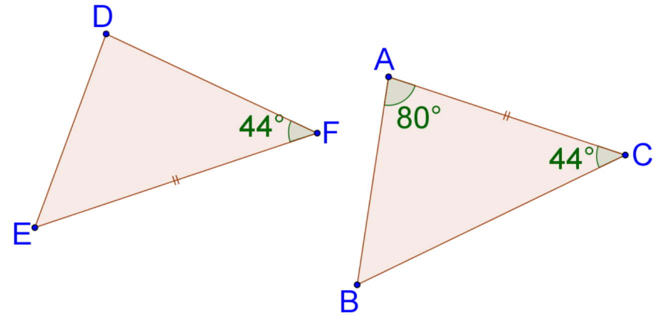

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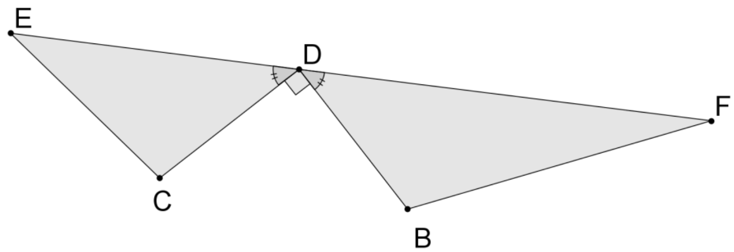

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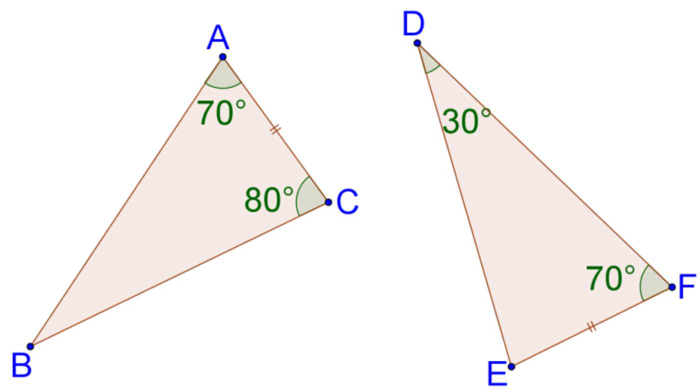

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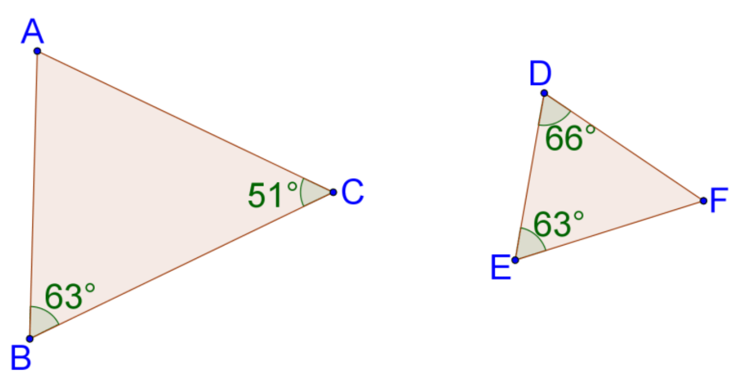


11.

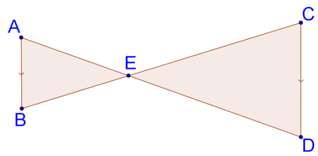


FIGURE 5.7

12.

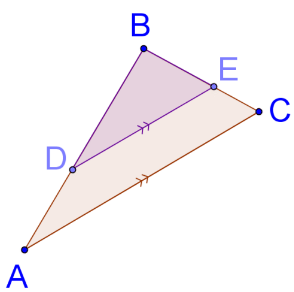
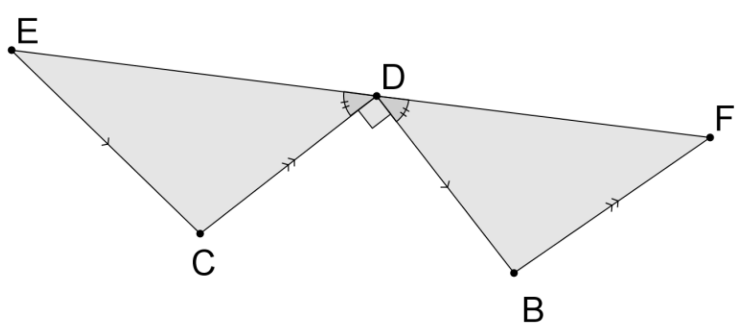


FIGURE 5.8

13.



14. Can you use the AA criteria to show that other shapes besides triangles are similar?

15. Use similarity transformations to explain in your own words why two triangles with two pairs of congruent angles must be similar. *Hint: Look at Examples A and B for help.*

16. Prove that if two triangles have two pairs of congruent angles, the third pair must be congruent.

17. For the two triangles below, corresponding angles are congruent. Is there a sequence of transformations that would map one to the other? Describe it. Are the triangles similar? Explain.

18. Given: $\overline{CF} \parallel \overline{BA}$; $\angle DFJ \cong \angle IBK$

Prove: $\Delta DFJ \cong \Delta IBK$.

19. Given: $\overline{DE} \parallel \overline{BC}$

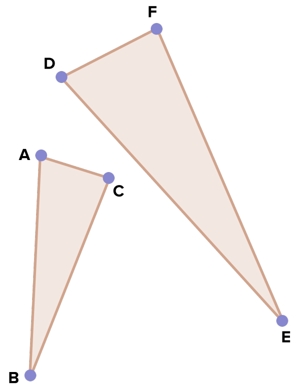


FIGURE 5.9

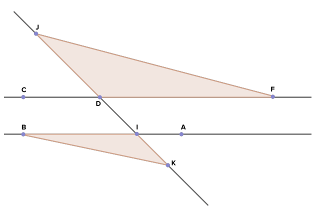


FIGURE 5.10

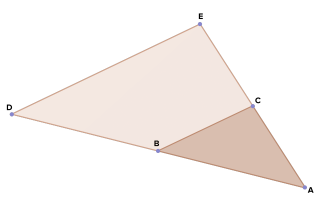


FIGURE 5.11

Prove: $\Delta ABC \cong \Delta ADE$.

20. Prove the three triangles in the diagram below are similar to each other.

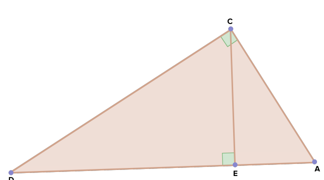


FIGURE 5.12

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.3.

5.3 SAS Triangle Similarity

Learning Objectives

Here you will explore the SAS criterion for triangle similarity.

SAS is a criterion for both triangle similarity and triangle congruence. What's the difference between the two criteria?

SAS Triangle Similarity

If two triangles are **similar** it means that all corresponding angle pairs are congruent and all corresponding sides are proportional. However, in order to be sure that two triangles are similar, you do not necessarily need to have information about all sides and all angles.

The SAS criterion for triangle similarity states that if two sides of one triangle are proportional to two sides of another triangle and their included angles are congruent, then the triangles are similar.

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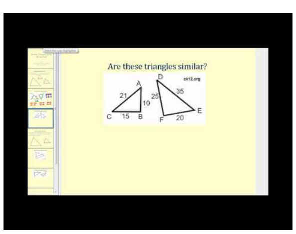
In the examples, you will use similarity transformations and criteria for triangle congruence to show why SAS is a criterion for triangle similarity.



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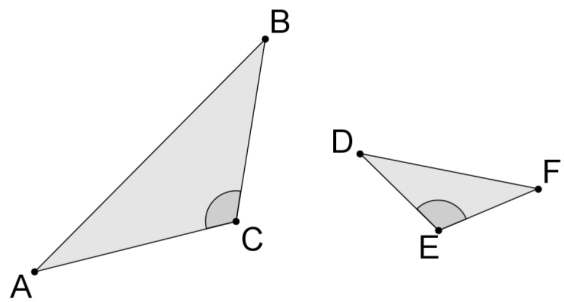
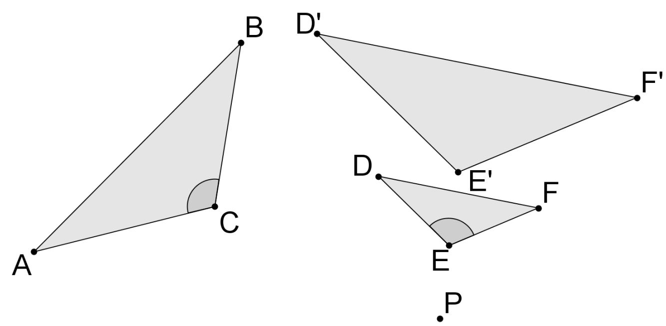
Let's take at a few example problems regarding SAS similarity.

1. Consider the triangles below with $\frac{AC}{DE} = \frac{BC}{FE} = k$ and $\angle C \cong \angle E$. Dilate ΔDEF with a scale factor of k to create $\Delta D'E'F'$. What do you know about the sides and angles of $\Delta D'E'F'$? How do they relate to the sides and angles of ΔABC?



Below, ΔDEF is dilated about point P with a scale factor of k to create $\Delta D'E'F'$.



Corresponding angles are congruent after a dilation is performed, so $\angle E \cong \angle E'$. Therefore, $\angle E' \cong \angle C$ as well. The scale factor was k, which is equal to $\frac{AC}{DE}$ and $\frac{BC}{FE}$. This means:

- $D'E' = k \cdot DE = \frac{AC}{DE} \cdot DE = AC$. Therefore, $\overline{D'E'} \cong \overline{AC}$.
- $F'E' = k \cdot FE = \frac{BC}{FE} \cdot FE = BC$. Therefore, $\overline{F'E'} \cong \overline{BC}$.

2. Use your work from #1 to prove that $\Delta ABC \sim \Delta DFE$.

From #1, you know that $\angle E' \cong \angle C$, $\overline{D'E'} \cong \overline{AC}$ and $\overline{F'E'} \cong \overline{BC}$. This means $\Delta ABC \cong \Delta D'F'E'$ by $SAS \cong$.

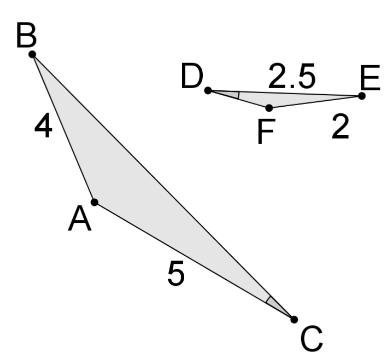
Therefore, there must exist a sequence of rigid transformations that will carry ΔABC to $\Delta D'F'E'$.

$\Delta ABC \sim \Delta DFE$ because a series of rigid transformations will carry ΔABC to $\Delta D'F'E'$, and then a dilation will carry to $\Delta D'F'E'$ to ΔDFE.

All that was known about the original two triangles in #1 was two pairs of proportional sides and included congruent angles. You have proved that SAS is a criterion for triangle similarity.

Now, let's take a look at determining if two triangles are similar.

Are the two triangles below similar? Explain.



First look at what is marked. $\angle C \cong \angle D$. Also, $\frac{AC}{DE} = 2$ and $\frac{AB}{FE} = 2$. Two sides are proportional but the congruent angle is *not* the included angle. This is SSA which is not a way to prove that triangles are similar (just like it is not a way to prove that triangles are congruent).

Look carefully at the two triangles. Notice that the longest side in ΔABC is $\overline{BC}$, which is unmarked. The longest side in ΔEFD appears to be $\overline{DE}$, which is marked. $\overline{BC}$ and $\overline{DF}$ will not be proportional to the other pairs of sides.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/sas-similarity/plix/Corresponding-Sides-of-Similar-Triangles-56fe9f418e0e083a7759e67e

Examples

Example 1

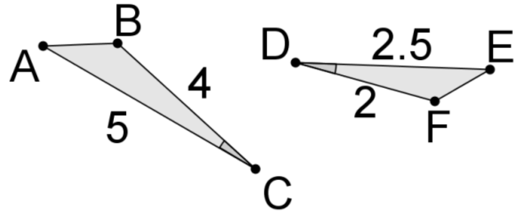
Earlier, you were asked what is the difference between triangle similarity and triangle congruence.

SAS is a criterion for both triangle similarity and triangle congruence. What's the difference between the two criteria?

With $SAS \cong$, you must show that two pairs of sides are **congruent** and their included angles are congruent as well. With $SAS \sim$, you must show that two pairs of sides are **proportional** and their included angles are congruent. Two triangles that are similar by $SAS \sim$ with a scale factor of 1 will be congruent.

Example 2

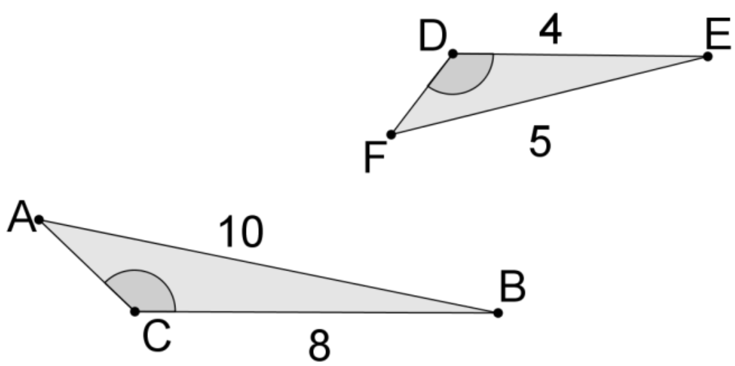
Are the triangles similar? Explain.



$\frac{AC}{ED} = \frac{BC}{FD} = 2$. $\angle C \cong \angle D$ so the included angles are congruent. Therefore, the triangles are similar by $SAS \sim$.

Example 3

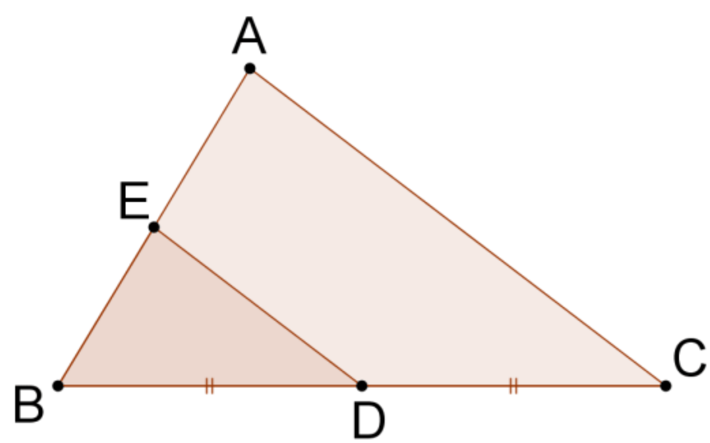
Are the triangles similar? Explain.



While two pairs of sides are proportional and one pair of angles are congruent, the angles are not the included angles. This is SSA, which is not a similarity criterion. Therefore, you cannot say for sure that the triangles are similar.

Example 4

What additional information would you need to be able to say that $\Delta ABC \sim \Delta EBD$?



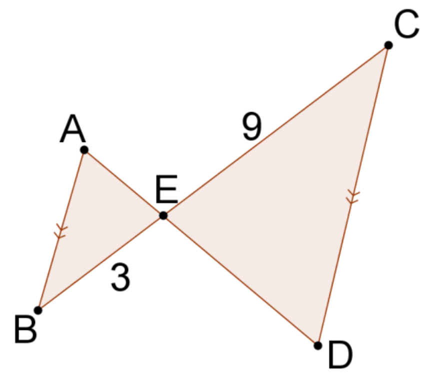
Because $\overline{BD} \cong \overline{DC}$, $\frac{BC}{BD} = 2$. The two triangles share $\angle B$, so that is a pair of congruent angles. To prove that the triangles are similar, you would need to know that $\frac{BA}{BE} = 2$ as well. If you knew that $\overline{BE} \cong \overline{EA}$ or E was the midpoint of $\overline{AB}$, then you could say that the triangles are similar.

Review

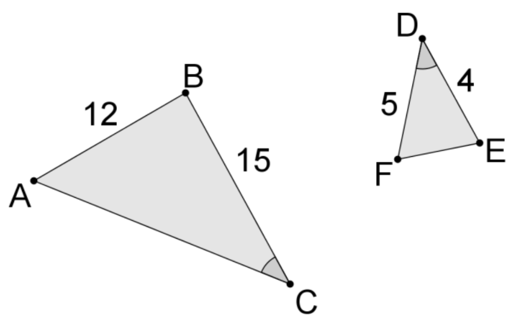
1. What does SAS stand for? What does it have to do with similar triangles?
2. What does SSA stand for? What does it have to do with similar triangles?
3. Draw an example of two triangles that must be similar due to SAS.
4. Draw an example of two triangles that **are not necessarily similar** because all you know is SSA.

For each pair of triangles below state if they are similar, congruent, or if there is not enough information to determine whether or not they are congruent. If they are similar or congruent, write a similarity or congruence statement.

5.



6.



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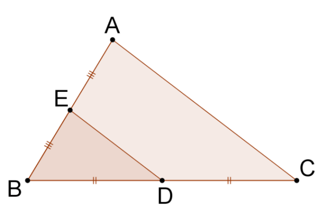


FIGURE 5.13

8.

9.

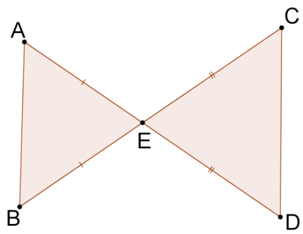
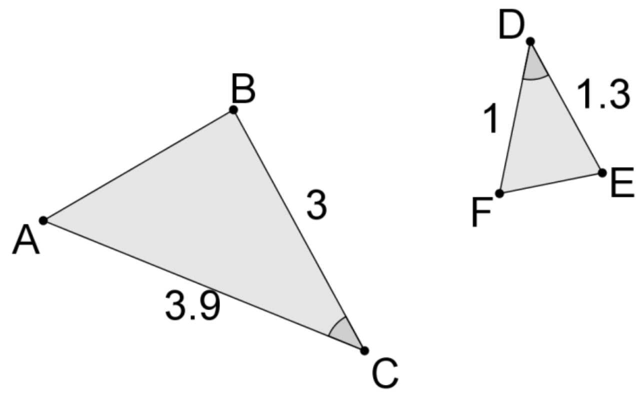


FIGURE 5.14



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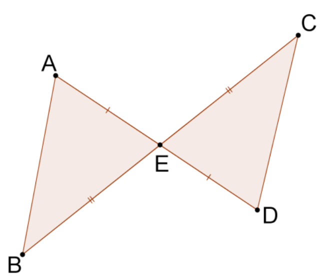
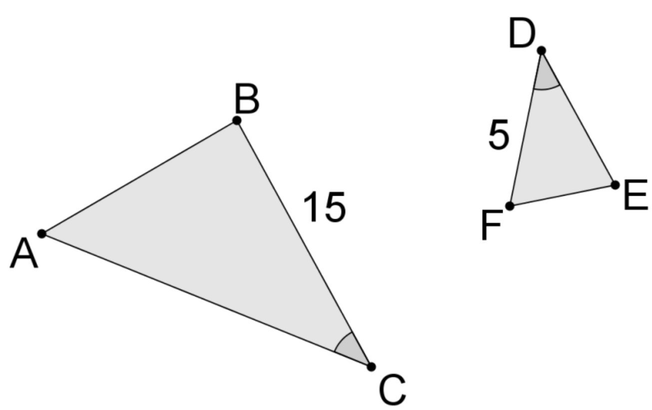
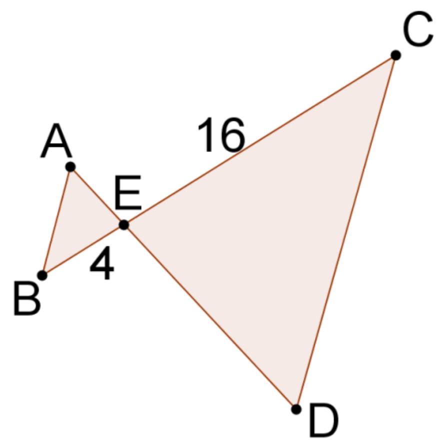


FIGURE 5.15

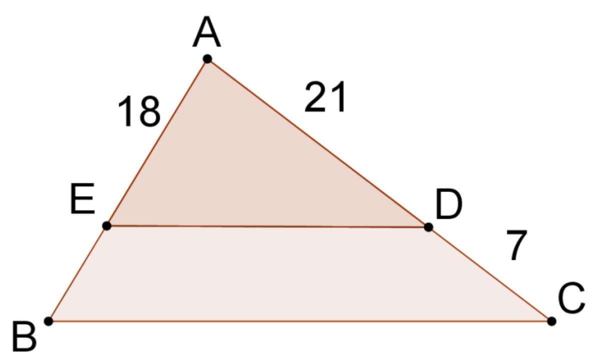
11. What is the minimum additional information you would need in order to be able to state that the triangles below are similar by SAS?



12. What is the minimum additional information you would need in order to be able to state that the triangles below are similar by SAS?

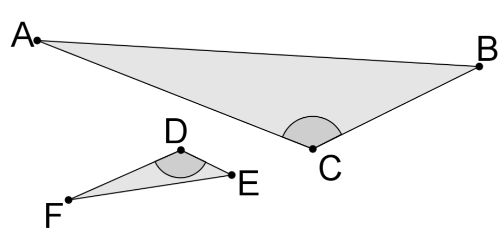


13. What is the minimum additional information you would need in order to be able to state that the triangles below are similar by SAS?



14. AAS and ASA are both criteria for triangle congruence. Are they also criteria for triangle similarity? Explain.

15. Show how the SAS criterion for triangle similarity works: use transformations to help explain why the triangles below are similar given that $\frac{AC}{FD} = \frac{CB}{DE}$. *Hint: See Examples A and B for help.*



16. Given two right triangles whose legs are proportional to each other, are the triangles similar? Diagram and explain.

17. Given two isosceles triangles such that the vertex angles are congruent, are the triangles similar? Diagram and explain.

18. Are all equilateral triangles similar? Why or why not?

19. Given the diagram below, are the two triangles similar? Why or why not?

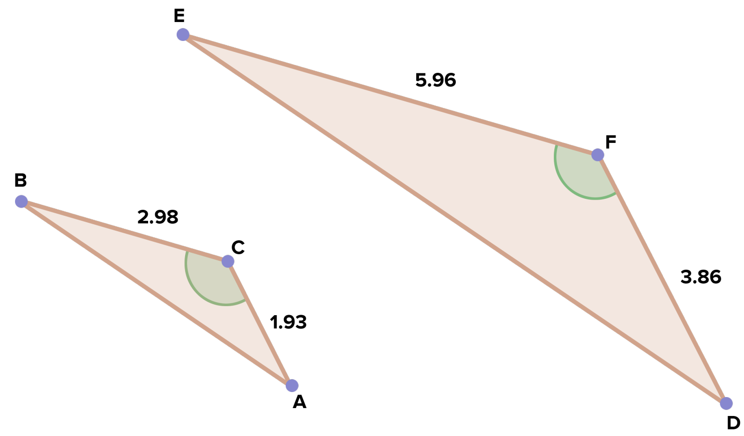


FIGURE 5.16

20. In the last problem it was necessary to write proportions to establish two pairs of proportional sides. What proportions did you write? How many ways can an equivalent proportion be written? Describe the ways these proportions can be written and the reasons one might choose to write them different ways.

21. ΔAEF is the image resulting from a dilation of ΔABC. Give the center of dilation and the approximate scale factor. Explain.

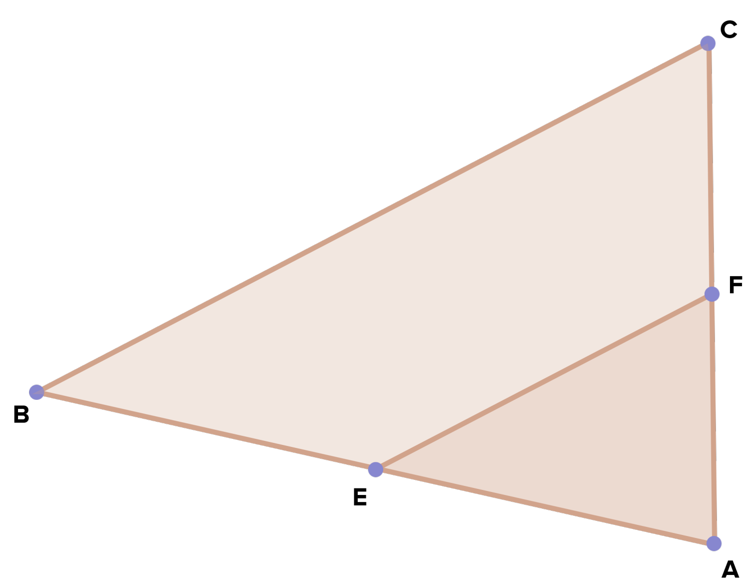


FIGURE 5.17

22. Write proportions for the above triangle expressing the ratio of corresponding sides.

23. Given: Diagram as shown; $\frac{a}{a+b} = \frac{c}{c+d}$

Prove: ΔAEF ΔABC

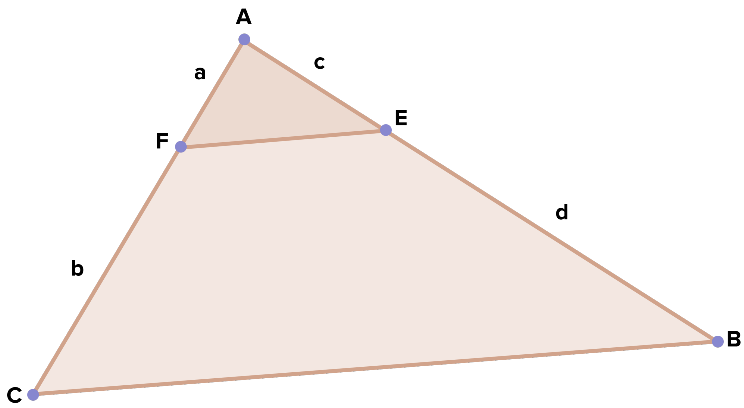


FIGURE 5.18

24. In the last proof, a proportion was given. What if the given proportion had been $\frac{a}{b}=\frac{c}{d}$? Is this equivalent to the proportion given above? If so, use algebra to show this is the case.

Review (Answers)

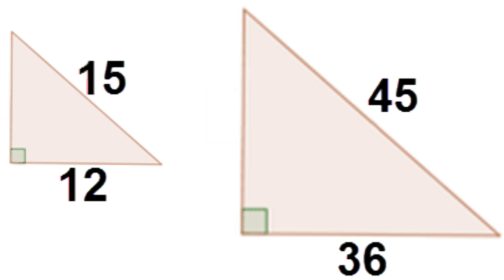
To see the Review answers, open this PDF file and look for section 6.4.

5.4 SSS Triangle Similarity

Learning Objectives

Here you will explore the SSS criterion for triangle similarity.

How can you use the SSS similarity criterion to show that the triangles below are similar?

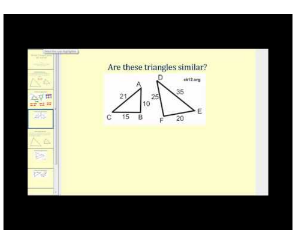


SSS Triangle Similarity

If two triangles are **similar** it means that all corresponding angle pairs are congruent and all corresponding sides are proportional. However, in order to be sure that two triangles are similar, you do not necessarily need to have information about all sides and all angles.

The SSS criterion for triangle similarity states that if three sides of one triangle are proportional to three sides of another triangle, then the triangles are similar.

In the examples, you will use similarity transformations and criteria for triangle congruence to show why SSS is a criterion for triangle similarity.



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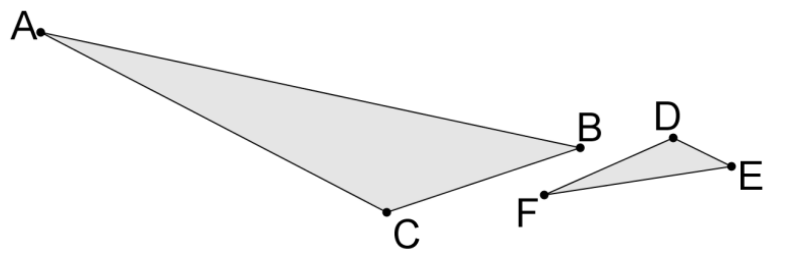
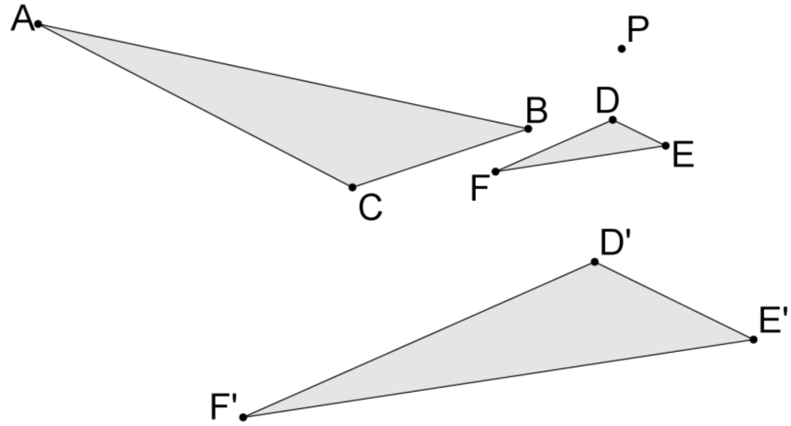
Let's take a look at few problems regarding SSS triangle similarity.

1. Consider the triangles below with $\frac{AB}{FE} = \frac{CA}{DF} = \frac{CB}{DE} = k$. Dilate ΔDEF with a scale factor of k to create $\Delta D'E'F'$. What do you know about the sides of $\Delta D'E'F'$? How do they relate to the sides of ΔABC?



Below, ΔDEF is dilated about point P with a scale factor of k to create $\Delta D'E'F'$.



The scale factor is k, which is equal to $\frac{AB}{FE}$ and $\frac{CA}{DF}$ and $\frac{CB}{DE}$. This means:

- $D'F' = k \cdot DF = \frac{CA}{DF} \cdot DF = CA$. Therefore, $\overline{D'F'} \cong \overline{CA}$.
- $D'E' = k \cdot DE = \frac{CB}{DE} \cdot DE = CB$. Therefore, $\overline{D'E'} \cong \overline{CB}$.
- $F'E' = k \cdot FE = \frac{AB}{FE} \cdot FE = AB$. Therefore, $\overline{F'E'} \cong \overline{AB}$.

2. Use your work from the previous problem to prove that $\Delta ABC \sim \Delta FED$.

You know that $\overline{D'F'} \cong \overline{CA}$, $\overline{D'E'} \cong \overline{CB}$ and $\overline{F'E'} \cong \overline{AB}$. This means $\Delta ABC \cong \Delta F'E'D'$ by $SSS \cong$.

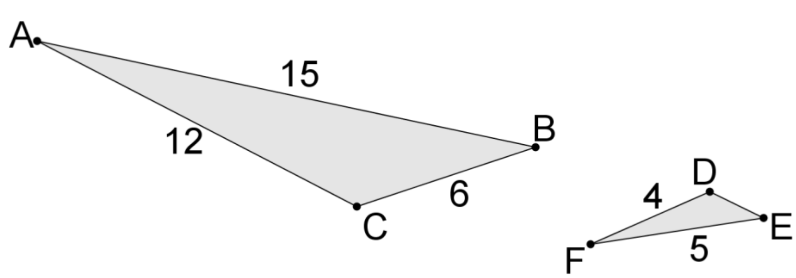
Therefore, there must exist a sequence of rigid transformations that will carry ΔABC to $\Delta F'E'D'$.

$\Delta ABC \sim \Delta FED$ because a series of rigid transformations will carry ΔABC to $\Delta F'E'D'$, and then a dilation will carry $\Delta F'E'D'$ to ΔABC.

All that was known about the original two triangles in previous problem was three pairs of proportional sides. You have proved that *SSS* is a criterion for triangle similarity.

Now, let's take a look at determining triangle similarity using SSS.

You want to show that the triangles below are similar by $SSS \sim$. What additional information do you need?



You have three side lengths for ΔABC with $\frac{AB}{FE} = \frac{AC}{FD} = 3$. Since $CB = 6$, you need to know that $DE = 2$ (so that $\frac{CB}{DE} = 3$) in order to show that the triangles are similar by $SSS \sim$.

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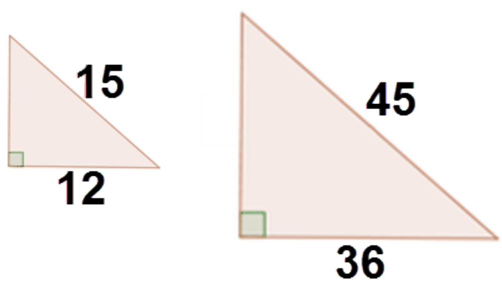
Examples

Example 1

Earlier, you were asked how can you use the SSS similarity criterion to show that the triangles below are similar.



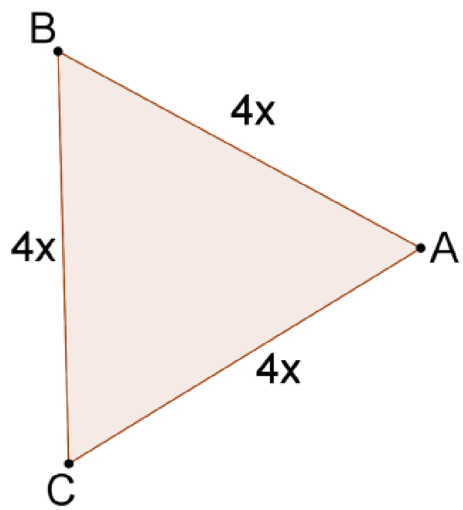
Only two pairs of sides and a pair of non-included angles are given. You can't say these triangles are similar by SSA because that is not a criterion for triangle similarity. However, because these are right triangles, you know that the third side of each triangle can be found with the Pythagorean Theorem.

- For the smaller triangle: $12^2 + x^2 = 15^2 \to x = 9$.
- For the larger triangle: $36^2 + x^2 = 45^2 \to x = 27$.

All three pairs of sides are proportional with a ratio of 3. Therefore, the triangles are similar due to $SSS \sim$. See the practice exercises to generalize this proof.

Example 2

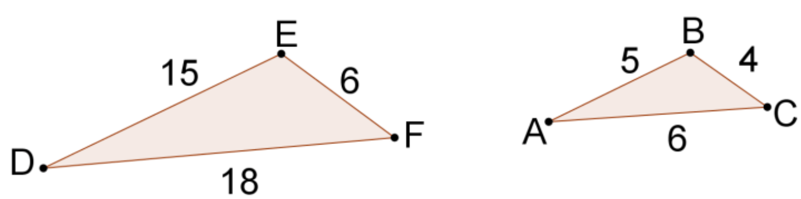
Are the triangles similar? Explain.



All three sides of each triangle are the same. The ratio between each pair of sides is $\frac{4x}{x} = 4$. Because three pairs of sides are proportional, the triangles are similar by $SSS \sim$.

Example 3

Are the triangles similar? Explain.



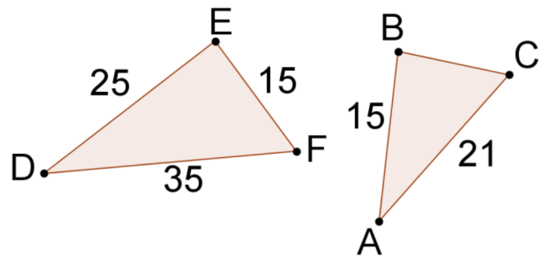
Match the longest side with the longest side and the shortest side with the shortest side. Check all three ratios:

- $\frac{DF}{AC} = \frac{18}{6} = 3$
- $\frac{DE}{AB} = \frac{15}{5} = 3$
- $\frac{EF}{BC} = \frac{6}{4} = 1.5$

Because the three pairs of sides are not proportional, the triangles are not similar.

Example 4

You want to show that the triangles below are similar by $SSS \sim$. What additional information do you need?



Look for two pairs of sides with lengths in the same ratio.

- $\frac{DE}{AB} = \frac{25}{15} = \frac{5}{3}$
- $\frac{DF}{AC} = \frac{35}{21} = \frac{5}{3}$

The common ratio is $\frac{5}{3}$. This means, $\frac{EF}{BC}$ must equal $\frac{5}{3}$ as well.

$$\frac{15}{BC} = \frac{5}{3}$$
$$45 = 5BC$$
$$BC = 9$$

For the triangles to be similar by $SSS \sim$, you need to know that $BC = 9$.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/sss-similarity/plix/Similar-Sides-Similar-Triangles-56eb2cd88e0e082bbff94739

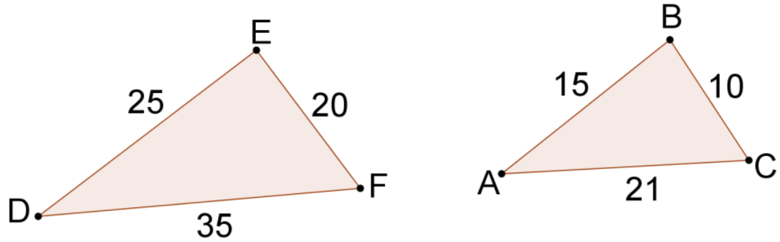
Review

1. What does SSS stand for? What does it have to do with similar triangles?

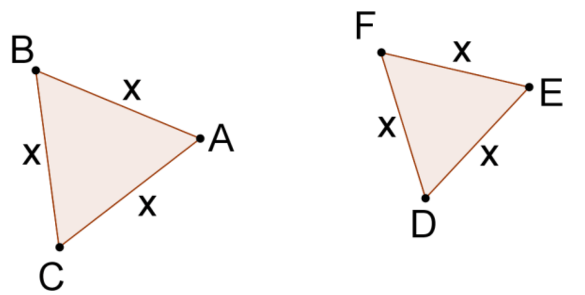
2. Draw an example of two triangles that must be similar due to SSS.

For each pair of triangles below state if they are similar, congruent, or if there is not enough information to determine whether or not they are congruent. If they are similar or congruent, write a similarity or congruence statement.

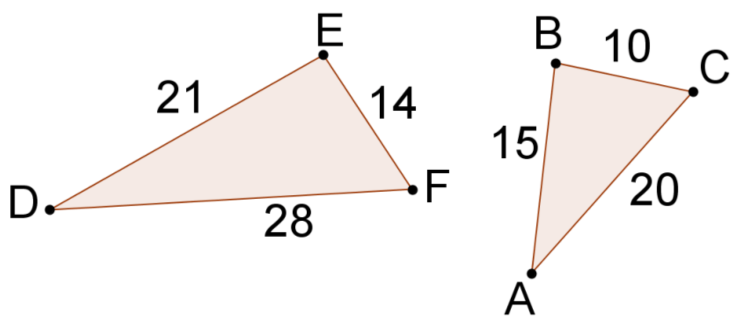
3.


E
25
20
D
35
F
B
15
10
A
21
C

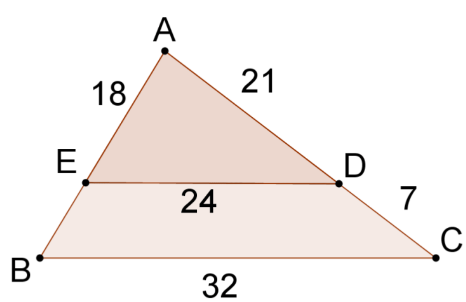
4.


B
x
x
A
x
C
F
x
E
x
x
D

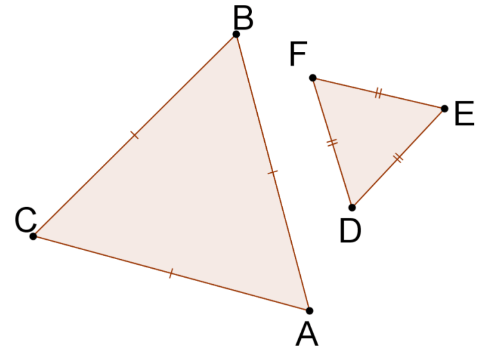
5.


E
21
14
D
28
F
B
10
C
15
20
A

6.


A
18
21
E
D
24
7
C
B
32

7.



8. What does the previous problem tell you about equilateral triangles?

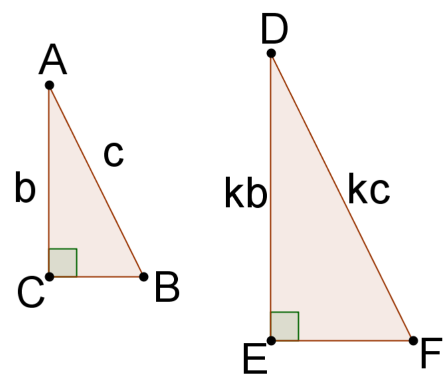
9. How are the perimeters of two similar triangles related?

10. One triangle has side lengths of 6, 8, and 10. A similar triangle has a perimeter of 60. What are the lengths of the sides of the similar triangle?

11. One triangle has side lengths of 7, 8, and 14. A similar triangle has a perimeter of 87. What is the ratio between corresponding sides?

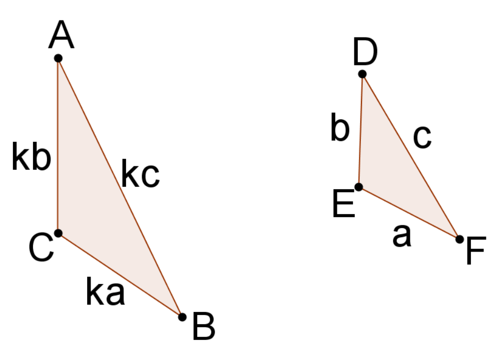
12. One triangle has side lengths of 2, 4, and 4. A similar triangle has a perimeter of 30. What are the lengths of the sides of the similar triangle?

13. Find the length of the unmarked side of each triangle in terms of c, b, and k.



14. Use your work from #13 to prove that the two triangles in #13 are similar. What does this tell you about one method for proving that right triangles are similar?

15. Show how the SSS criterion for triangle similarity works: use transformations to help explain why the triangles below are similar. *Hint: See Examples A and B for help.*



16. Are all congruent triangles similar? Are all similar triangles congruent? Explain.

17. We use the abbreviations SSS and SAS for similarity and congruence. What do these expressions mean in each case?

18. Given the diagram as shown below, determine if the triangles are similar or not. Explain.

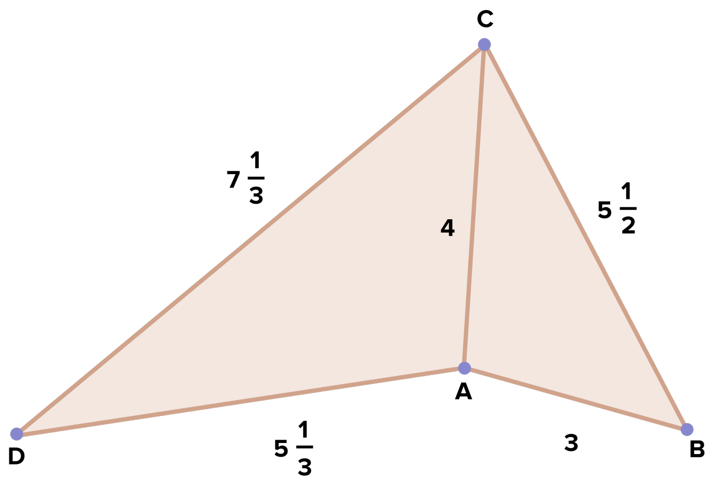


FIGURE 5.19

Review (Answers)

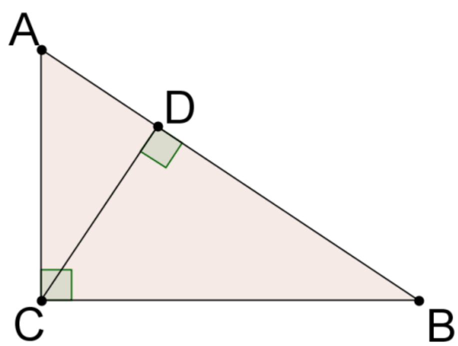
To see the Review answers, open this PDF file and look for section 6.5.

5.5 Theorems Involving Similarity

Learning Objectives

Here you will use similar triangles to prove new theorems about triangles.

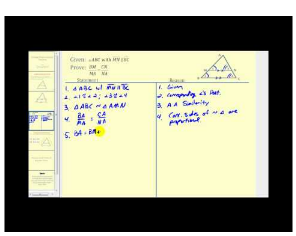
Can you find any similar triangles in the picture below?



Theorems on Similar Triangles

If two triangles are similar, then their corresponding angles are congruent and their corresponding sides are proportional. There are many theorems about triangles that you can prove using similar triangles.

1. **Triangle Proportionality Theorem:** A line parallel to one side of a triangle divides the other two sides of the triangle proportionally. *This theorem and its converse will be explored and proved in #1 and #2, and the Review exercises.*
2. **Triangle Angle Bisector Theorem:** The angle bisector of one angle of a triangle divides the opposite side of the triangle into segments proportional to the lengths of the other two sides of the triangle. *This theorem will be explored and proved in #3.*
3. **Pythagorean Theorem:** For a right triangle with legs a and b and hypotenuse c, $a^2+b^2=c^2$. *This theorem will be explored and proved in the Examples problems.*



MEDIA

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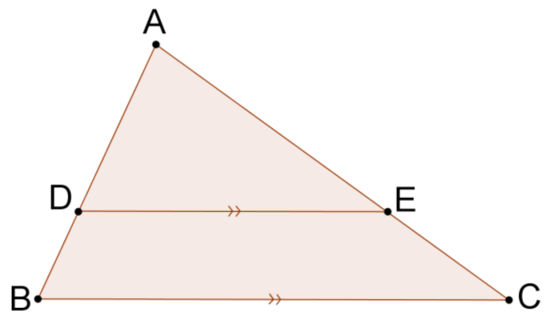
Let's take a look at some problems about proving triangle similarity.

1. Prove that $\Delta ADE \sim \Delta ABC$.



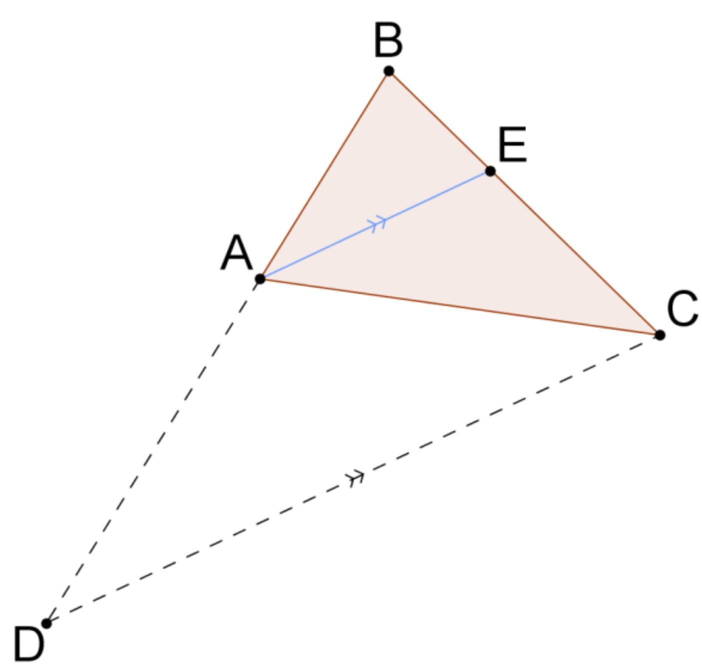
The two triangles share $\angle A$. Because $\overline{DE} \| \overline{BC}$, corresponding angles are congruent. Therefore, $\angle ADE \cong \angle ABC$. The two triangles have two pairs of congruent angles. Therefore, $\Delta ADE \sim \Delta ABC$ by $AA \sim$.

2. Use your result from #1 to prove that $\frac{AB}{AD} = \frac{AC}{AE}$. Then, use algebra to show that $\frac{DB}{AD} = \frac{EC}{AE}$.

$\Delta ADE \sim \Delta ABC$ which means that corresponding sides are proportional. Therefore, $\frac{AB}{AD} = \frac{AC}{AE}$. Now, you can use algebra to show that the second proportion must be true. Remember that $AB = AD + DB$ and $AC = AE + EC$.

You have now proved the triangle proportionality theorem: a line parallel to one side of a triangle divides the other two sides of the triangle proportionally.

3. Consider ΔABC with $\overline{AE}$ the angle bisector of $\angle BAC$ and point D constructed so that $\overline{DC} \| \overline{AE}$. Prove that $\frac{EB}{BA} = \frac{EC}{CA}$.



By the triangle proportionality theorem, $\frac{EB}{EC} = \frac{BA}{AD}$. Multiply both sides of this proportion by $\frac{EC}{BA}$.

$$\left(\frac{EC}{BA}\right) \cdot \frac{EB}{EC} = \frac{BA}{AD} \cdot \left(\frac{EC}{BA}\right)$$
$$\rightarrow \frac{EB}{BA} = \frac{EC}{AD}$$

Now all you need to show is that $AD = CA$ in order to prove the desired result.

- Because $\overline{AE}$ is the angle bisector of $\angle BAC$, $\angle BAE \cong \angle EAC$.
- Because $\overline{DC} \| \overline{AE}$, $\angle BAE \cong \angle BDC$ (corresponding angles).
- Because $\overline{DC} \| \overline{AE}$, $\angle EAC \cong \angle DCA$ (alternate interior angles).
- Thus, $\angle BDC \cong \angle DCA$ by the transitive property.

Therefore, ΔADC is isosceles because its base angles are congruent and it must be true that $\overline{AD} \cong \overline{CA}$. This means that $AD = CA$. Therefore:

$\frac{EB}{BA} = \frac{EC}{CA}$

This proves the triangle angle bisector theorem: the angle bisector of one angle of a triangle divides the opposite side of the triangle into segments proportional to the lengths of the other two sides of the triangle.

Multimedia

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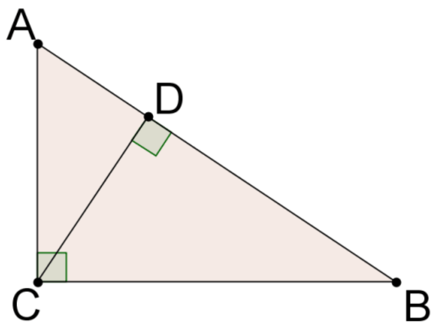
TABLE 5.1:

Statement	***Reason***
1. $\overrightarrow{BD}$ bisects $\angle ABC, \overrightarrow{BA} \perp \overline{AD}, \overrightarrow{BC} \perp \overline{DC}$	Given
2. $\angle ABD \cong \angle DBC$	Definition of an angle bisector
3. $\angle DAB$ and $\angle DCB$ are right angles	Definition of perpendicular lines
4. $\angle DAB \cong \angle DCB$	All right angles are congruent
5. $\overline{BD} \cong \overline{BD}$	Reflexive PoC
6. $\triangle ABD \cong \triangle CBD$	AAS
7. $\overline{AD} \cong \overline{DC}$	CPCTC

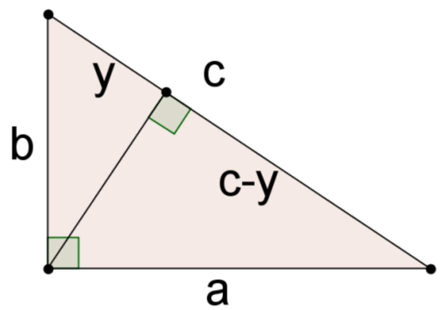
Examples

Example 1

Earlier, you were asked can you find any similar triangles in the picture below.



There are three triangles in this picture: ΔBAC, ΔBCD, ΔCAD. All three triangles are right triangles so they have one set of congruent angles (the right angle). ΔBAC and ΔBCD share $\angle B$, so $\Delta BAC \sim \Delta BCD$ by $AA \sim$. Similarly, ΔBAC and ΔCAD share $\angle C$, so $\Delta BAC \sim \Delta CAD$ by $AA \sim$. By the transitive property, all three triangles must be similar to one another.



The large triangle above has sides a, b, and c. Side c has been divided into two parts: y and $c-y$. In the Concept Problem Revisited you showed that the three triangles in this picture are similar.

Example 2

Explain why $\frac{a}{c} = \frac{c-y}{a}$.

When triangles are similar, corresponding sides are proportional. Carefully match corresponding sides and you see that $\frac{a}{c} = \frac{c-y}{a}$.

Example 3

Explain why $\frac{b}{c} = \frac{y}{b}$.

When triangles are similar, corresponding sides are proportional. Carefully match corresponding sides and you see that $\frac{b}{c} = \frac{y}{b}$.

Example 4

Use the results from #2 and #3 to show that $a^2 + b^2 = c^2$.

Cross multiply to rewrite each equation. Then, add the two equations together.

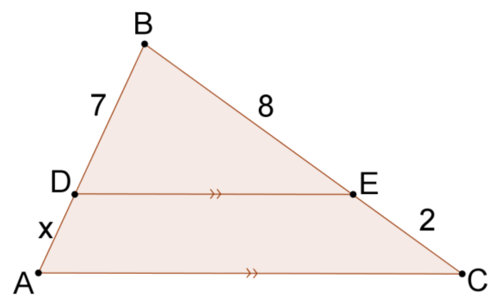
$$\frac{a}{c} = \frac{c-y}{a} \rightarrow a^2 = c^2 - cy$$
$$\frac{b}{c} = \frac{y}{b} \rightarrow b^2 = cy$$
$$\rightarrow a^2 + b^2 = c^2 - cy + cy$$
$$\rightarrow a^2 + b^2 = c^2$$

You have just proved the Pythagorean Theorem using similar triangles.

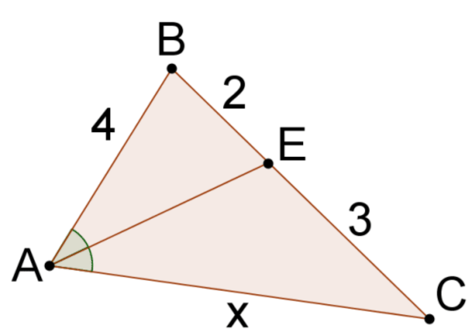
Review

Solve for x in each problem.

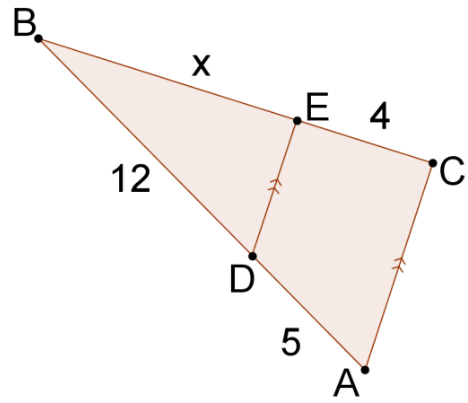
1.



2.



3.



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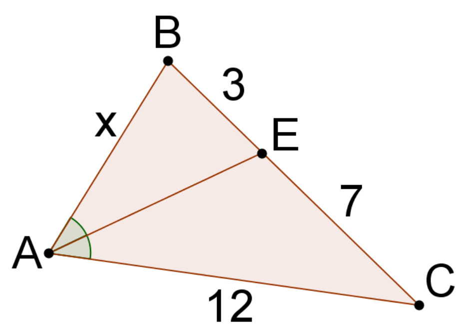

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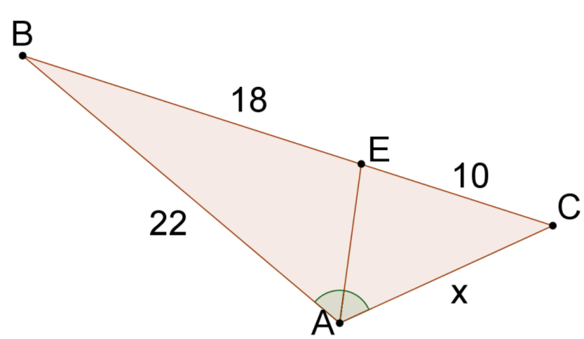

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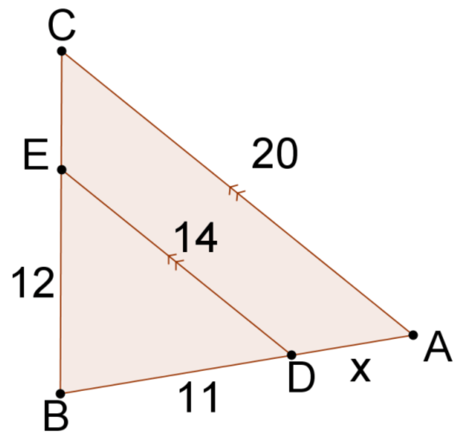

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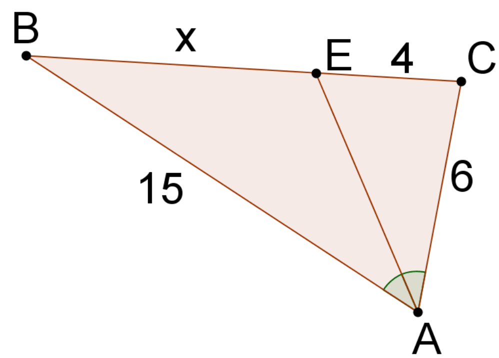
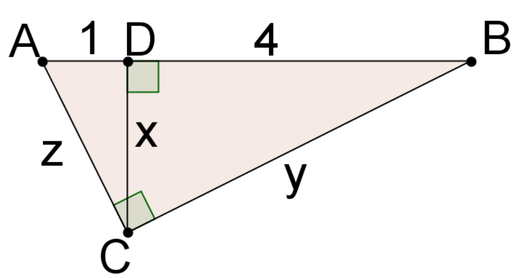


Use the picture below for #8-#10.

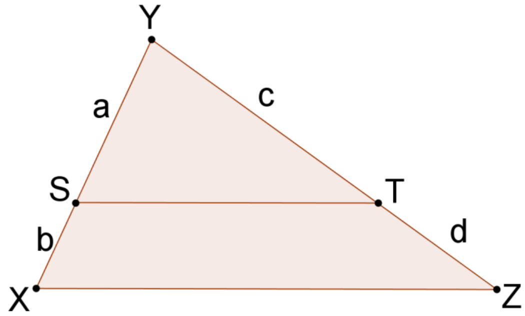


8. Solve for x.

9. Solve for z.

10. Solve for y.

Use the picture below for #11-#13.



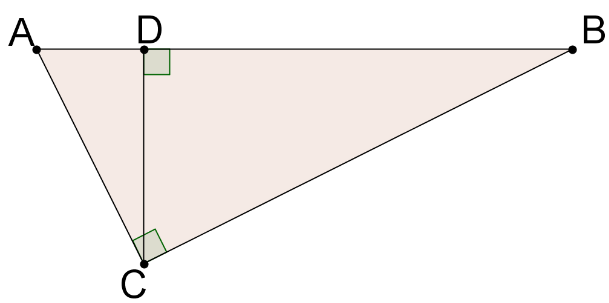
11. Assume that $\frac{b}{a} = \frac{d}{c}$. Use algebra to show that $\frac{b+a}{a} = \frac{d+c}{c}$.

12. Prove that $\Delta YST \sim \Delta YXZ$

13. Prove that $\overline{ST} \| \overline{XZ}$

14. Prove that a segment that connects the midpoints of two sides of a triangle will be parallel to the third side of the triangle.

15. Prove the Pythagorean Theorem using the picture below.



16. Refer to the diagram below to prove the triangle angle bisector theorem by a slightly different method.

Given: $\angle 1 \cong \angle 2$; $\overline{DC} \parallel \overline{BA}$ by construction

Prove: $\frac{m}{n} = \frac{p}{q}$

Here are some questions to guide you in your proof:

a. What can you prove relating ΔDEC to ΔAEB?

b. In that case, what labeled sides are proportional?

c. It's given that $\angle 1 \cong \angle 2$. Is it also true that $\angle 1 \cong \angle 5$? What can you conclude?

d. What can you conclude about ΔDCA?

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.6.

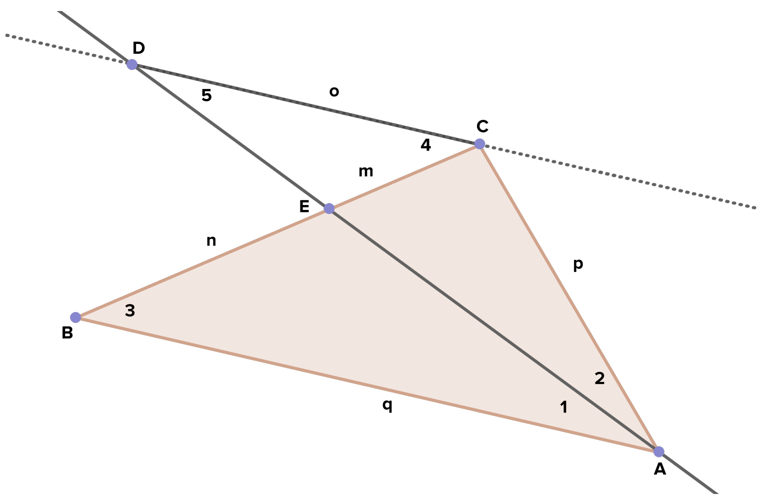


FIGURE 5.20

5.6 Applications of Similar Triangles

Learning Objectives

Here you will solve problems involving similar triangles and learn about special right triangles.

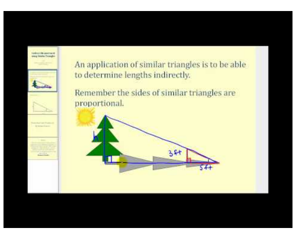
Michael is 6 feet tall and is standing outside next to his younger sister. He notices that he can see both of their shadows and decides to measure each shadow. His shadow is 8 feet long and his sister's shadow is 5 feet long. How tall is Michael's sister?

Applications of Similar Triangles

If two triangles are similar, then their corresponding angles are congruent and their corresponding sides are proportional. There are three criteria for proving that triangles are similar:

1. **AA:** If two triangles have two pairs of congruent angles, then the triangles are similar.
2. **SAS:** If two sides of one triangle are proportional to two sides of another triangle and their included angles are congruent, then the triangles are similar.
3. **SSS:** If three sides of one triangle are proportional to three sides of another triangle, then the triangles are similar.

Once you know that two triangles are similar, you can use the fact that their corresponding sides are proportional and their corresponding angles are congruent to solve problems.



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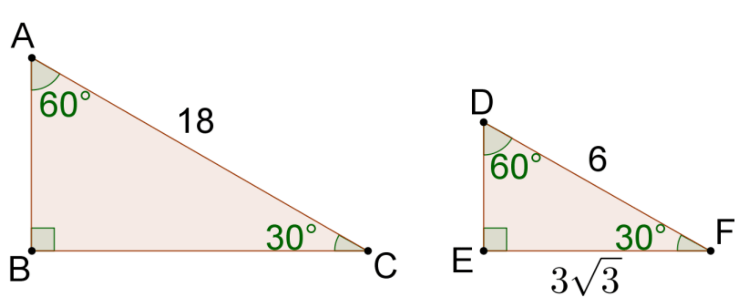
Let's take a look at some example problems.

1. Prove that the two triangles below are similar.



The triangles are similar by $AA\sim$ because they have at least two pairs of congruent angles.

Use the Pythagorean Theorem to find DE.

Use the fact that the triangles are similar to find the missing sides of ΔABC.

$\frac{AC}{DF}=\frac{18}{6}=3$, so the scale factor is 3.

- $\frac{AB}{DE}=3\rightarrow\frac{AB}{3}=3\rightarrow AB=9$
- $\frac{BC}{EF}=3\rightarrow\frac{BC}{3\sqrt{3}}=3\rightarrow BC=9\sqrt{3}$

The triangles are called 30-60-90 triangles because of their angles measures.

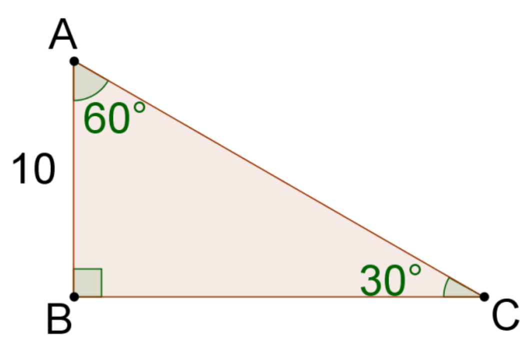
Explain why all 30-60-90 triangles are similar.

All 30-60-90 triangles are similar by $AA\sim$ because they will all have at least two pairs of congruent angles.

2. Use the triangles from #1 to find the ratios between the three sides of any 30-60-90 triangle.

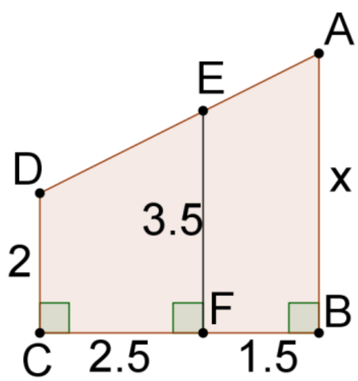
ΔDEF had sides $3, 3\sqrt{3}, 6$. This ratio of $3:3\sqrt{3}:6$ reduces to $1:\sqrt{3}:2$. The three sides of any 30-60-90 triangle will be in this ratio.

3. Find the missing sides of the triangle below.

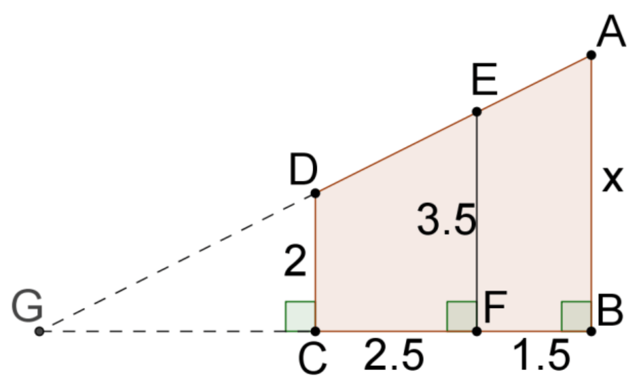


The side opposite the 30° angle is the smallest side because 30° is the smallest angle. Therefore, the length of 10 corresponds to the length of "1" in the ratio $1:\sqrt{3}:2$. The scale factor is 10. The other sides of the triangle will be $10\sqrt{3}$ and 20, because $10:10\sqrt{3}:20$ is equivalent to $1:\sqrt{3}:2$. $BC=10\sqrt{3}$ and $AC=20$.

4. Create similar triangles in order to solve for x.



Extend $\overline{AD}$ and $\overline{BC}$ to create point G.



$\Delta DGC \sim \Delta EGF \sim \Delta AGB$ by $AA \sim$ because angles $\angle DCG, \angle EFG, \angle ABG$ are all right angles and are therefore congruent and all triangles share $\angle G$. This means that their corresponding sides are proportional. First, solve for GC by looking at ΔDGC and ΔEGF.

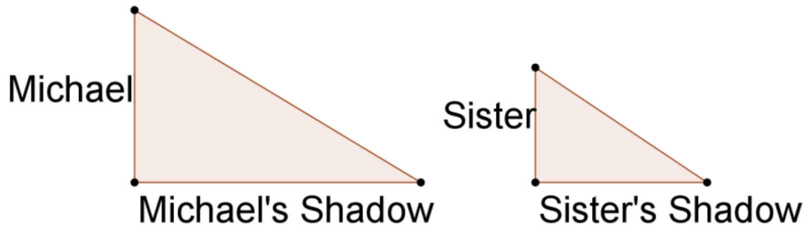
Next, solve for x by looking at ΔDGC and ΔAGB.

Examples

Example 1

Earlier, you were asked how tall Michael's sister is. You can answer this question using applications of similar triangles.

The sun creates shadows at the same angle for both Michael and his sister. Assuming they are both standing up straight and making right angles with the ground, similar triangles are created.



Corresponding sides are proportional because the triangles are similar.

Cross multiply and solve for his sister's height. His sister is 3.75 feet tall.

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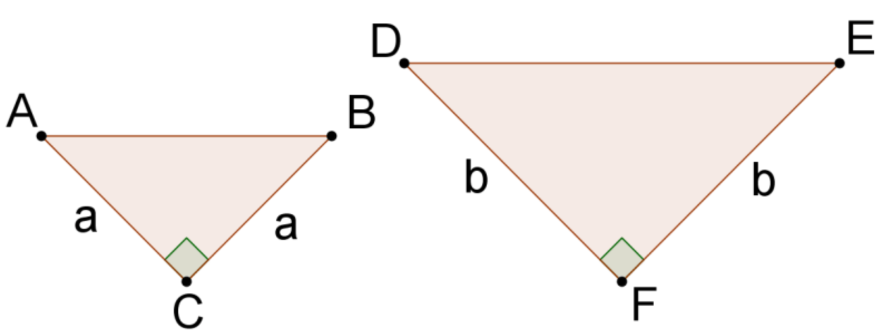
Example 2

Prove that all isosceles right triangles are similar.

Consider two generic isosceles right triangles:



Two pairs of sides are proportional with a ratio of $\frac{b}{a}$. Also, $\angle C \cong \angle F$. Therefore, the two triangles are similar by $SAS \sim$.

Example 3

Find the measures of the angles of an isosceles right triangle. Why are isosceles right triangles called 45-45-90 triangles?

The base angles of an isosceles triangle are congruent. If the vertex angle is 90°, each base angle is $\frac{180^\circ - 90^\circ}{2} = 45^\circ$. The measures of the angles of an isosceles right triangle are 45, 45, and 90. An isosceles right triangle is called a 45-45-90 triangle because those are its angle measures.

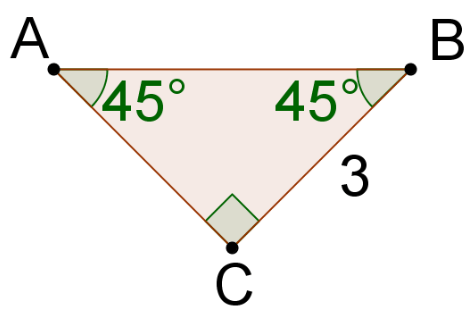
Example 4

Use the Pythagorean Theorem to find the missing side of an isosceles right triangle whose legs are each length x.

The missing side is the hypotenuse of the right triangle, c. By the Pythagorean Theorem, $c^2 = x^2 + x^2$. This means $c^2 = 2x^2$ and therefore $c = x\sqrt{2}$. The ratio of the sides of any isosceles right triangle will be $x : x : x\sqrt{2}$ which simplifies to $1 : 1 : \sqrt{2}$.

Example 5

Use what you have learned in #1-#3 to find the missing sides of the right triangle below without using the Pythagorean Theorem.



If one of the legs is 3, then the other leg is also 3, so $AC = 3$. The hypotenuse will be $3\sqrt{2}$ following the pattern from #3, so $AB = 3\sqrt{2}$.

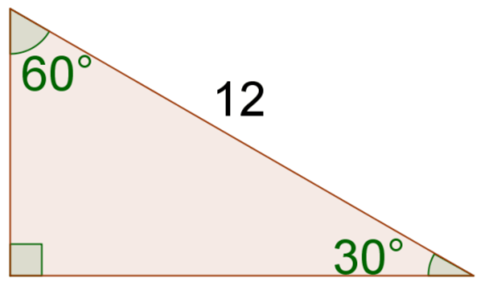
Review

1. Explain why all 30-60-90 triangles are similar.

2. The ratio between the sides of any 30-60-90 triangle is ________:________:________.

Find the missing sides of each triangle:

3.



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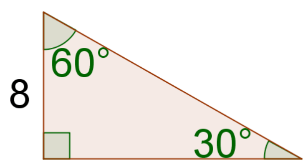

FIGURE 5.21

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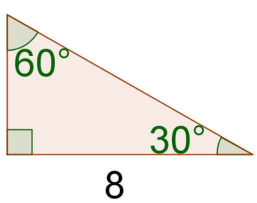

FIGURE 5.22

6. Explain why all 45-45-90 triangles are similar.

7. The ratio between the sides of any 45-45-90 triangle is ________:________:________.

Find the missing sides of each triangle:

8.

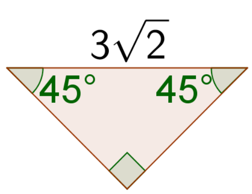

FIGURE 5.23

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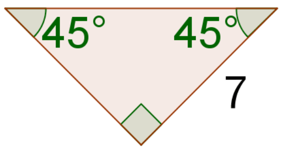


FIGURE 5.24

10.

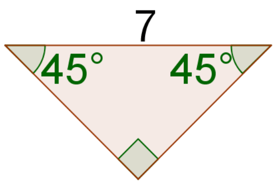
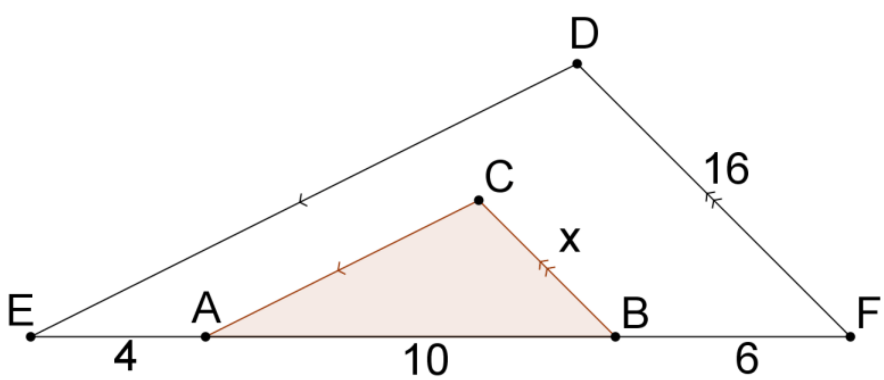


FIGURE 5.25

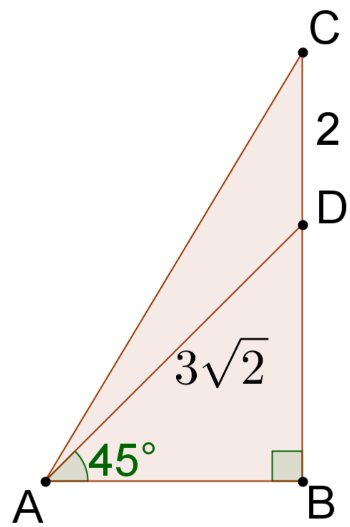
Use the figure below for #11 and #12.



11. Prove that $\Delta ABC \sim \Delta EFD$.

12. Find the value of x.

Use the figure below for #13-#15.



13. Find $m\angle ADB$. What type of triangle is ΔADB?

14. Find BD and AB.

15. Find AC.

16. A forest ranger sticks a walking stick that's 2 meters long vertically in the flat ground. The stick casts a shadow that's 3 meters long. She measures the shadow of a redwood tree nearby. It's 20 meters long. Create a diagram of the situation, showing the two right triangles which model the two objects and their shadows. Are these triangles similar? Why or why not? Are their sides proportional? If so, write and solve a proportion to find the height of the tree.

17. True or false: On a sunny day on flat ground, all vertical segments near me cast shadows that form right triangles. All these triangles are similar. As the sun rises or sets, the acute angles of these triangles change accordingly. Explain with a diagram.

18. Use the triangle angle bisector theorem to prove that the angle bisector of the vertex angle of an isosceles triangle bisects the base.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.7.

5.7 References

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CHAPTER 6 Polygons

Chapter Outline

6.1 Polygons

Learning Objectives

Here you will review the names and basic properties of polygons.

A polygon is a shape bounded by a number of straight lines. A polygon is usually classified by its number of sides, as shown in the table below.

TABLE 6.1:

Number of Sides	Name of Polygon
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon (or enneagon)
10	Decagon

For a polygon with more than 10 sides, most people prefer to name it by the number of sides and the suffix "gon." For example, a 40 sided polygon would be a "40-gon."

Convex and Concave Polygon

A diagonal is a line segment that connects any two non-adjacent vertices of a polygon. A polygon is convex if all diagonals remain inside the polygon. Most polygons that you study in high school geometry will be convex.

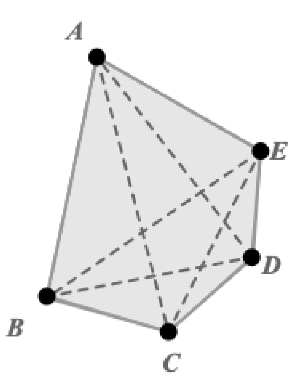

FIGURE 6.1

If a polygon is not convex then it is concave (or non-convex). Some diagonals of a concave polygon lie partly or wholly outside the polygon.

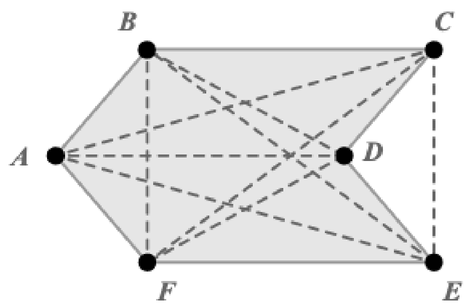

FIGURE 6.2

Regular Polygon

A polygon is equilateral if all of its sides are the same length. A polygon is equiangular if all of its angles are the same measure. A polygon is regular if it is both equilateral and equiangular.

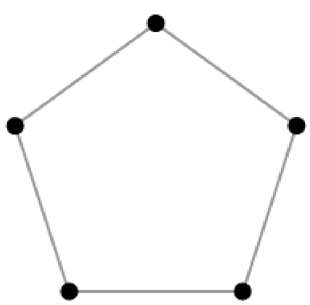

FIGURE 6.3

Exploring Polygons

Play with a polygon by adding or removing sides. See different polygons from triangle up. Explore the angles and lengths for regular and irregular polygons by dragging any of the vertices of the polygon.

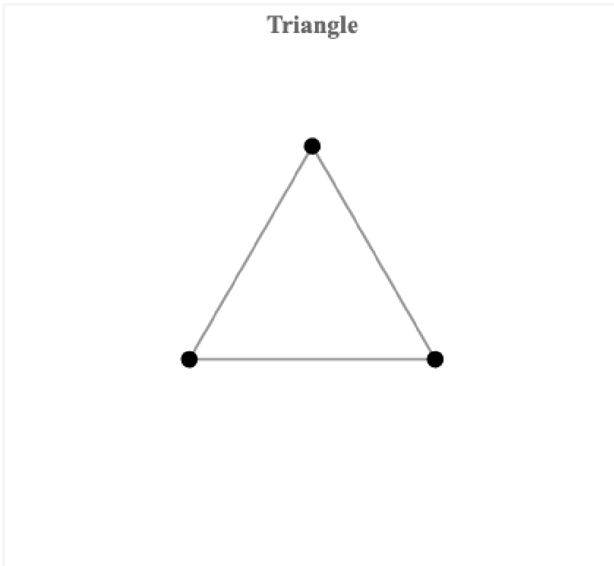

FIGURE 6.4

Sum of Interior Angles

The sum of the measures of the three angles in a triangle is 180°. You can use this fact to find the sum of the measures of the angles in any polygon.

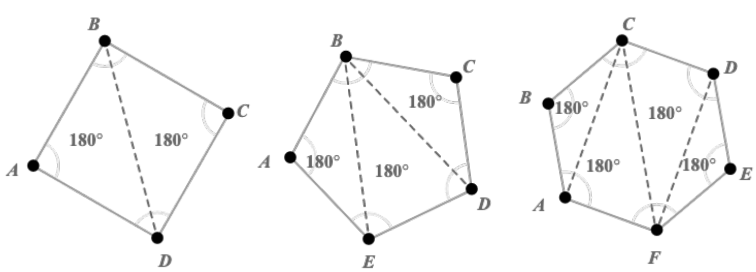

FIGURE 6.5

The pentagon above has been divided into three triangles, such that all the triangles are non-overlapping pieces and the interior angles have been marked. The sum of the measures of the angles of each triangle is 180°. Therefore, the sum of the interior angles of the pentagon is $180^\circ \cdot 3 = 540^\circ$.

General Rule: A polygon of n-sides can be divided into (n-2) triangles.

$\therefore$ Sum of all the interior angles of an n-sided polygon $= (n-2) \cdot 180^\circ$

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Click image to the left or use the URL below.

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If the polygon is regular (and thus equiangular), you can figure out the measure of each interior angle.

Naming Polygons

For the heptagon given below, find the sum of the interior angles.

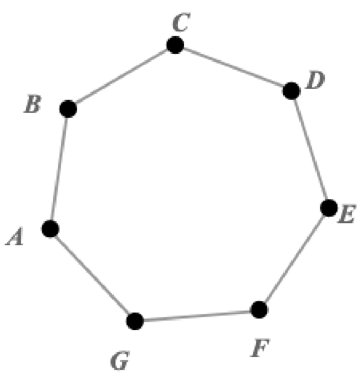


FIGURE 6.6

The sum of the interior angles of a heptagon is _____.

Measuring Interior Angles

Find the measure of an interior angle of a regular heptagon.

Because this is a regular polygon, it is equiangular. This means that each of the seven interior angles has the same measure. The sum of the interior angles was 900°. This means that each of the seven interior angles is $\frac{900^\circ}{7} \approx 128.6^\circ$.

General Rule: The measure of each interior angle of an equiangular n-gon is $\frac{(n-2)\cdot 180^\circ}{n}$.

Examining the Relationship Between Angles

An exterior angle is the angle between one side of a polygon and the extension of an adjacent side. In the polygon below, an exterior angle has been marked at vertex G. How are exterior angles related to interior angles? What is the measure of the exterior angle at G?

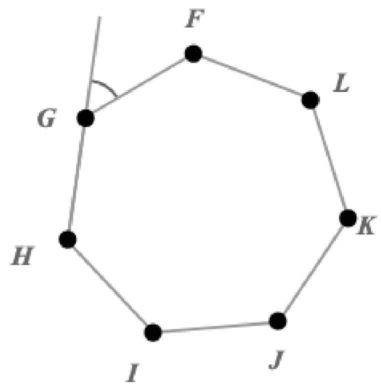
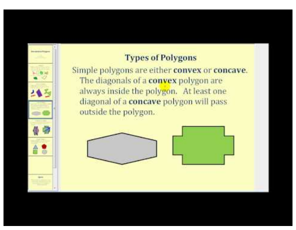


FIGURE 6.7

The exterior angle and interior angle at the same vertex will always be supplementary because together they form a straight angle. In this case, the interior angle at point G was approximately 128.6°, therefore the exterior angle is $180^\circ - 128.6^\circ = 51.4^\circ$.



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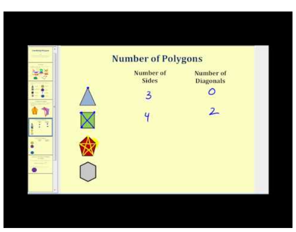

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Sum of Exterior Angles

The sum of the exterior angles of any polygon is 360°. How is it possible?

There are many ways to think about the sum of the exterior angles of a polygon. One way is to first consider that the sum of all the straight angles through the vertices is $180n^\circ$ (where n is the number of sides of the polygon).

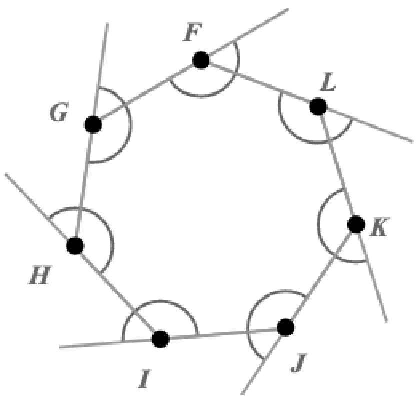

FIGURE 6.8

If you only want the sum of the exterior angles, you must subtract the sum of the interior angles. Recall from above that the sum of the interior angles is $180(n-2)^\circ$.

The sum of the exterior angles = sum of straight angles - sum of interior angles

$$\begin{aligned} & 180n - 180(n-2) \\ &= 180n - 180n + 360 \\ &= 360^\circ \end{aligned}$$

FIGURE 6.9

Multimedia

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Examples

Example 1

Find the sum of the interior angles of a nonagon.

Answer: Sum of the interior angles of a nonagon
$= (n-2) \cdot 180^\circ$
$= (9-2) \cdot 180^\circ$
$= 7 \cdot 180^\circ$
$= 1260^\circ$

Example 2

Find the measure of one interior angle of a regular nonagon.

Answer: Each interior angle of a nonagon

$= \frac{\text{sum of interior angles}}{\text{number of sides}}$

$= \frac{1260^\circ}{9}$

$= 140^\circ$

Example 3

Find the measure of one exterior angle of a regular decagon.

Answer: Each exterior angle of a decagon

$= \frac{\text{sum of exterior angles}}{\text{number of sides}}$

$= \frac{360^\circ}{10}$

$= 36^\circ$

CK-12 PLIX Interactive

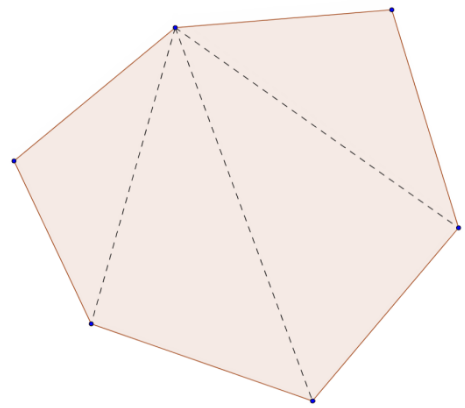
PLIX

Click image to the left or use the URL below.

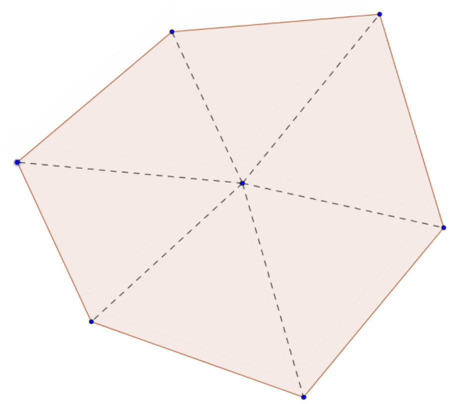
URL: http://www.ck12.org/geometry/exterior-angles-in-convex-polygons/plix/Exterior-Angles-in-Convex-Polygons-Pentagon-55785ed3da2cfe25c60683a5

Review

1. What is the measure of an exterior angle of a regular 45-gon?
2. What is the sum of the interior angles of a 35-gon?
3. Draw an example of a convex polygon and a concave polygon.
4. What is the name of a polygon with 8 sides?
5. What is the name of a polygon with 10 sides?
6. What is the name of a polygon with 4 sides?
7. How could you use the dissection shown in the picture below to show why the sum of the interior angles of a hexagon is 720°?



8. How could you use the dissection shown in the picture below to show why the sum of the interior angles of a hexagon is 720°?



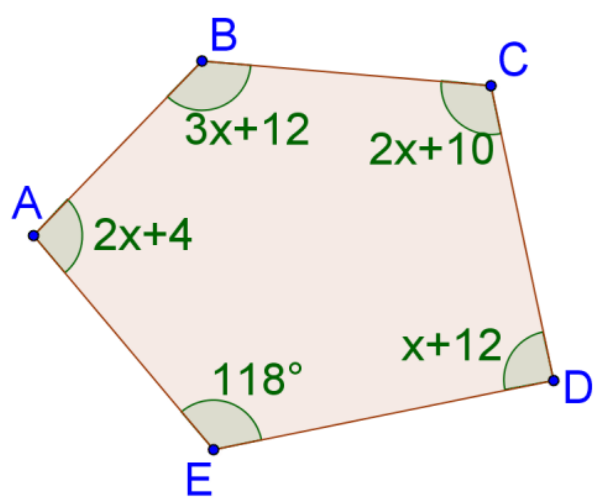
9. A regular polygon has an interior angle of 150°. How many sides does the polygon have?

10. How could you use exterior angles to help you find the answer to question 9?

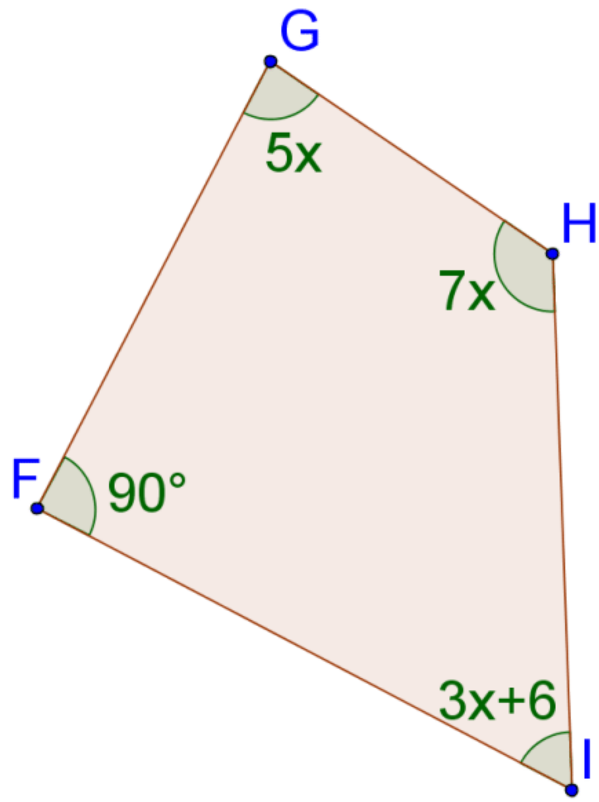
11. What is the sum of the exterior angles of an 11-gon?

12. What is the sum of the interior angles of an 11-gon?

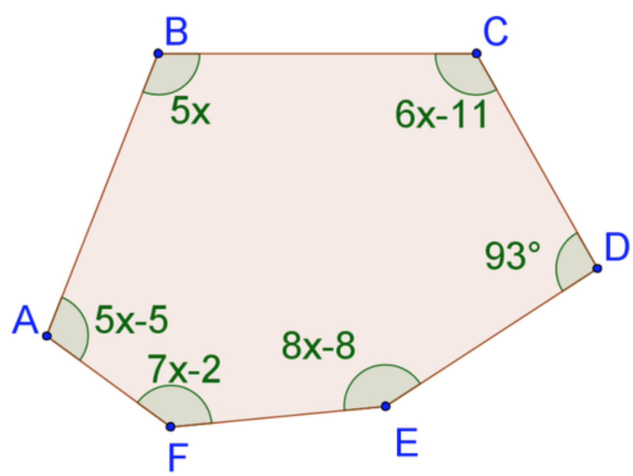
13. Solve for x :



14. Solve for x :



15. Solve for x :



16. Draw a regular polygon that has more than 5 sides. Find the measure of one interior angle using two different methods. Explain which one you prefer and why.

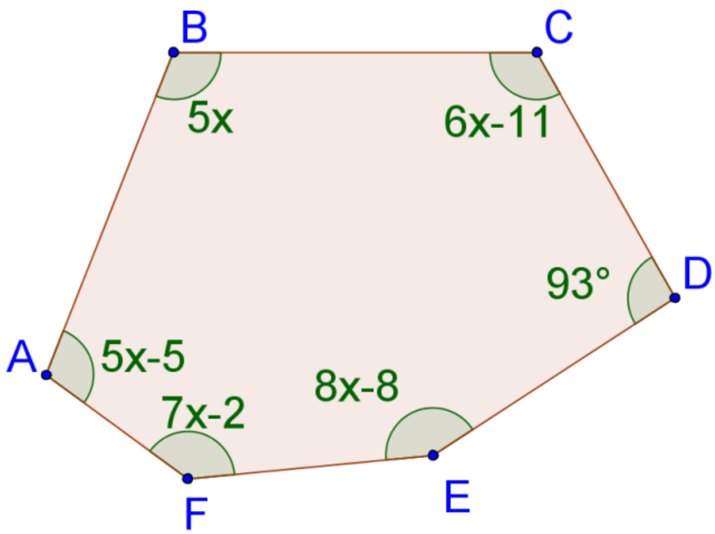
17. Is it possible to draw a polygon that **cannot** be decomposed into triangles? Why or why not?

18. What is the shape of a polygon that has an infinite number of sides? Why?

19. How many diagonals can be drawn in a triangle? A quadrilateral? A pentagon? Continue the experiment and explain the pattern.

20. Jerome claims that he can completely cover his bathroom floor with rectangular tiles, without any gaps or overlaps. Halle claims she can do the same with equilateral triangles. Who is correct and why? Experiment to see if there are other shapes that can be used to tile the floor without gaps or overlaps. Explain your results.

21. Daniel claims that he's created a regular polygon whose interior angles measure 142°. Is his claim true? Explain.



Review (Answers)

To see the Review answers, open this PDF file and look for section 1.3.

6.2 The Pythagorean Theorem

Learning Objectives

Here you will review the Pythagorean Theorem for right triangles.

The Pythagorean Theorem states that for right triangles with legs of lengths a and b and hypotenuse of length c, $a^2+b^2=c^2$. Let $a=3$, $b=4$ be the legs and $c=5$ be the hypotenuse of a right triangle.

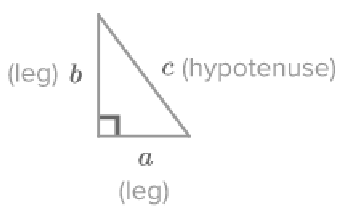


FIGURE 6.10

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \end{aligned}$$

FIGURE 6.11

The geometric interpretation of the Pythagorean theorem states that the sum of the area with side a, and the area of the square with side b, is equal to the area of the square with side c.

Geometric Proof

There are many different proofs of the Pythagorean Theorem. The following picture leads to one of those proofs.

Rearranging the triangles, we can also form another square with the same side length $(a+b)$. Hence, this geometric proof shows that the area of the square with side c is equal to the sum of the areas of the squares with side a and side b.

Algebraic Proof

First, you can verify that the quadrilateral in the center is a square. All sides are the same length and each angle must be 90°. The angles must be 90°because the three angles that make a straight angle at each corner of the interior quadrilateral are the same three angles that make up each of the triangles. This relationship is shown with the angle markings in the picture below.

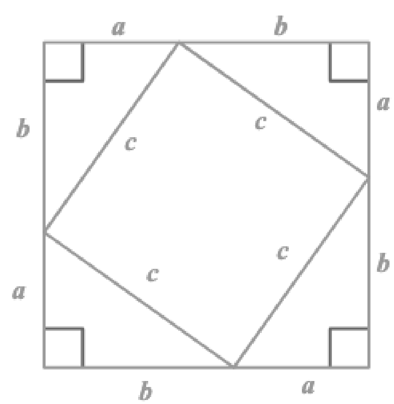


FIGURE 6.12

In order to prove the Pythagorean Theorem, find the area of the interior square in two ways. First, find the area directly:

Area of square $= c^2$

Next, find the area as the difference between the area of the large square and the area of the triangles.

Area of square $=(a+b)^2 - 4\left(\frac{1}{2}ab\right)$

Since you are referring to the same square each time, those two areas must be equal.

$$
\begin{aligned}
(a+b)^2 - 4\left(\frac{1}{2}ab\right) &= c^2 \\
(a+b)(a+b) - 2ab &= c^2 \\
a^2 + ab + ba + b^2 - 2ab &= c^2 \\
a^2 + 2ab + b^2 - 2ab &= c^2 \\
a^2 + b^2 &= c^2
\end{aligned}
$$

FIGURE 6.13

Converse of Pythagorean Theorem

The converse of the Pythagorean Theorem is also true. The converse switches the "if" and "then" parts of the theorem. The converse says that if $a^2+b^2=c^2$, then the triangle is a right triangle.

With the Pythagorean Theorem and its converse , you can solve many types of problems. You can:

1. Find the missing side of a right triangle when you know the other two sides.
2. Determine whether a triangle is right, acute, or obtuse.
3. Find the distance between two points.

Finding the Length of the Hypotenuse

The two legs of a right triangle have lengths 3 and 4. What is the length of the hypotenuse?

Because it is a right triangle, you can use the Pythagorean Theorem. It doesn't matter whether you assign a as 3 or b as 3.

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
3^2 + 4^2 &= c^2 \\
9 + 16 &= c^2 \\
25 &= c^2 \\
\sqrt{25} &= c \\
\pm 5 &= c
\end{aligned}
$$

FIGURE 6.14

Because length must be positive, the hypotenuse has a length of 5. Side lengths of 3, 4 and 5 are common in geometry. You should remember that they are the lengths of a right triangle. Triples of whole numbers that satisfy the Pythagorean Theorem are called Pythagorean triples . "3, 4, 5" is an example of a Pythagorean triple.

Classifying Triangles

A triangle has side lengths of 4, 8 and 9. What type of triangle is this?

If the numbers satisfy the Pythagorean Theorem (in other words, if the lengths of the sides form a Pythagorean Triple), then it is a right triangle. If $a^2+b^2>c^2$, then c is shorter than it would be in a right triangle, so the angle opposite it is smaller and it is an acute triangle. If $a^2+b^2<c^2$, then c is longer than it would be in a right triangle, so the angle opposite it is larger and it is an obtuse triangle.

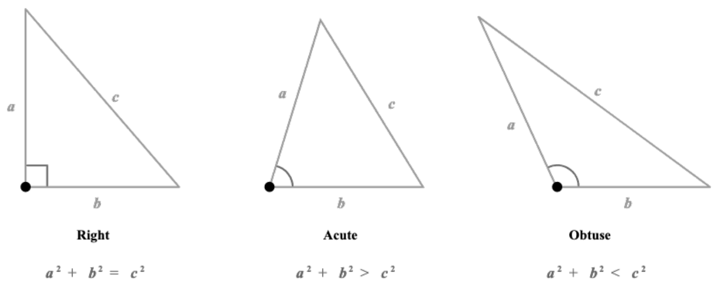


FIGURE 6.15

In this case, $4^2+8^2=80$ and $9^2=81$. So $a^2+b^2<c^2$, hence the triangle is obtuse.

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Distance Formula

Use the Pythagorean Theorem to derive the distance formula : $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

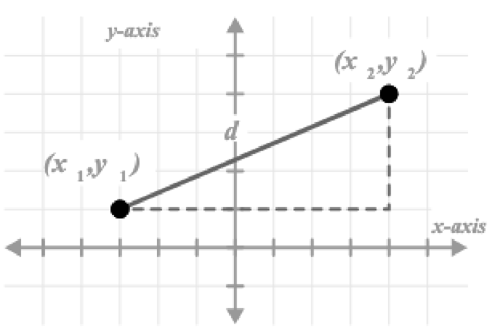


FIGURE 6.16

It can help to draw in a right triangle, where d is the hypotenuse. Then, find the lengths of the sides in terms of x_1, y_1, x_2, y_2. Finally, use the Pythagorean Theorem to find d.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

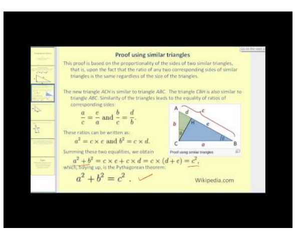
FIGURE 6.17

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/distance-formula-and-the-pythagorean-theorem/plix/Pythagorean-Theorem-to-Determine-Distance-Tree-Shadows-53d147578e0e0876d4df82f1



MEDIA

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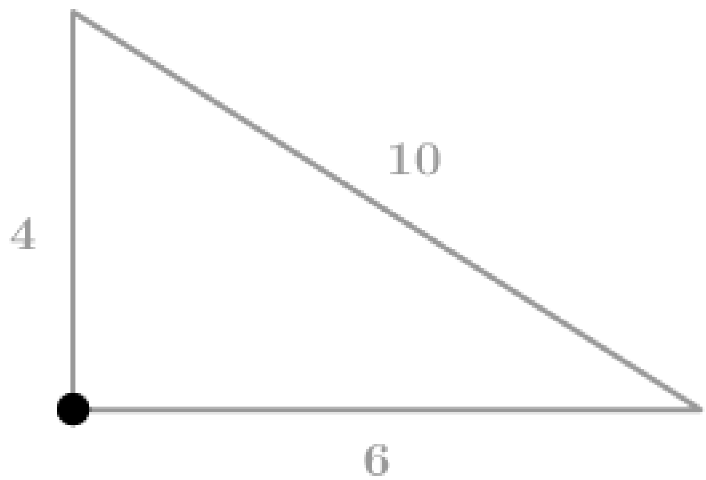
Examples

Example 1

The lengths of the three sides of the triangle are 4, 6 and 10. Is this a right triangle?



The lengths of the three sides of the triangle are 4, 6, and 10.

$4^2 + 6^2 = 52$
$10^2 = 100$
$52 < 100$

$a^2 + b^2 < c^2$ so this is not a right triangle, it is an obtuse triangle.

Example 2

Will a multiple of a Pythagorean triple always also be a Pythagorean triple? For example, "6, 8, 10" is a multiple of "3, 4, 5". Is "6, 8, 10" a Pythagorean triple? Is any multiple of "3, 4, 5" (or any other Pythagorean triple) also a Pythagorean triple?

Yes. Assume "a, b, c" is a Pythagorean triple, so $a^2 + b^2 = c^2$. "ka, kb, kc" where k is a whole number is a multiple of this Pythagorean triple.

$(ka)^2 + (kb)^2 = k^2a^2 + k^2b^2$
$= k^2(a^2 + b^2)$
$= k^2c^2$
$= (kc)^2$

Since $(ka)^2 + (kb)^2 = (kc)^2$, "ka, kb, kc" is also a Pythagorean triple.

Example 3

Find the distance between $(3, -4)$ and $(-1, 5)$.

Use the Pythagorean Theorem or the distance formula .

$$d = \sqrt{(3-(-1))^2 + (-4-5)^2}$$
$$d = \sqrt{4^2 + (-9)^2}$$
$$d = \sqrt{16 + 81}$$
$$d = \sqrt{97}$$
$$d \approx 9.85$$

FIGURE 6.18

Example 4

The length of one leg of a triangle is 5 and the length of the hypotenuse is 8. What is the length of the other leg?

Use the Pythagorean Theorem.

$a^2 + 5^2 = 8^2$
$a^2 + 25 = 64$
$a^2 = 64 - 25 = 39$
$a = \sqrt{39} \approx 6.24$

CK-12 PLIX Interactive

Multimedia

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/the-pythagorean-theorem/plix/Pythagorean-Theorem-and-Pythagorean-Triples-Phone-Pixels-53601a395aa41325a50d173d

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Review

Use the Pythagorean Theorem to solve for x in each right triangle below.

1.

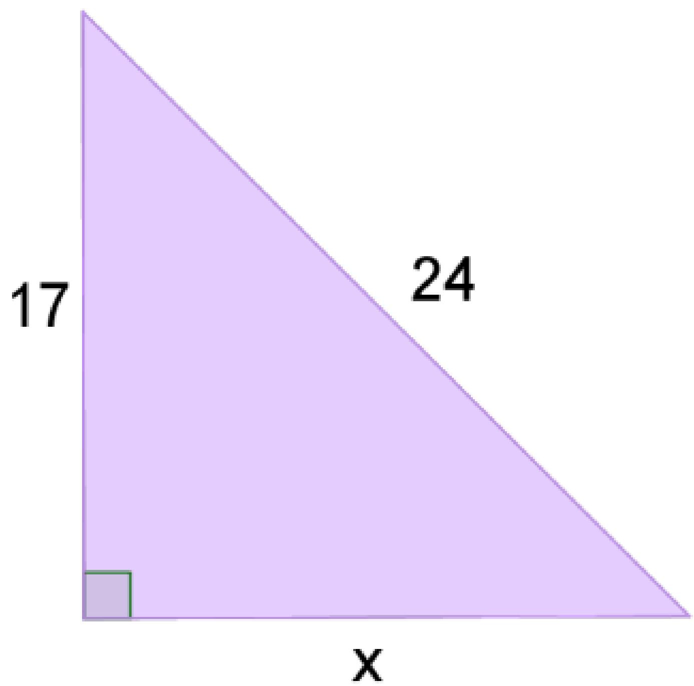


FIGURE 6.19

2.

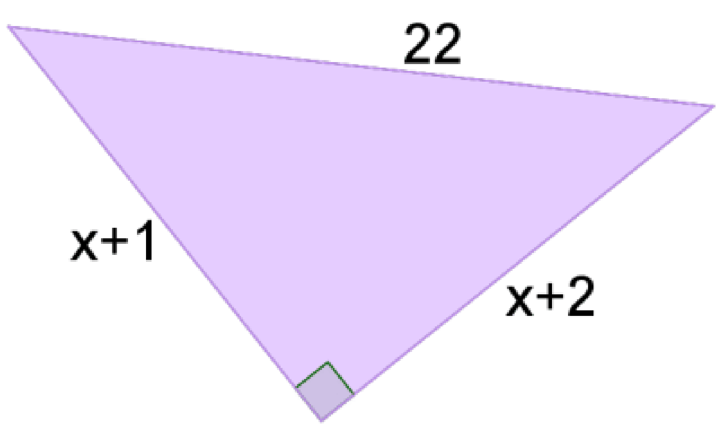


FIGURE 6.20

3.

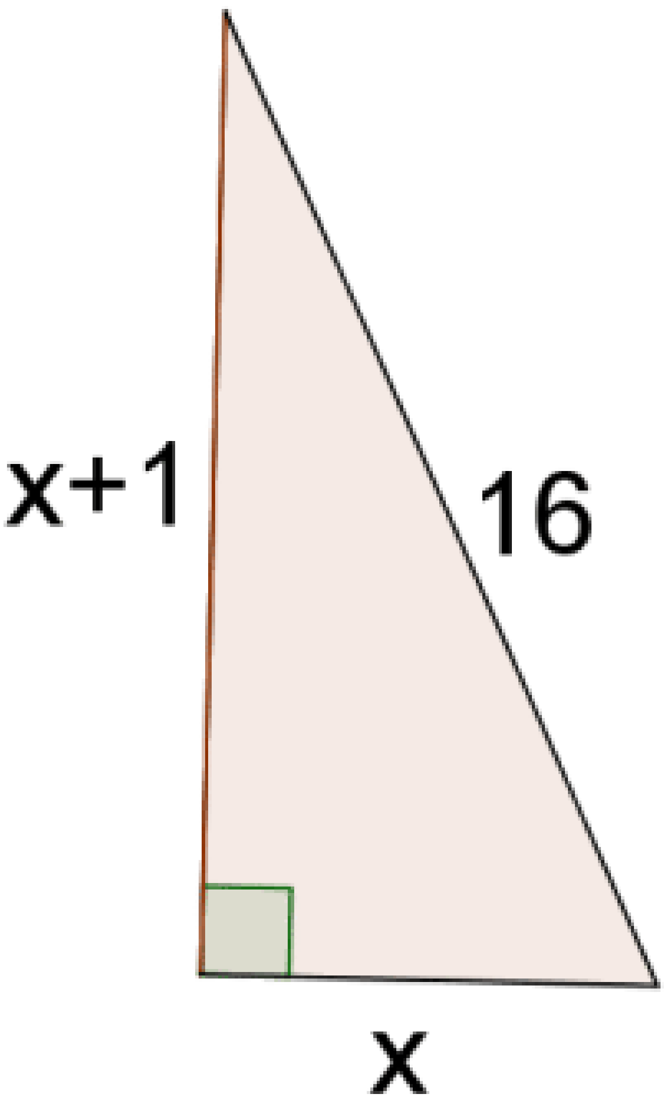


FIGURE 6.21

4.

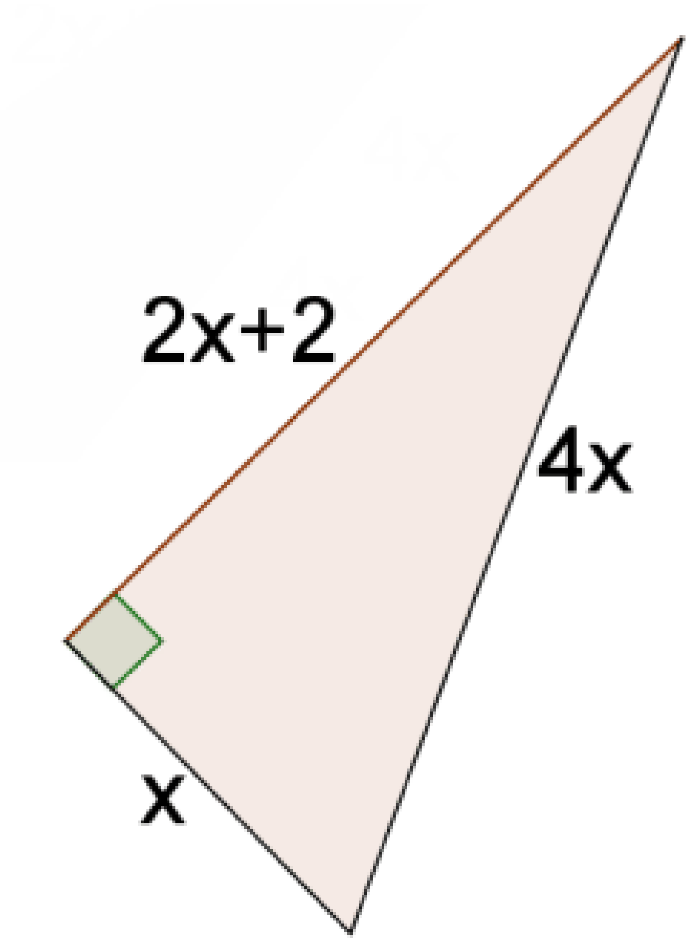


FIGURE 6.22

5.

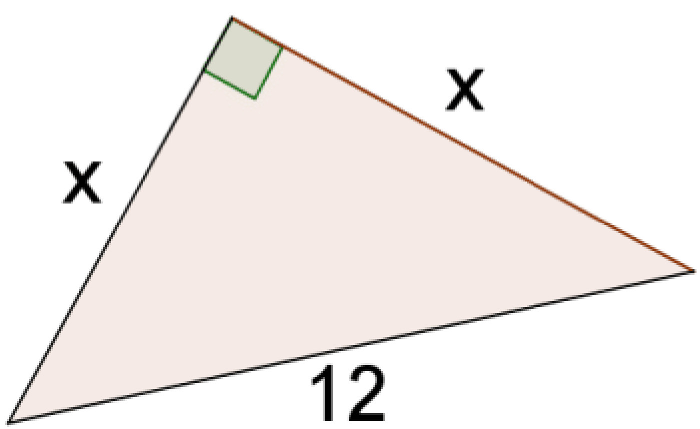


FIGURE 6.23

Three side lengths for triangles are given. Determine whether or not each triangle is right, acute, or obtuse.

6. 2, 5, 6

7. 4, 7, 8

8. 6, 8, 10

9. 6, 9, 10

Find the distance between each pair of points.

10. $(2,5)$ and $(1,-3)$

11. $(-4.5,2)$ and $(1.6,5)$

12. $(-3.7,2.1)$ and $(-3.2,-1.5)$

13. $(-3,-5)$ and $(5,6)$

14. Find two more Pythagorean triples that are *not* multiples of "3, 4, 5".

15. Pick any two whole numbers m and n with $n > m$. Then $n^2 - m^2$, $2mn$, and $n^2 + m^2$ will be a Pythagorean triple. Test this with a few values of n and m and then show why this process works using algebra.

16. There are many ways to prove the Pythagorean Theorem. In the diagram below, ΔBCE is given as shown. How can this triangle be rotated and translated to create ΔDEF? Is $DFCB$ a trapezoid? How do you know? Is ΔDEB a right triangle? How do you know? Find the area of the trapezoid two different ways. Set the results equal to each other. Then perform algebra to establish the Pythagorean Theorem. Do you prefer this proof or the one in the notes above? Why?

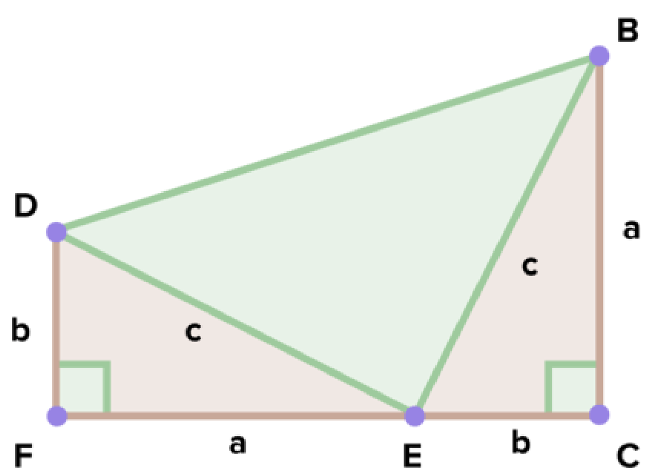


FIGURE 6.24

17. The diagram below is one visual representation of the squares of the sides of a right triangle. Explore the relationship between the area of the squares using the interactive in the discussion above. Is it true that the area of a square of side length 50 has the same area as the sum of two squares having side lengths 30 and 40? How about the area of a circle whose radius is 13, compared with two circles whose radii are 5 and 12? Use this relationship to find the radii of three different circles such that areas of the first two sum to that of the third.

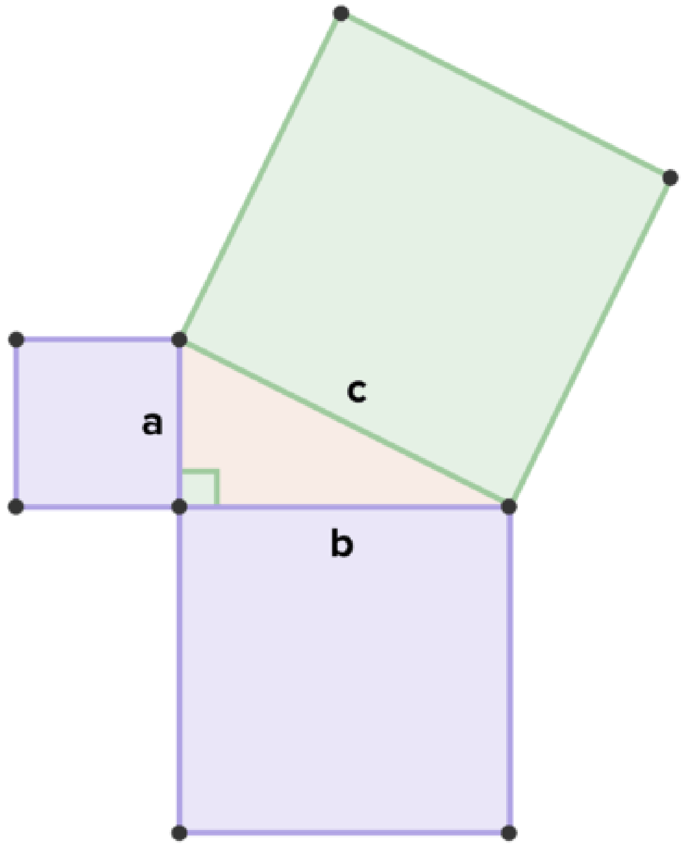


FIGURE 6.25

18. Find the dimensions of three cubes such that the surface areas of the first two add to the surface area of the third.

19. Will the relationship in the last problem hold true for the volume ? Why or why not?

20. In the diagram below there is a line with two slope triangles. Find the dimensions of the large one. Use these to find the dimensions of the smaller one. Round the value for the hypotenuse to the hundredths place. Now imagine creating more slope triangles, increasing x by 1 at a time. For every increase of x by 1, how much does the hypotenuse increase by? How long would the hypotenuse be for an x of 29?

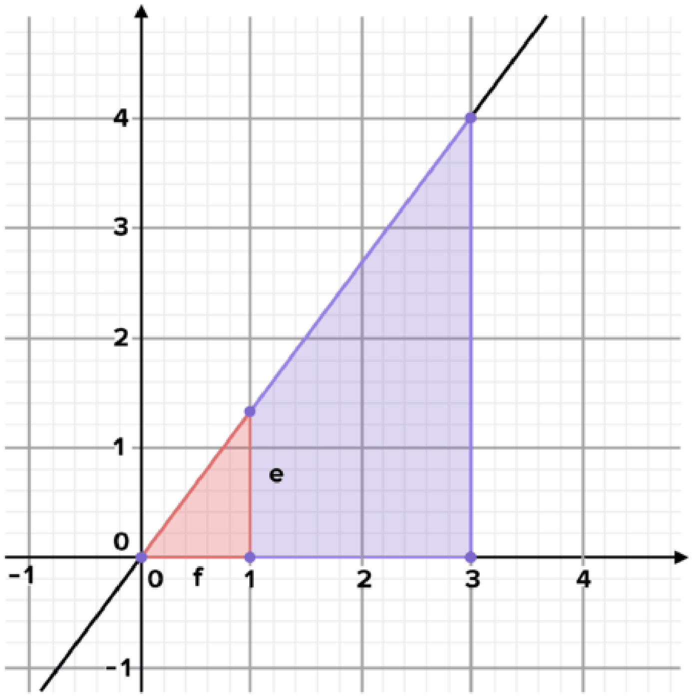


FIGURE 6.26

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.7.

6.3 Quadrilaterals

Learning Objectives

Here you will review different kinds of quadrilaterals and their properties.

The prefix "quad-" means "four", and "lateral" is derived from the Latin word for "side". So a quadrilateral is a four-sided polygon. Since it is a polygon, you know that it is a two-dimensional figure made up of straight sides. A quadrilateral also has four angles formed by its four sides.

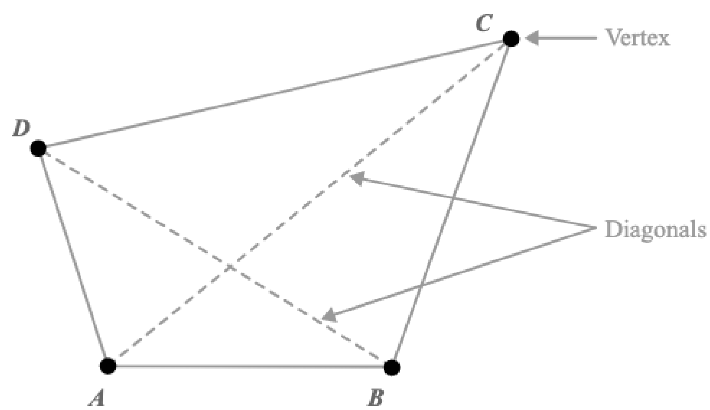


FIGURE 6.27

AB, *BC*, *CD* and *DA* are the sides and *A*, *B*, *C* and *D* are the vertices of the quadrilaterals.

Line segments *AC* and *BD* joining two non-consecutive vertices are called diagonals.

Two sides like *AB* and *AD* having a common endpoint are called adjacent sides.

There are many common special quadrilaterals that you should be familiar with. Below, these special quadrilaterals are described with their definitions and some properties.

Kite

A kite is a convex quadrilateral with two pairs of adjacent congruent sides such that not all sides are congruent.

The angles between the congruent sides are called vertex angles. The other angles are called non-vertex angles.

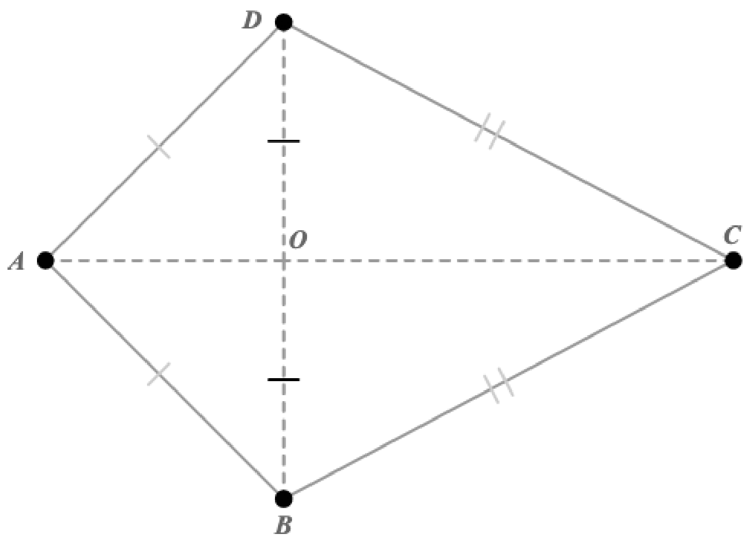


FIGURE 6.28

The Properties of a Kite:

1. Two pairs of adjacent sides are congruent, i.e., $AB = AD$ and $BC = CD$.

2. Non-vertex angles are congruent, i.e., $\angle ABC = \angle ADC$.

3. Diagonals intersect each other at right angles, i.e., $\angle AOB$=$\angle BOC$=$\angle COD$=$\angle DOA$= $90°$.

4. The longer diagonal of a kite bisects the shorter one, i.e., $BO = OD$.

5. The diagonal through the vertex angles is the angle bisector for both angles, i.e., $\angle BAC = \angle DAC$ and $\angle BCA = \angle DCA$.

6. One of the diagonals bisects the kite, i.e., divides it into two congruent triangles, i.e., $\Delta ABC \cong \Delta ADC$.

Trapezoid

A trapezoid is a quadrilateral with exactly one pair of parallel sides.

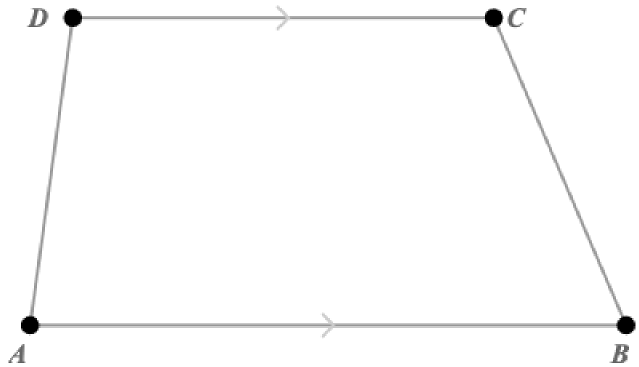


FIGURE 6.29

The Properties of a Trapezoid:

1. One pair of opposite sides are parallel, i.e., $AB \parallel CD$.

2. The two pairs of adjacent angles along the sides are supplementary, i.e., $\angle ABC + \angle BCD = 180^\circ$and $\angle CDA + \angle DAB = 180^\circ$.

Note: Some texts leave out the word "exactly", which means quadrilaterals with two pairs of parallel sides are sometimes considered trapezoids. Here, assume trapezoids have exactly one pair of parallel sides.

Isosceles Trapezoid

An isosceles trapezoid is a trapezoid with the non-parallel sides congruent. An additional property of isosceles trapezoids is base angles are congruent.

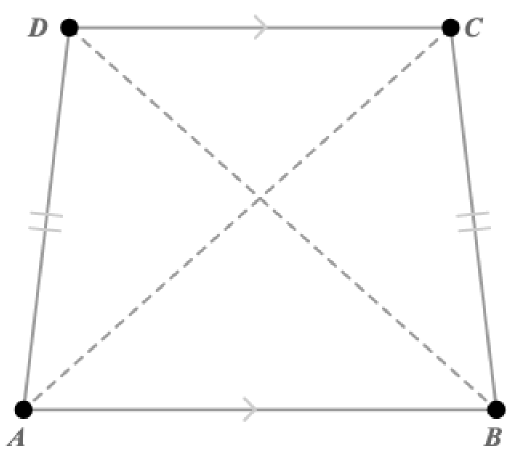


FIGURE 6.30

The Properties of an Isosceles Trapezoid:

1. One pair of opposite sides are parallel, i.e., $AB \parallel CD$.

2. Two pairs of adjacent angles are supplementary, i.e., $\angle ABC + \angle BCD = 180°$and $\angle CDA + \angle DAB = 180°$.

3. Base angles are congruent, i.e., $\angle ABC = \angle DAB$ and $\angle BCD = \angle CDA$.

4. The diagonals are congruent, i.e., $AC = BD$.

Parallelogram

A parallelogram is a quadrilateral with two pairs of parallel sides.

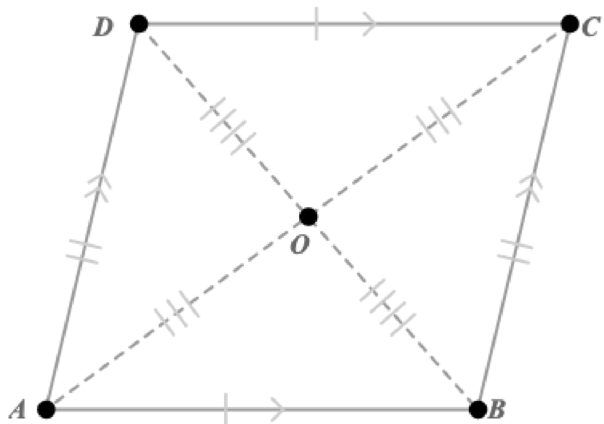


FIGURE 6.31

The Properties of a Parallelogram:

1. Opposite sides are parallel, i.e., $AB \parallel CD$ and $BC \parallel DA$.

2. Opposite sides are congruent, i.e., $AB = CD$ and $BC = DA$.

3. Opposite angles are congruent, i.e., $\angle ABC = \angle CDA$ and $\angle DAB = \angle BCD$.

4. Adjacent angles are supplementary, i.e., $\angle ABC + \angle BCD = 180°$, $\angle BCD + \angle CDA = 180°$, $\angle CDA + \angle DAB = 180°$and $\angle DAB + \angle ABC = 180°$.

5. Diagonals bisect each other, i.e., $BO = OD$ and $AO = OC$.

6. Each diagonal bisects the parallelogram, i.e., divides it into two congruent triangles $\Delta ABC \cong \Delta ADC$ and $\Delta BCD \cong \Delta BAD$.

Rectangle

A rectangle is a quadrilateral with four right angles. All rectangles are parallelograms.

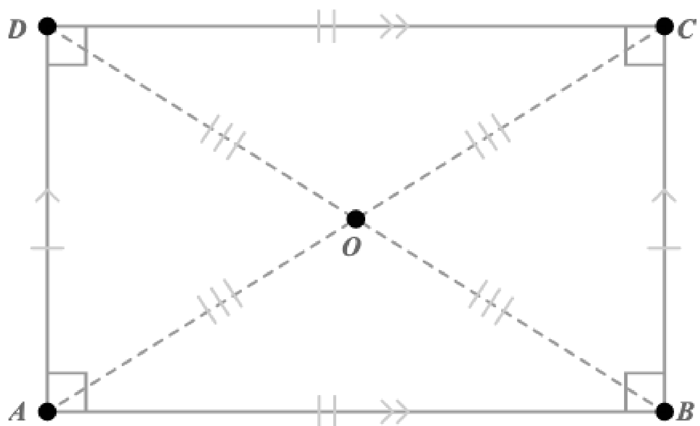


FIGURE 6.32

The Properties of a Rectangle:

1. Opposite sides are parallel, i.e., $AB \parallel CD$ and $BC \parallel DA$.

2. Opposite sides are congruent, i.e., $AB = CD$ and $BC = DA$.

3. All the four angles are congruent and measure 90°, i.e., $\angle ABC = \angle BCD = \angle CDA = \angle DAB = 90^\circ$.

4. Diagonals are congruent, i.e., $AC = BD$.

5. Diagonals bisect each other, i.e., $BO = OD$ and $AO = OC$.

Rhombus

A rhombus is a quadrilateral with four congruent sides. All rhombuses are parallelograms.

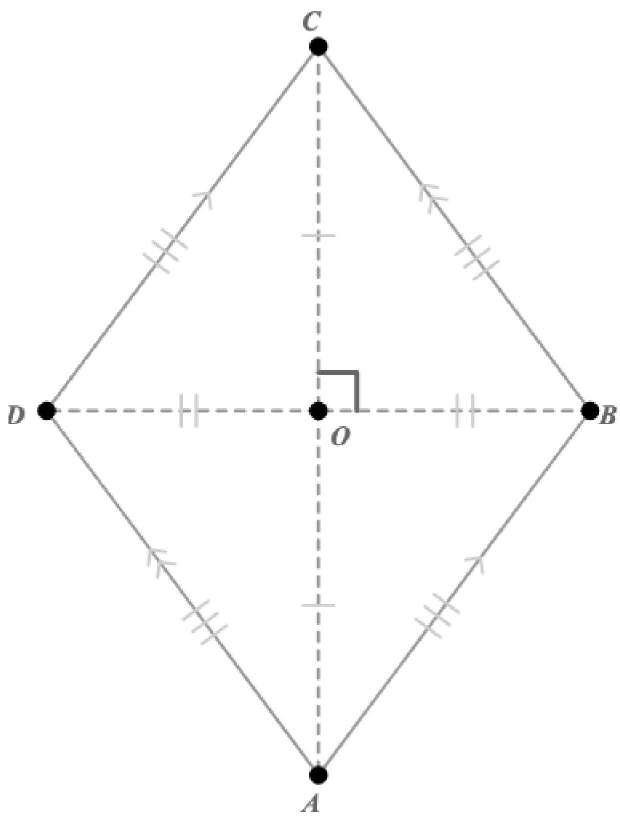


FIGURE 6.33

The Properties of a Rhombus:

1. Opposite sides are parallel, i.e., $AB \parallel CD$ and $BC \parallel DA$.

2. All four sides are congruent, i.e., $AB = BC = CD = DA$.

3. Opposite angles are congruent, i.e., $\angle ABC = \angle CDA$ and $\angle DAB = \angle BCD$.

4. Diagonals are the interior angle bisectors, i.e., $\angle BAC = \angle DAC$, $\angle BDC = \angle BDA$, $\angle DBC = \angle DBA$ and $\angle BCA = \angle DCA$.

5. Diagonals intersect each other at right angles, i.e., $\angle AOB = \angle BOC = \angle COD = \angle DOA = 90^\circ$.

Square

A square is a quadrilateral with four right angles and four congruent sides. All squares are rectangles and rhombuses.

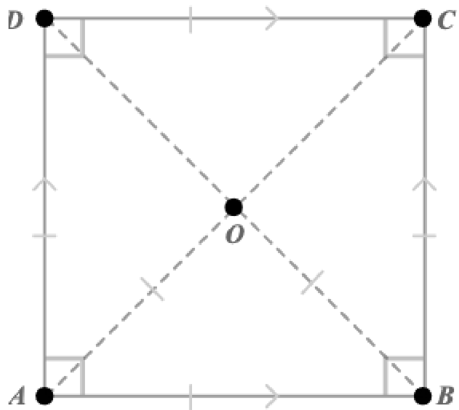


FIGURE 6.34

The Properties of a Square:

1. All four sides are congruent, i.e., $AB = BC = CD = DA$.
2. All the four angles are congruent and measures 90°, i.e., $\angle ABC = \angle BCD = \angle CDA = \angle DAB = 90^\circ$.
3. Diagonals are congruent, i.e., $AC = BD$.
4. Diagonals bisect each other at right angles, i.e., $\angle AOB = \angle BOC = \angle COD = \angle DOA = 90^\circ$.

All the angles of a quadrilateral are congruent and make right angles. What type of quadrilateral is this?

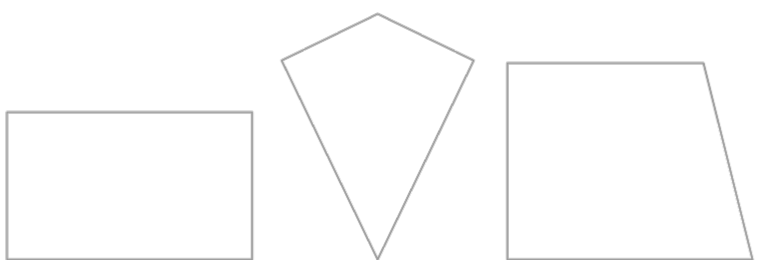


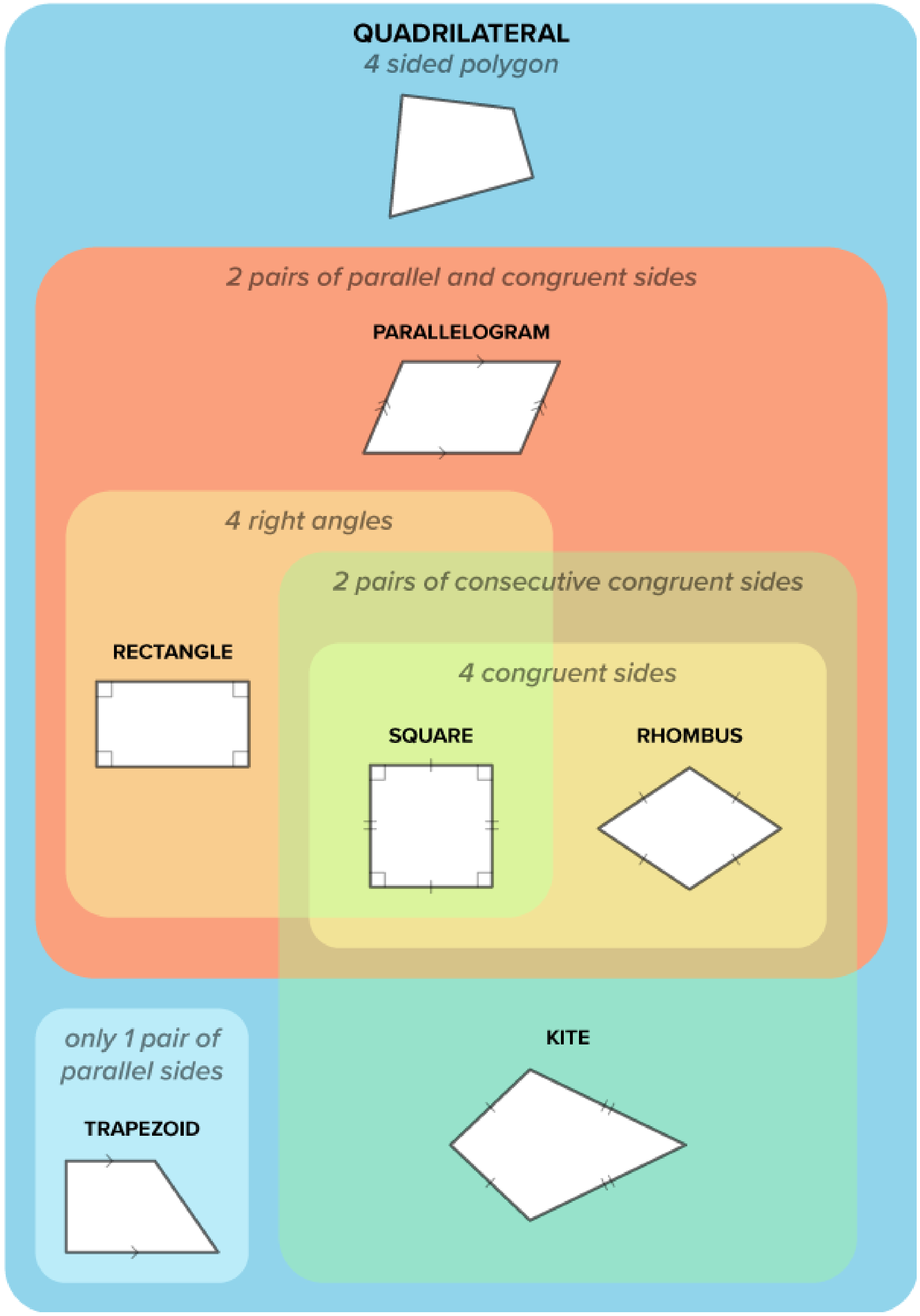
FIGURE 6.35

All the angles of a _____ are congruent and make right angles.

Hierarchy of quadrilaterals

Notice that properties of quadrilaterals overlap. A square is not only a square, but also a rhombus, a rectangle, a parallelogram, and a quadrilateral. This means that a square will have all the same properties as rhombuses, rectangles, parallelograms, and quadrilaterals.

The following diagram shows the hierarchy of quadrilaterals.



CK-12: Parallelogram Classification

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/parallelogram-classification/plix/Parallelogram-Classification-53f783465aa4130b5baa4b92

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/224135

Exploring Quadrilaterals

Play with a quadrilateral by dragging the vertex. Explore the angles and lengths for the given quadrilaterals.

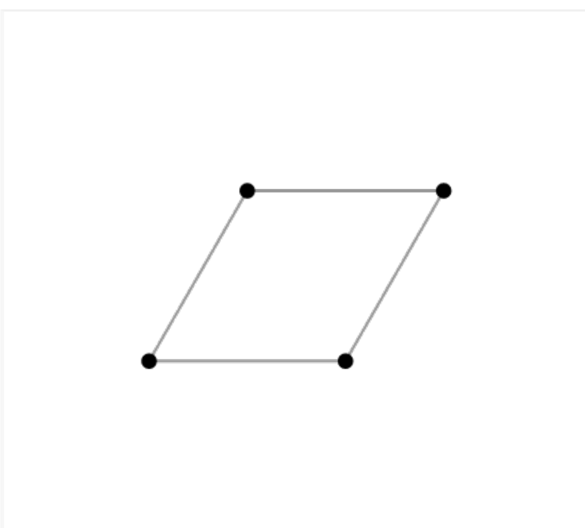


FIGURE 6.36

Solving for Unknown Values

Solve for x (picture not drawn to scale).

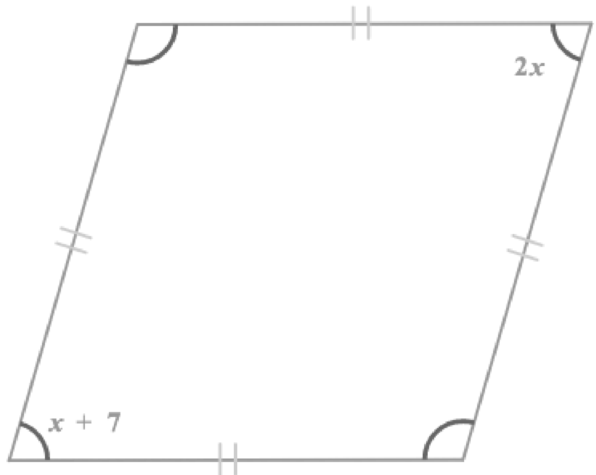


FIGURE 6.37

This quadrilateral is marked as having four congruent sides, so it is a rhombus. Rhombuses are parallelograms, so they have all the same properties as parallelograms. One property of parallelograms is that opposite angles are congruent. This means that the marked angles in this rhombus must be congruent.

$x+7=2x$
$x=7$

Examples

Example 1

All squares are rectangles, but not all rectangles are squares. How is this possible?

Answer: Rectangles are defined as quadrilaterals with four right angles. Squares are defined as quadrilaterals with four right angles and four congruent sides. Because all squares have four right angles and satisfy the definition for rectangles, they can all also be called rectangles. On the other hand, not all rectangles have four congruent sides, so not all rectangles can also be called squares.

Example 2

Draw a square. Draw in the diagonals of the square. Make at least one conjecture about the diagonals of the square.

Answer: To make a conjecture means to make an educated guess. There are a few conjectures you might make about the diagonals of a square. In other lessons, you will learn how these conjectures may be proven true.

Here are some possible conjectures:

1. diagonals of a square are congruent

2. diagonals of a square are perpendicular

3. diagonals of a square bisect each other (cut each other in half)
4. diagonals of a square bisect the angles (cut the 90°angles in half)

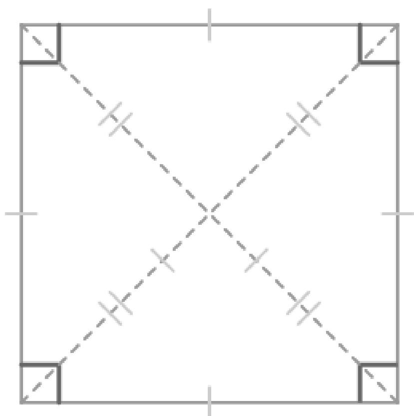


FIGURE 6.38

Example 3

A quadrilateral has four congruent sides. What type of quadrilateral must it be? What type of quadrilateral could it be?

Answer: It must be a rhombus and therefore also a parallelogram. It could be a square.

Example 4

Solve for x (picture not drawn to scale).

FIGURE 6.39

Answer: This is a parallelogram so opposite sides are congruent.

$3x+1 = 5x-12$
$2x = 13$
$x = 6.5$

CK-12 PLIX: Quadrilateral Classification

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/quadrilaterals/plix/Quadrilateral-Classification-528a53585aa41325fb80100e

Review

Decide whether each statement is always, sometimes, or never true. Explain your answer.

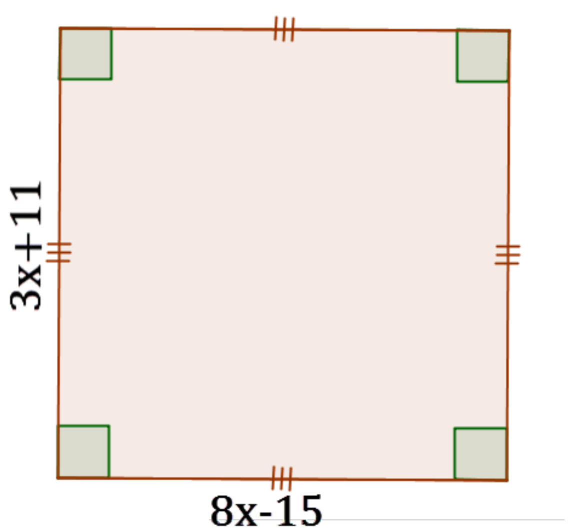
1. A square is a rectangle.
2. A rhombus is a square.
3. An isosceles trapezoid is a trapezoid.
4. A parallelogram is a quadrilateral.
5. A square is a parallelogram.
6. A trapezoid is a parallelogram.

Decide what type of quadrilateral it **must** be and what type of quadrilateral it **could** be based on the description.

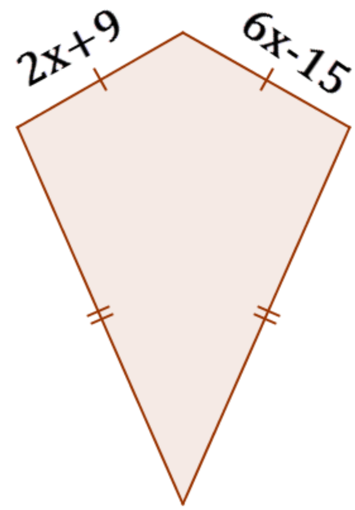
7. A quadrilateral has 4 congruent angles.
8. A quadrilateral has 2 pairs of congruent sides.
9. Draw a kite. Draw in its diagonals. Make at least one conjecture about the diagonals of kites.
10. Draw a rectangle. Draw in its diagonals. Make at least one conjecture about the diagonals of rectangles.
11. Draw a rhombus. Draw in its diagonals. Make at least one conjecture about the diagonals of rhombuses.
12. Draw a kite. Make a conjecture about the opposite angles of kites.

Use the markings on the shapes below to identify the shape. Then, solve for x. *Note: pictures are not drawn to scale.*

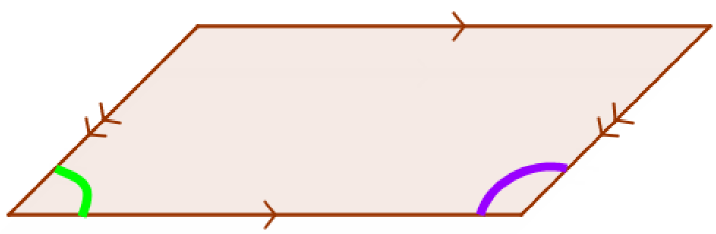
13.



14.



15. Make a conjecture about the adjacent angles of a parallelogram (such as the ones marked in the picture below). How must they be related?



16. If a quadrilateral is a rhombus, does it have all the properties of a parallelogram? Explain.

17. Sketch several different parallelograms. A parallelogram has many properties beyond its definition. Here are some **possible** properties of a parallelogram. Through experimentation with a variety of parallelograms, decide which are true or false, and explain why. Afterwards, write down a list of properties that are very likely to be true.

- Diagonals are the same length.
- Opposite angles are the same measure.
- Consecutive angles are the same measure.
- Consecutive sides are the same length.
- Diagonals bisect each other.
- Diagonals bisect the vertex angles at their endpoints.
- Opposite sides are congruent.
- Diagonals are perpendicular.
- All the angles are right.

18. Perform similar experiments with sketches of rectangles, rhombuses, squares, trapezoids, and isosceles trapezoids. Write down candidates for properties of these figures.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.5.

6.4 Area or Perimeter of Triangles and Quadrilaterals

Learning Objectives

Here you will review finding the area and perimeter of triangles and special quadrilaterals.

Perimeter

Perimeter is the distance around a shape. In other words, the total boundary length of a closed two-dimensional figure is called its perimeter. The perimeter of a circle or ellipse is called its circumference. To find the perimeter of any two dimensional shape, find the sum of the lengths of all the sides.

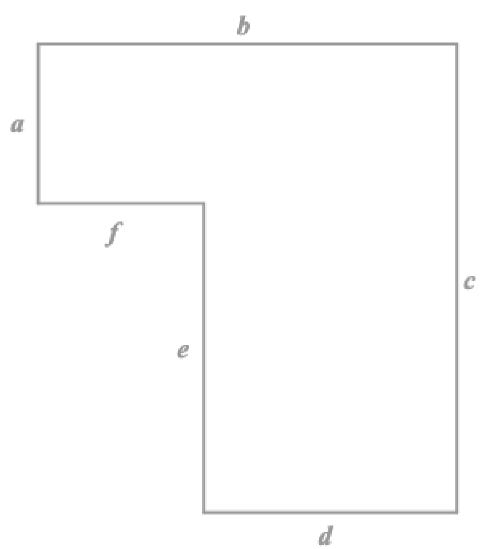


FIGURE 6.40

Area

Area is the amount of surface enclosed by a closed two-dimensional figure. It is measured by the number of unit squares it takes to cover a two dimensional shape. For example, if you count the small squares, you will find there are 15 of them. Therefore, the area is $3 \cdot 5$ or 15 unit^2.

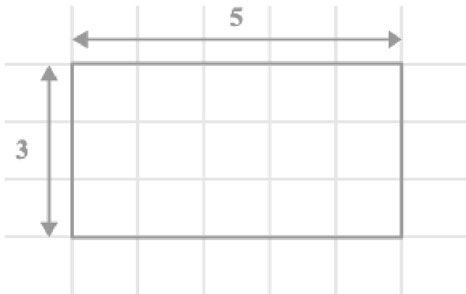


FIGURE 6.41

Area of Rectangle

A rectangle is a very basic shape for area calculation. The area of a rectangle is base times height.

$\text{Area}_{\text{rectangle}} = bh$

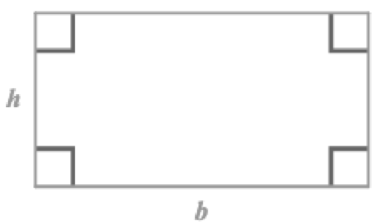


FIGURE 6.42

Area of Parallelogram

Any parallelogram can be turned into a rectangle without changing its area.

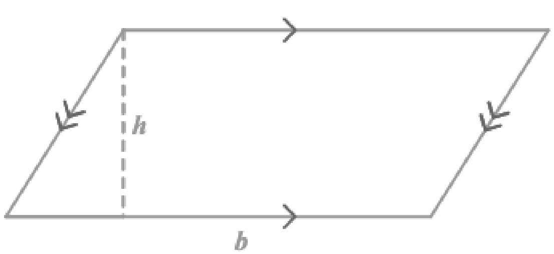


FIGURE 6.43

Therefore, the area of a parallelogram is also base times height.

$\text{Area}_{\text{parallelogram}} = bh$

Is it possible to find the area of a parallelogram by just multiplying the lengths of two adjacent sides?

It is _____ to find the area of a parallelogram by just multiplying the lengths of two adjacent sides.

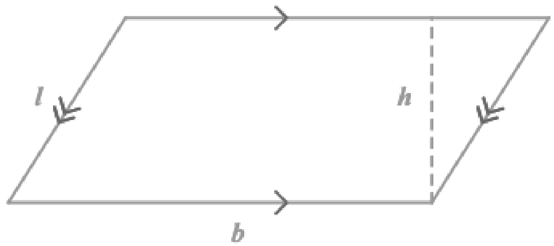


FIGURE 6.44

Area of Triangle

You can think of any triangle as half a parallelogram. If you rotate a triangle 180°about the midpoint of one of its sides, the original triangle and the new triangle will be a parallelogram.

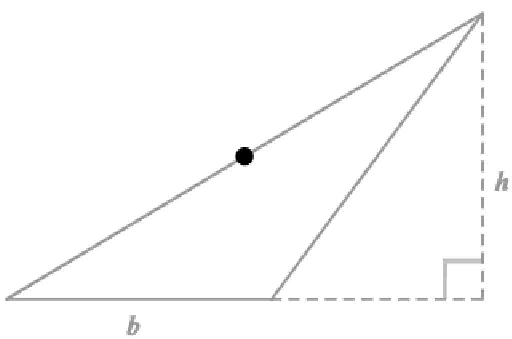


FIGURE 6.45

Therefore, the area of a triangle is base times height divided by two.

Remember that any of the three sides can be the base. Also remember that the height must be perpendicular to the base and extend to the highest point of the triangle.

$Area_{triangle} = \frac{bh}{2} = \frac{1}{2}bh$

Area of Trapezoid

A trapezoid can be thought of as half a parallelogram by rotating it 180°about the midpoint of one of its non-parallel sides.

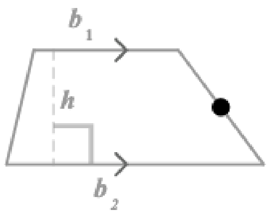


FIGURE 6.46

The base of this parallelogram is b_1+b_2 and the height is h. The area of the parallelogram is $(b_1+b_2)h$. Therefore, the area of the trapezoid is $\frac{(b_1+b_2)h}{2}$.

These and other area formulas that are good to know are shown below.

TABLE 6.2:

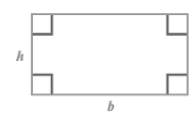
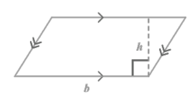
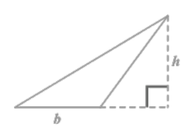
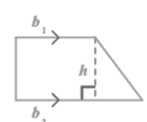
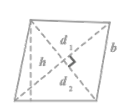
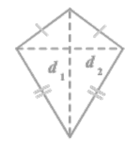
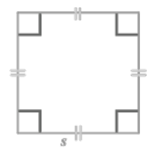
Shape	**Area Formula**	**Picture**
Rectangle	$A = bh$	
Parallelogram	$A = bh$	
Triangle	$A = \frac{bh}{2}$	
Trapezoid	$A = \frac{(b_1+b_2)h}{2}$	
Rhombus	$A = bh$ or $A = \frac{d_1 d_2}{2}$	

TABLE 6.2: (continued)

Kite	$A = \frac{d_1 d_2}{2}$	
Square	$A = s^2$	

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Calculating Area and Perimeter

Adjust the dimensions of the quadrilateral or the triangle given below by dragging the vertex and observe how the perimeter and area changes.

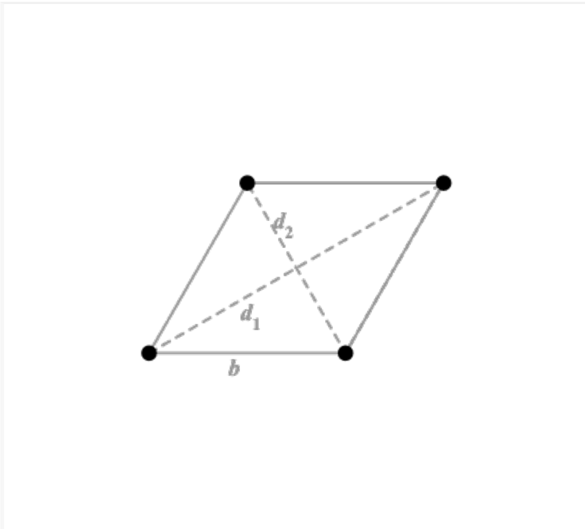


FIGURE 6.47

Finding the Area and Perimeter

1. Find the area of the trapezoid.

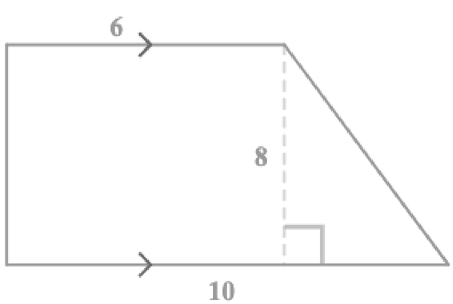


FIGURE 6.48

Given : height$(h) = 8$, base$(b_1) = 6$, base$(b_2) = 10$

Area of a trapezoid
$= \frac{1}{2}h(b_1 + b_2)$
$= \frac{1}{2}(8)(6+10)$
$= \frac{1}{2}(8)(16)$
$= 64$ units^2

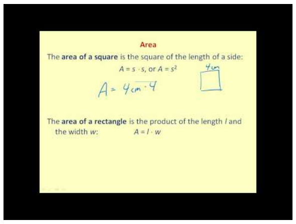
2. Find the area and perimeter of a square with side length 10 inches.

Perimeter is the distance around the shape. A square has four congruent sides, so each side is 10 inches.

Perimeter of a square $= 4 \cdot$ sides
Perimeter of a square $= 4(10) = 40$ in

Area is the number of square units it takes to cover the shape.

Area of a square $=$ sides^2
Area of a square $= 10^2 = 100\text{ in}^2$.



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Examples

Example 1

For both a rhombus and a kite, area can be found if you know the lengths of the diagonals. What is special about the diagonals of these shapes?

The diagonals of both rhombuses and kites are perpendicular. This means that when the shapes are broken down into triangles, one diagonal can be the base of the triangle and a portion of the other diagonal will be the height.

Example 2

The diagonals of a rhombus bisect each other. This means that they cut each other in half. The diagonals are also perpendicular. Derive the area formula for a rhombus $A = \frac{d_1 d_2}{2}$.

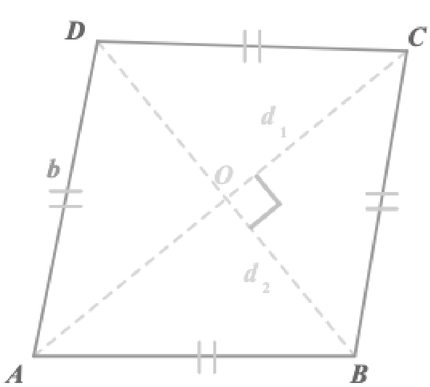

FIGURE 6.49

The diagonals divide the rhombus into four congruent triangles. For each of the triangles, the base and height are $\frac{d_1}{2}$ and $\frac{d_2}{2}$.

Area of rhombus $ABCD$

$= \Delta AOB + \Delta BOC + \Delta COD + \Delta DOA$

$= \left(\frac{1}{2} \cdot AO \cdot BO\right) + \left(\frac{1}{2} \cdot BO \cdot CO\right) + \left(\frac{1}{2} \cdot CO \cdot DO\right) + \left(\frac{1}{2} \cdot DO \cdot AO\right)$

$= \left(\frac{1}{2} \cdot \frac{d_1}{2} \cdot \frac{d_2}{2}\right) + \left(\frac{1}{2} \cdot \frac{d_1}{2} \cdot \frac{d_2}{2}\right) + \left(\frac{1}{2} \cdot \frac{d_1}{2} \cdot \frac{d_2}{2}\right) + \left(\frac{1}{2} \cdot \frac{d_1}{2} \cdot \frac{d_2}{2}\right)$

The rhombus is made up of four triangles with the same area $4\left(\frac{d_1 d_2}{8}\right) = \frac{d_1 d_2}{2}$.

Example 3

Explain why the formula $A = bh$ works to find the area of a rhombus.

$A = bh$ is the formula for the area for any parallelogram. Since a rhombus is a parallelogram, this area formula works for a rhombus.

CK-12 PLIX Interactive

Multimedia

PLIX

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URL: http://www.ck12.org/geometry/area-of-a-parallelogram/plix/Area-of-a-Parallelogram-542c8fb58e0e084496320733

Review

1. Find the area and perimeter of a rectangle with a length of 12 inches and a width of 15 inches.

2. Find the area and perimeter of a right triangle with legs 3 cm and 4 cm and hypotenuse 5 cm.

3. Find the area of a trapezoid with bases 4 cm and 12 cm and height 9 cm.

4. The perimeter of a rectangle is 150 cm. The length is 4 times more than the width. What is the length of the rectangle?

5. The area of a triangle is 30 cm^2. The base is twice as long as the height. What is the height of the triangle?

6. The perimeter of a right triangle is 24 in. The area is 24 in^2. The hypotenuse is 4 inches longer than the base. What are the lengths of the sides of the triangle?

7. Why is it necessary to use square units (such as in^2) when referring to area?

8. Why don't you use square units when referring to perimeter?

9. When finding the area of a trapezoid, does it matter which sides you label as b_1 and b_2?

10. Find the area of a square with a diagonal of 12 in.

11. Explain in your own words why you divide by two in the formula for the area of a triangle.

12. Explain in your own words why you divide by two in the formula for the area of a trapezoid.

13. Find the area of a kite with diagonals 10 cm and 18 cm.
14. Derive the formula $A = s^2$ for the area of a square.

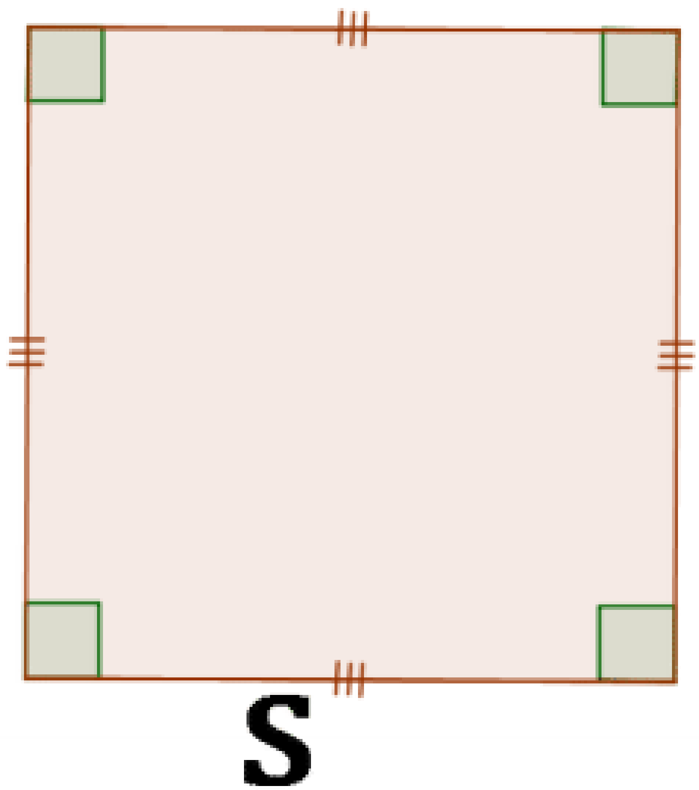


FIGURE 6.50

15. The diagonals of a kite are perpendicular and one diagonal bisects the other diagonal. Derive the formula $A = \frac{d_1 d_2}{2}$ for the area of a kite.

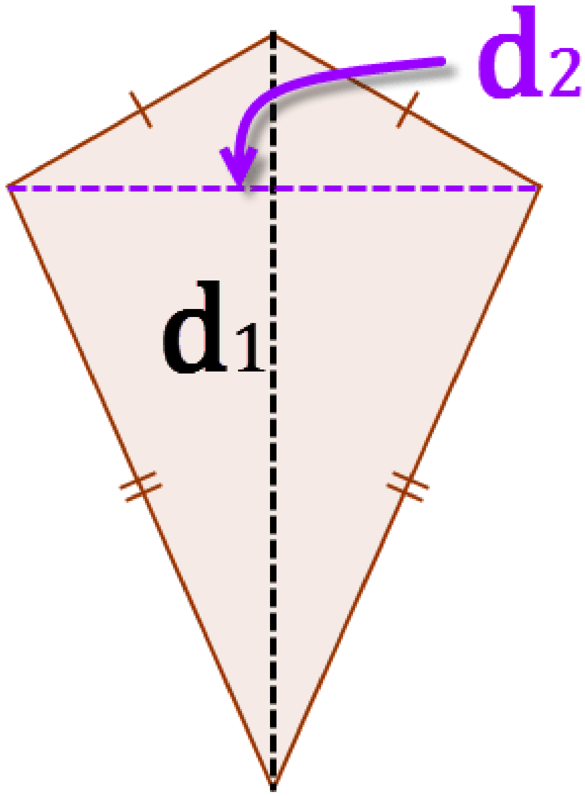


FIGURE 6.51

16. Sketch and assign dimensions to a rectangle and parallelogram (one that isn't a rectangle) such that they have the same area.

17. Sketch and assign dimensions to a rhombus and a rectangle (neither of which are squares) and give them dimensions such that they have the same area.

18. Sketch and assign dimensions to a trapezoid and a triangle such that they have the same area.

19. Sketch and assign dimensions to a rectangle. Find its area and perimeter. Double the dimensions to create a new rectangle, and find the area and perimeter. What do you observe? Why is this the case? Experiment with other figures and other scale factors to see if these results hold true.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.6.

6.5 Theorems about Quadrilaterals

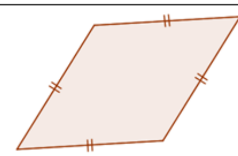
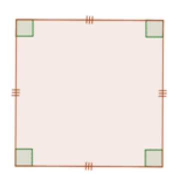
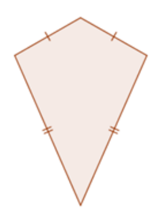
Learning Objectives

Here you will learn theorems about quadrilaterals and how to prove them.

Is a rectangle a parallelogram? If so, what does this mean about the properties of a rectangle?

Theorems about Quadrilaterals

A quadrilateral is a polygon with four sides. Five special quadrilaterals are shown below, along with their definitions and pictures.

Quadrilateral	Definition	Picture
Parallelogram	A quadrilateral with two pairs of parallel sides.	
Rectangle	A quadrilateral with four right angles.	
Rhombus	A quadrilateral with four congruent sides.	
Square	A quadrilateral with four right angles and four congruent sides.	
Kite	A convex quadrilateral with two pairs of adjacent congruent sides such that not all sides are congruent.	

The formal definitions of these quadrilaterals only give some information about them. Each quadrilateral has other properties that can be proved. For example, while a parallelogram is defined as a quadrilateral with two pairs of parallel sides, it can be proved that the opposite sides of a parallelogram must be congruent.

Suppose you are given a quadrilateral and believe that it is a parallelogram. You can prove that it is a parallelogram by showing that it has two pairs of parallel sides; however, you can also use other properties that are unique to parallelograms to prove that the given shape is a parallelogram.

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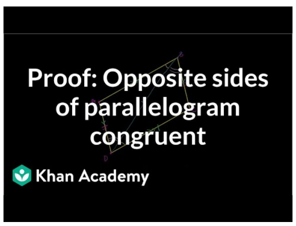

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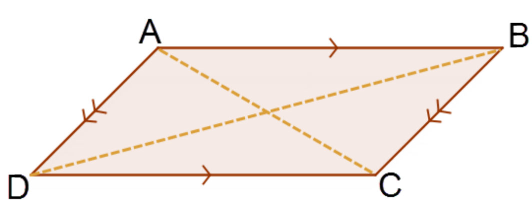
Let's prove that the diagonals of a parallelogram divide the parallelogram into congruent triangles. Then, we'll use this to prove that the opposite sides of a parallelogram are congruent.

1. Start by drawing a generic parallelogram and previewing this proof.

All you can assume in this proof is the definition of a parallelogram. This means that all you know is that the shape is a quadrilateral with two pairs of parallel sides. Your first goal is to prove that the diagonals divide the parallelogram into congruent triangles. You can use the parallel lines to give you congruent angles, which will help you to prove that the triangles are congruent.

Your second goal is to prove that the opposite sides of the parallelogram are congruent. Since the opposite sides of the parallelogram are corresponding parts of the congruent triangles, you can use CPCTC to show that they must be congruent.



Given: Parallelogram $ABCD$

Prove: $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$

Here is a two-column proof:

TABLE 6.3:

Statements	Reasons

TABLE 6.3: (continued)

Parallelogram $ABCD$	Given
$\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$	Definition of a parallelogram
$\angle BAC \cong \angle DCA$ $\angle ABD \cong \angle BDC$ $\angle DAC \cong \angle BCA$ $\angle ADB \cong \angle DBC$	Alternate interior angles are congruent if lines are parallel
$\overline{BD} \cong \overline{BD}$ $\overline{AC} \cong \overline{AC}$	Reflexive Property
$\Delta ADB \cong \Delta CBD$ $\Delta ADC \cong \Delta CBA$	$ASA \cong$
$\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$	CPCTC

You have now proven two theorems about parallelograms. You can use these theorems in future proofs without proving them again.

Parallelogram Theorem #1: Each diagonal of a parallelogram divides the parallelogram into two congruent triangles.

Parallelogram Theorem #2: The opposite sides of a parallelogram are congruent.

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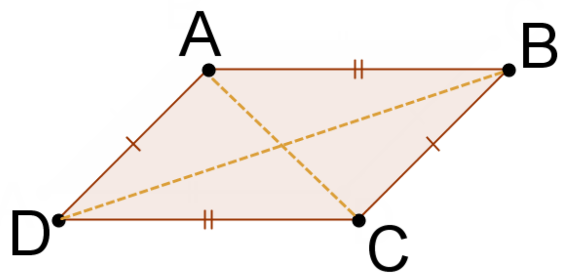
2. Now, let's prove that if a quadrilateral has opposite sides congruent, then its diagonals divide the quadrilateral into congruent triangles. Use this to prove that the quadrilateral must be a parallelogram.

These are the **converses** of the parallelogram theorems proved in Example A. Draw a generic quadrilateral with two pairs of congruent sides and preview the proof.

Your first goal is to show that the set of triangles created by each diagonal must be congruent. You can use the congruent opposite sides as well as the reflexive property to show that the triangles are congruent with $SSS \cong$.

Your second goal is to prove that the opposite sides must be parallel and so therefore the quadrilateral is a parallelogram. You can show that alternate interior angles are congruent and hence lines are parallel for this part of the proof.



Given: Quadrilateral $ABCD$ with $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$.

Prove: $ABCD$ is a parallelogram

Here is a two-column proof:

TABLE 6.4:

Statements	Reasons
$\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$	Given
$\overline{BD} \cong \overline{BD}$ $\overline{AC} \cong \overline{AC}$	Reflexive Property
$\Delta ADB \cong \Delta CBD$ $\Delta ADC \cong \Delta CBA$	$SSS \cong$
$\angle BAC \cong \angle DCA$ $\angle ABD \cong \angle BDC$ $\angle DAC \cong \angle BCA$ $\angle ADB \cong \angle DBC$	CPCTC
$\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$	If alternate interior angles are congruent then lines are parallel
$ABCD$ is a parallelogram	Definition of a parallelogram

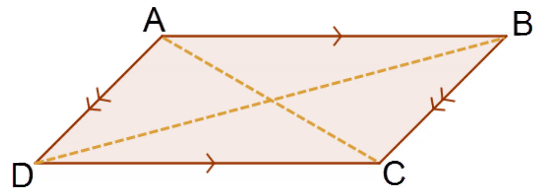
You have now proven the converses of the first two parallelogram theorems. These are two additional ways to show that a quadrilateral is a parallelogram besides showing that the quadrilateral satisfies the definition of a parallelogram.

Parallelogram Theorem #1 Converse: If each of the diagonals of a quadrilateral divide the quadrilateral into two congruent triangles, then the quadrilateral is a parallelogram.

Parallelogram Theorem #2 Converse: If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

3. To get another theorem for parallelograms, let's prove that the opposite angles of a parallelogram are congruent.

Draw a generic parallelogram and preview the proof. Your goal is to prove that the opposite angles are congruent. Remember that you can use the theorems that have already been proven about parallelograms, so you can use the fact that the triangles created by the diagonals must be congruent.



Given: Parallelogram $ABCD$.

Prove: $\angle A \cong \angle C$ and $\angle B \cong \angle D$

Here is a paragraph proof:

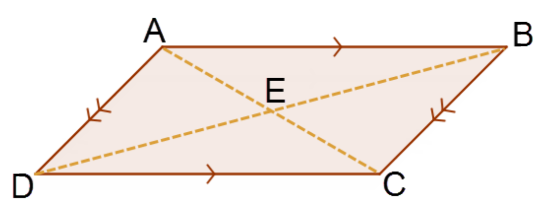
Because each diagonal of a parallelogram divides the parallelogram into two congruent triangles, $\Delta ADB \cong \Delta CBD$ and $\Delta ADC \cong \Delta CBA$. $\angle A$ and $\angle C$ are corresponding parts of ΔADB and ΔCBD. Similarly, $\angle B$ and $\angle D$ are corresponding parts of ΔADC and ΔCBA. This means that $\angle A \cong \angle C$ and $\angle B \cong \angle D$ because corresponding parts of congruent triangles are congruent.

You have now proven a third theorem about parallelograms. You can use this theorem in future proofs without proving it again.

Parallelogram Theorem #3: The opposite angles of a parallelogram are congruent.

4. Finally, let's prove that the diagonals of a parallelogram bisect each other.

Draw a generic parallelogram and preview the proof.



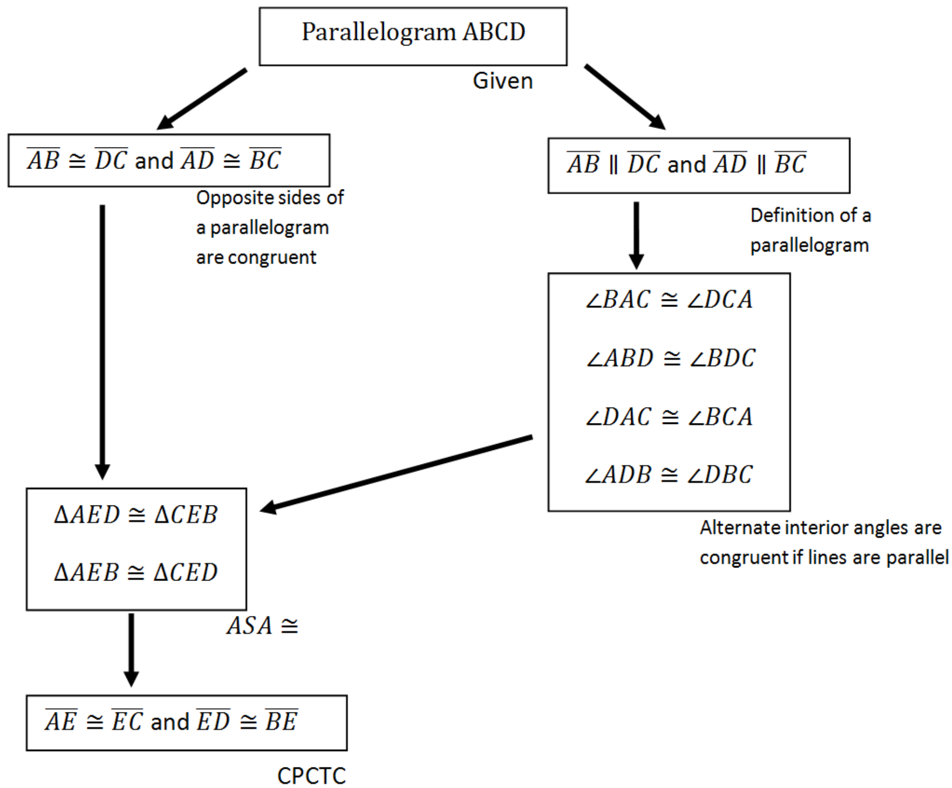
What does it mean for the diagonals to bisect each other? If the diagonals bisect each other then each diagonal crosses the other diagonal at its midpoint. This would mean that $\overline{AE} \cong \overline{EC}$ and $\overline{ED} \cong \overline{BE}$.

Your goal is to use the parallelogram definition and theorems to show that $\overline{AE} \cong \overline{EC}$ and $\overline{ED} \cong \overline{BE}$. First, try to prove that the two diagonals divide the parallelogram into four small triangles, and each pair of these triangles is congruent.

Given: Parallelogram $ABCD$

Prove: $\overline{AE} \cong \overline{EC}$ and $\overline{ED} \cong \overline{BE}$ (the diagonals bisect each other)

Here is a flow diagram proof.



You have now proven a fourth theorem about parallelograms. You can use this theorem in future proofs without proving it again.

Parallelogram Theorem #4: The diagonals of a parallelogram bisect each other.

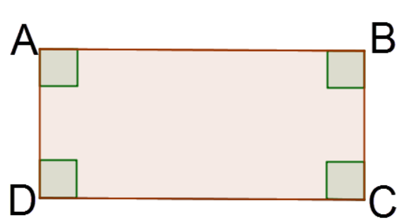
Examples

Example 1

Earlier, you were asked if a rectangle is a parallelogram.

To prove that a rectangle is a parallelogram, you must prove that it either satisfies **the definition of a parallelogram** or satisfies **any of the theorems that prove that quadrilaterals are parallelograms**.

A rectangle is a quadrilateral with four right angles. You can use these angles to show that the opposite sides of a rectangle must be parallel.



Given: Rectangle $ABCD$

Prove: $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$

Here is a paragraph proof:

A rectangle has four right angles by definition, so $m\angle A = m\angle B = m\angle C = m\angle D = 90^\circ$. $\angle A$ and $\angle D$ are same side interior angles. $m\angle A + m\angle D = 180^\circ$, which means that $\angle A$ and $\angle D$ are supplementary. If same side interior angles are supplementary, then lines are parallel. This means that $\overline{AB} \parallel \overline{DC}$. Similarly, $\angle A$ and $\angle B$ are same side interior angles. $m\angle A + m\angle B = 180^\circ$, which means that $\angle A$ and $\angle B$ are supplementary. If same side interior angles are supplementary, then lines are parallel. This means that $\overline{AD} \parallel \overline{BC}$.

You have proven that a rectangle is a parallelogram.

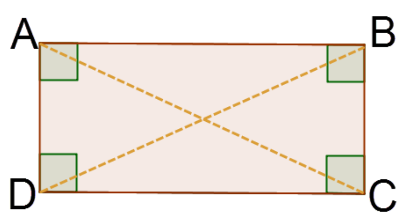
Rectangle Theorem #1: A rectangle is a parallelogram.

This means that rectangles have all the same properties as parallelograms. Like parallelograms, rectangles have opposite sides congruent and parallel and diagonals that bisect each other.

Example 2

Prove that the diagonals of a rectangle are congruent.

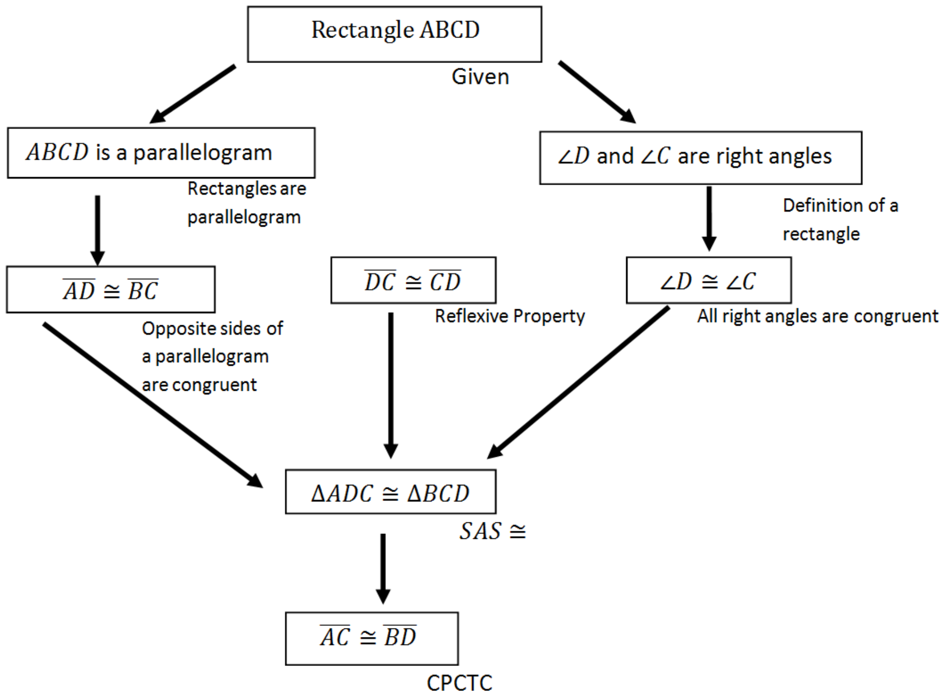
Draw a rectangle with its diagonals and preview the proof. To prove that the diagonals are congruent, you will first want to prove that $\Delta ADC \cong \Delta BCD$. These are two right triangles and their hypotenuses are the diagonals of the rectangle. If you can prove that these two right triangles are congruent, then you can prove that the diagonals are congruent.



Given: Rectangle $ABCD$

Prove: $\overline{AC} \cong \overline{BD}$

Here is a flow diagram proof:



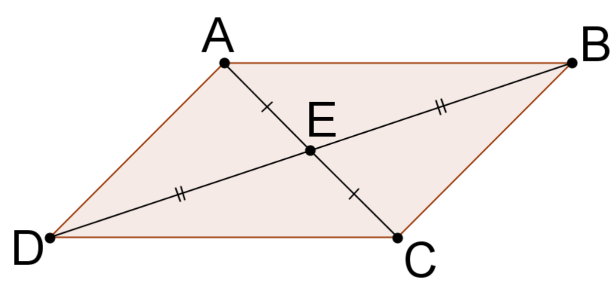
You have proven that a rectangle has congruent diagonals. You can now use this theorem in future proof.

Rectangle Theorem #2: A rectangle has congruent diagonals.

Example 3

Prove that if a quadrilateral has diagonals that bisect each other, then it is a parallelogram.

This is the converse of parallelogram theorem #4 from Example C. Draw a quadrilateral with diagonals that bisect each other and preview the proof.

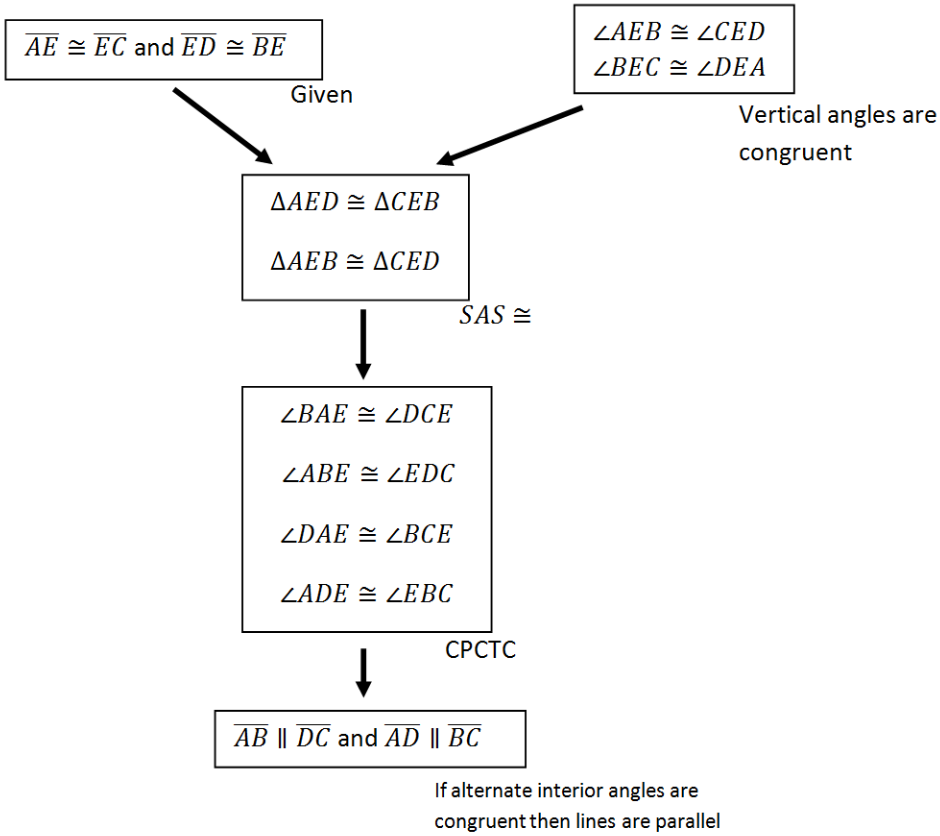


Your goal will be to show that there are two pairs of congruent triangles created by these diagonals. Then, you can show that alternate interior angles are congruent and therefore the opposite sides must be parallel.

Given: $\overline{AE} \cong \overline{EC}$ and $\overline{ED} \cong \overline{BE}$

Prove: $\overline{AB} \| \overline{DC}$ and $\overline{AD} \| \overline{BC}$ ($ABCD$ is a parallelogram)

Here is a flow diagram proof:



You have now proven the converse of the fourth parallelogram theorem. This is one additional way to show that a quadrilateral is a parallelogram.

Parallelogram Theorem #4 Converse: If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Example 4

Prove that a rhombus is a parallelogram. What does this tell you about the properties of a rhombus?

Is a rhombus a parallelogram? This would mean that a rhombus has opposite sides that are parallel. To prove that a rhombus is a parallelogram, you must prove that it either satisfies **the definition of a parallelogram** or satisfies **any of the theorems that prove that quadrilaterals are parallelograms**.

Here is a paragraph proof:

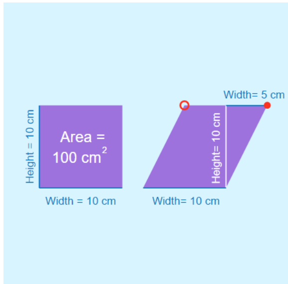
A rhombus is a quadrilateral with four congruent sides, therefore opposite sides of a rhombus are congruent. Parallelogram theorem #2 converse states that "if the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram". Therefore, a rhombus is a parallelogram.

You have proven that a rhombus is a parallelogram.

Rhombus Theorem #1: A rhombus is a parallelogram.

This means that rhombuses have all the same properties as parallelograms. Like parallelograms, rhombuses have opposite sides parallel, opposite angles congruent and diagonals that bisect each other.

CK-12 PLIX Interactive



PLIX

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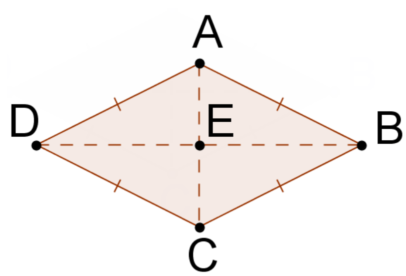
URL: http://www.ck12.org/geometry/area-of-a-parallelogram/plix/Area-of-a-Parallelogram-542c8fb58e0e084496320733

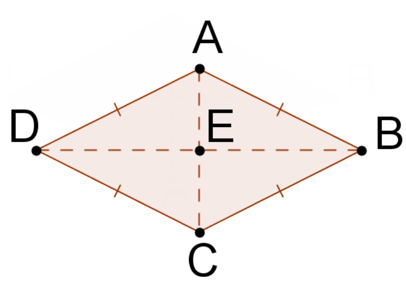
Review

1. State the definition of a parallelogram and three additional properties of a parallelogram.
2. State the definition of a rectangle and four additional properties of a rectangle.
3. Use the rhombus below to prove that a rhombus has perpendicular diagonals.



   1. Use the definition of a rhombus and the theorems about parallelograms to prove that $\Delta AED \cong \Delta AEB$.
   2. Prove that $m\angle AED = m\angle AEB$.
   3. Prove that $\angle AED$ is a right angle.
   4. Prove that $AC \perp DB$.

4. Use the rhombus below to prove that a rhombus has diagonals that bisect its angles.



   1. Use the definition of a rhombus and the theorems about parallelograms to prove that $\Delta ADC \cong \Delta ABC$ and $\Delta ADB \cong \Delta CDB$.

2. Prove that there are congruent angles and therefore the diagonals have bisected the angles.

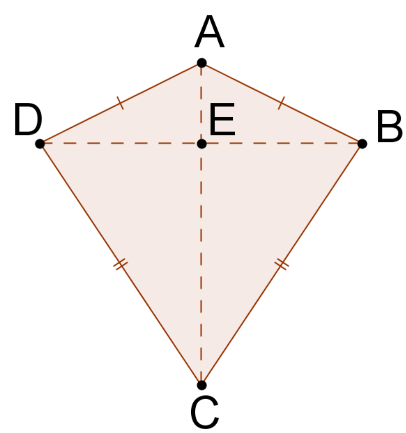
5. State the definition of a rhombus and five additional properties of a rhombus.

6. Is a square a rectangle? Is a square a rhombus? Explain.

7. Is a square a parallelogram? Explain.

8. State the definition of a square and five additional properties of a square.

Use the kite below for the proofs in #9-#13.



9. Prove that one diagonal of a kite divides it into two congruent triangles.

10. Prove that a kite has one pair of opposite angles congruent. *Hint: Use the result to #9!*

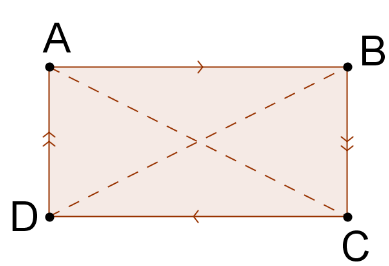
11. Prove that one diagonal of a kite bisects its angles. *Hint: Draw the diagonal that bisects its angles. Use the result of #9 and then CPCTC.*

12. Prove that one diagonal of a kite is bisected by another diagonal. *Hint: Draw both diagonals. Prove that two of the four small triangles are congruent and then use CPCTC. Use the result of #11 to help.*

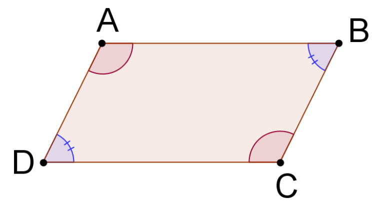
13. Prove that the diagonals of a kite are perpendicular. *Hint: Use the result of #11 and a similar method to the one that was used in #3!*

14. State the definition of a kite and four additional properties of a kite.

15. Use the picture below with $\overline{AC} \cong \overline{BD}$ to help prove that a parallelogram with congruent diagonals is a rectangle.



16. Use the picture below to help prove that a quadrilateral with opposite angles congruent is a parallelogram. *Hint: Use the fact that the sum of the interior angles of a quadrilateral is* 360°.



17. An isosceles trapezoid has two congruent legs. One of the properties of an isosceles trapezoid is that base angles (formed with either base) are congruent. You may use that property in the following proof. Prove that diagonals in an isosceles trapezoid are congruent.

18. Given: ABCD is a parallelogram; $\overline{AF} \cong \overline{CE}$

Prove: $\overline{FB} \cong \overline{DE}$

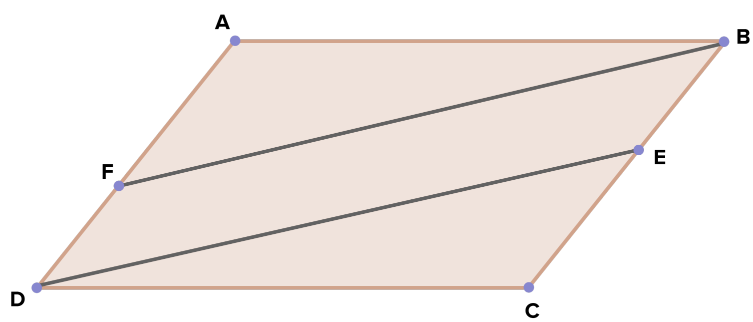


FIGURE 6.52

19. Prove that if one pair of sides in a quadrilateral is both congruent and parallel, then the quadrilateral is a parallelogram.

Review (Answers)

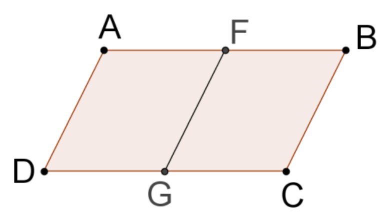
To see the Review answers, open this PDF file and look for section 4.7.

6.6 Applications of Quadrilateral Theorems

Learning Objectives

Here you will use quadrilateral theorems to solve problems.

Consider parallelogram $ABCD$ below.



F is the midpoint of $\overline{AB}$ and G is the midpoint of $\overline{DC}$. Make at least one conjecture about how $\overline{FG}$ is related to $\overline{AD}$.

Applications of Quadrilateral Theorems

Recall that many properties of special quadrilaterals can be proved from the definitions of these quadrilaterals. Below, quadrilateral definitions and the properties that you have previously proved are summarized.

Parallelogram: A quadrilateral with two pairs of parallel sides.

- **Parallelogram Theorem #1:** Each diagonal of a parallelogram divides the parallelogram into two congruent triangles.
- **Parallelogram Theorem #1 Converse:** If each of the diagonals of a quadrilateral divide the quadrilateral into two congruent triangles, then the quadrilateral is a parallelogram.
- **Parallelogram Theorem #2:** The opposite sides of a parallelogram are congruent.
- **Parallelogram Theorem #2 Converse:** If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
- **Parallelogram Theorem #3:** The opposite angles of a parallelogram are congruent.
- **Parallelogram Theorem #3 Converse:** If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
- **Parallelogram Theorem #4:** The diagonals of a parallelogram bisect each other.
- **Parallelogram Theorem #4 Converse:** If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Rectangle: A quadrilateral with four right angles.

- **Rectangle Theorem #1:** A rectangle is a parallelogram.
- **Rectangle Theorem #2:** A rectangle has congruent diagonals.
- **Rectangle Theorem #2 Converse:** If a parallelogram has congruent diagonals, then it is a rectangle.

Rhombus: A quadrilateral with four congruent sides.

- **Rhombus Theorem #1:** A rhombus is a parallelogram.
- **Rhombus Theorem #2:** The diagonals of a rhombus are perpendicular.
- **Rhombus Theorem #3:** The diagonals of a rhombus bisect its angles.

Square: A quadrilateral with four right angles and four congruent sides.

Kite: A quadrilateral with two pairs of adjacent, congruent sides.

- **Kite Theorem #1:** One diagonal of a kite bisects the other diagonal.
- **Kite Theorem #2:** The diagonals of a kite are perpendicular.
- **Kite Theorem #3:** One diagonal of a kite bisects its angles.
- **Kite Theorem #4:** A kite has one pair of opposite angles congruent.

You can use the definitions and proved properties of each of these quadrilaterals to solve problems about the quadrilaterals.

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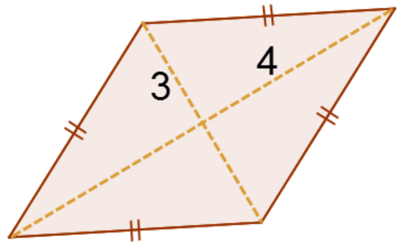
Finding the Perimeter

Find the perimeter of the quadrilateral below.



The markings indicate that the quadrilateral has four congruent sides. This means that the quadrilateral is a rhombus. One property of a rhombus is that its diagonals are perpendicular. This means that the four inner triangles are actually right triangles. You can use the Pythagorean Theorem to find the hypotenuse of one of these triangles, which will be the length of each side of the rhombus.

$$
\begin{aligned}
3^2+4^2 &= c^2 \\
9+15 &= c^2 \\
5 &= c
\end{aligned}
$$

Since each side of the rhombus is 5 units long, the perimeter of the rhombus is $P = 5 \cdot 4 = 20$ units.

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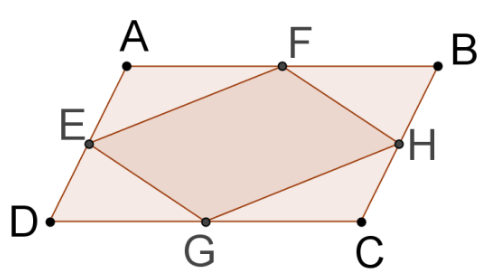
Making and Proving Conjectures

The midpoints of each of the sides of parallelogram *ABCD* have been connected, as shown below. Make a conjecture about the inner quadrilateral that has been formed. Then, prove the conjecture.



The inner quadrilateral looks to be a parallelogram. To prove it is a parallelogram, prove that its opposite sides are congruent. (Parallelogram Theorem #2 Converse states that if the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.)

Given: Parallelogram *ABCD* with midpoints E, F, H, G.

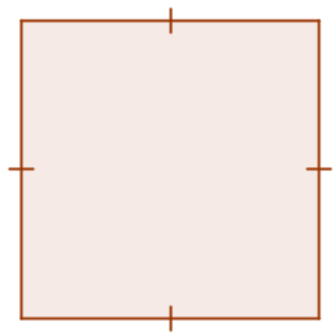
Prove: $EFHG$ is a parallelogram

Here is paragraph proof:

Because $ABCD$ is a parallelogram, its opposite sides and angles are congruent. This means that $\angle C \cong \angle A$, $\angle B \cong \angle D$, $\overline{AB} \cong \overline{DC}$, and $\overline{AD} \cong \overline{BC}$. E, F, G, and H are midpoints, so they divide each segment into two congruent segments. Because opposite sides of $ABCD$ are congruent, it must be true that $\overline{AE} \cong \overline{CH}$, $\overline{AF} \cong \overline{GC}$, $\overline{FB} \cong \overline{GD}$, and $\overline{ED} \cong \overline{BH}$. Therefore, $\Delta AEF \cong \Delta CHG$ and $\Delta BFH \cong \Delta DGE$ by $SAS \cong$. This means that corresponding parts of these triangles are congruent and therefore $\overline{EF} \cong \overline{DH}$ and $\overline{FH} \cong \overline{DG}$. Because opposite sides of $EFHG$ are congruent, $EFHG$ is a parallelogram.

Naming Shapes

Name the shape below based on its markings as precisely as you can. Don't assume that the shape is drawn to scale.

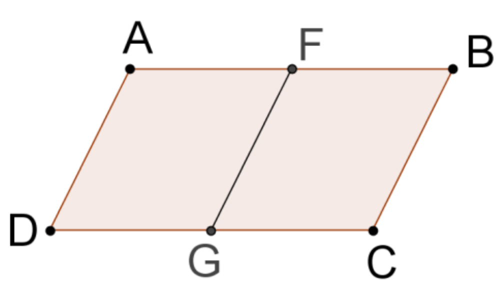


All that is marked is that the shape has four congruent sides. This is the definition of a rhombus so it must be a rhombus. Note that even though it might look like a square, if you don't know for sure that the shape has right angles, you cannot be sure it is a square.

Examples

Example 1

Earlier, you were asked to make at least one conjecture how $\overline{FG}$ is related to $\overline{AD}$.

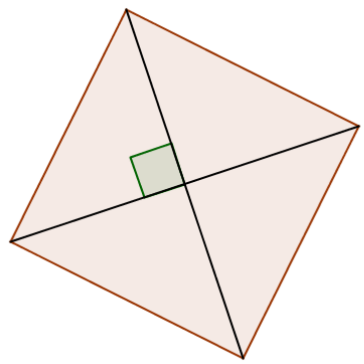


Two possible conjectures are:

1. $\overline{FG} \parallel \overline{AD}$
2. $\overline{FG} \cong \overline{AD}$

Example 2

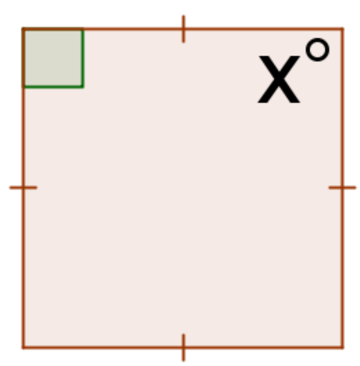
Name the shape below based on its markings as precisely as you can. Don't assume that the shape is drawn to scale.



It is marked that the diagonals are congruent. Shapes with congruent diagonals are rhombuses, kites, and squares. Without additional information, you cannot say whether this is a rhombus, a kite, or a square. Therefore, all you can say for certain is that this shape is a quadrilateral.

Example 3

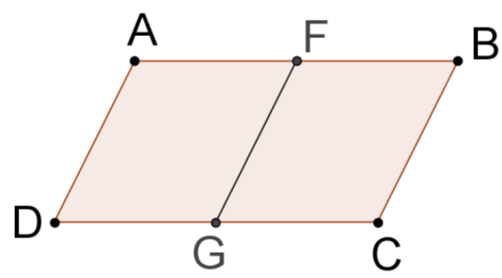
Use the markings on the shape to name the shape. Then, solve for x.



The shape has four congruent sides which makes it a rhombus. It also has one right angle. Opposite sides of a rhombus are parallel, so same side interior angles are supplementary. This means that $x = 90$. Because opposite angles of a rhombus are congruent, all four angles must be right angles. Therefore, a more precise name for this shape is a square.

Example 4

In parallelogram $ABCD$, F is the midpoint of $\overline{AB}$ and G is the midpoint of $\overline{DC}$. Prove that $\overline{FG} \cong \overline{AD}$.



Draw in $\overline{AG}$. Prove that the two triangles formed are congruent, and therefore corresponding parts (the desired sides) are congruent.

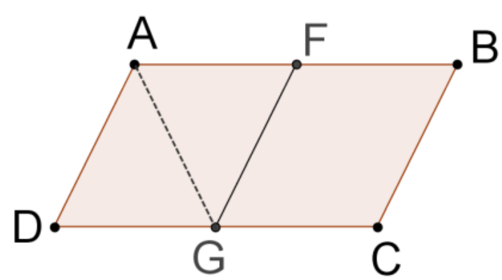


TABLE 6.5:

Statements	Reasons
Parallelogram $ABCD$ with midpoints F and G	Given
$\overline{AB} \cong \overline{DC}$, $\overline{AB} \parallel \overline{DC}$	Opposite sides of a parallelogram are congruent and parallel
$\overline{AF} \cong \overline{FB}$, $\overline{DG} \cong \overline{GC}$	Definition of midpoint
$\overline{AF} \cong \overline{DG}$	Half of congruent segments are congruent
$\overline{AG} \cong \overline{AG}$	Reflexive Property
$\angle GFA \cong \angle ADG$	If lines are parallel then alternate interior angles are congruent
$\Delta GFA \cong \Delta ADG$	$SAS \cong$
$\overline{FG} \cong \overline{DA}$	CPCTC

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

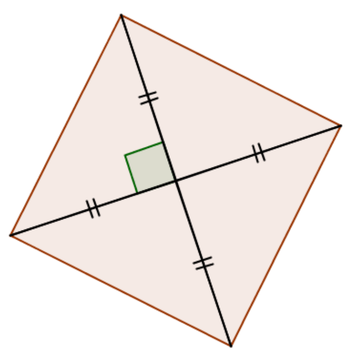
URL: http://www.ck12.org/geometry/area-and-perimeter-of-rhombuses-and-kites/plix/The-Perimeter-and-Area-of-a-Kite-580108c85aa4133109d8488d

Review

1. True or false: A rectangle is always a square.
2. True or false: A rhombus is always a parallelogram.
3. True or false: A square is always a rectangle.
4. True or false: A kite is always a quadrilateral.

Name each shape below based on its markings as precisely as you can. Don't assume that the shape is drawn to scale.

5.



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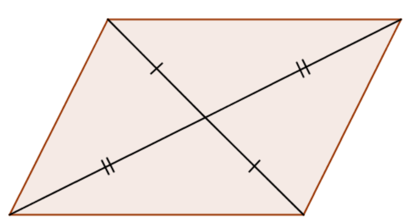

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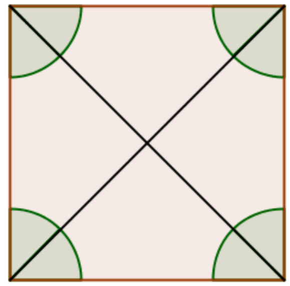

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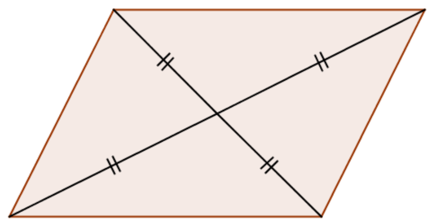
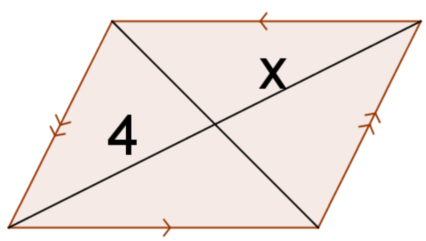


For each quadrilateral below, name the shape. Then, solve for *x*.

9.



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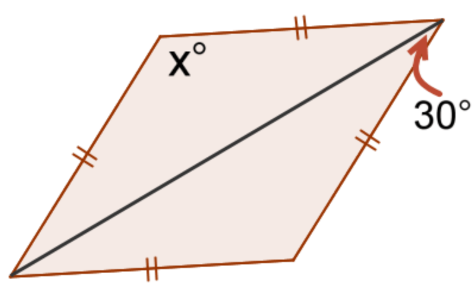
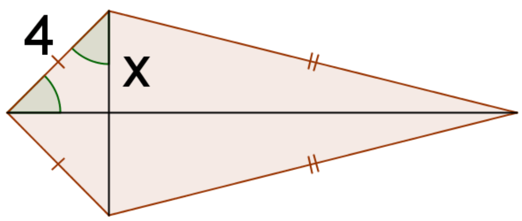
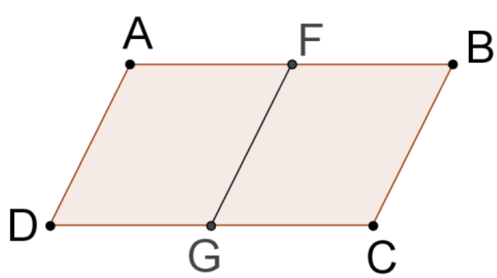


11.

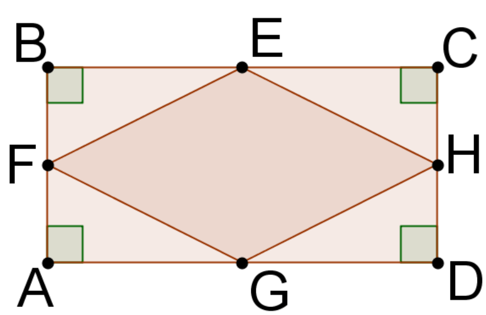


12.

13. Continue the work from guided practice #3 to prove that $\overline{FG} \parallel \overline{AD}$.



14. The midpoints of each of the sides of rectangle $ABCD$ have been connected, as shown below. Make a conjecture about the inner quadrilateral that has been formed.



15. Prove your conjecture from #14.

16. Given: $\overline{AH} \cong GC$; $\overline{DG} \cong \overline{HB}$; $\overline{DG} \parallel \overline{HB}$

Prove: ABCD is a parallelogram

17. Given: ABCD is a parallelogram; $\overline{AG} \cong \overline{CH}$

Prove: ABCD is a parallelogram

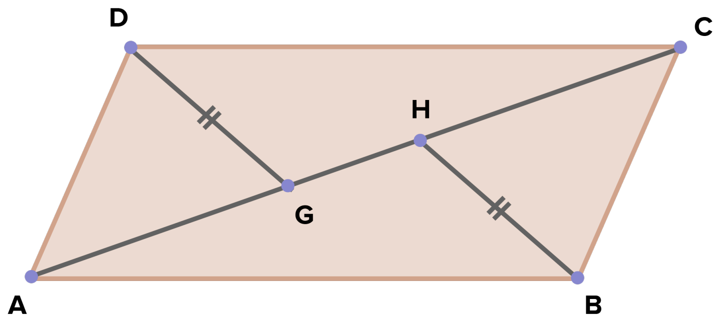


FIGURE 6.53

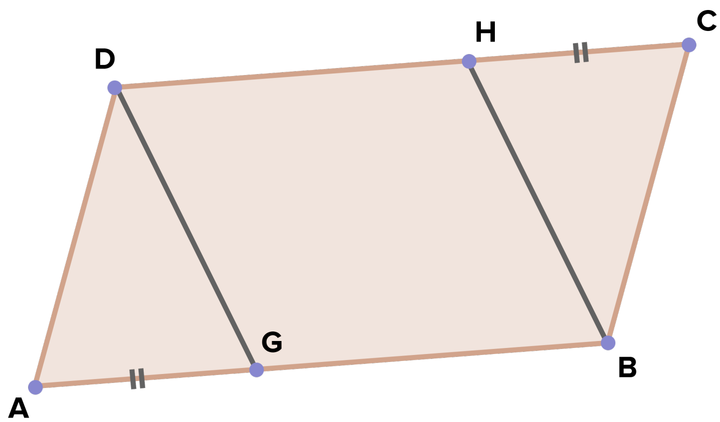


FIGURE 6.54

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.8.

6.7 Circles

Learning Objectives

Here you will review the parts of a circle and how to find the area and circumference of a circle.

A circle is a simple closed curve, with a set of all points at a constant distance from a fixed (center) point, A, in the same plane. Since the circle has only one center, you can name the circle by naming its center. In this way, you can name this as circle A.

The distance from the center point to the circle is called the radius(r). In other words, a line segment joining the center of a circle with any point on the circle is called a radius (plural: radii) of that circle. AB is a radius of circle A.

The distance from one side of the circle to the other through the center point is called the diameter(d). In other words, a line segment joining any two points on a circle and passing through the center of the circle is called a diameter of that circle. The diameter of a circle is twice its radius. PQ is a diameter of circle A.

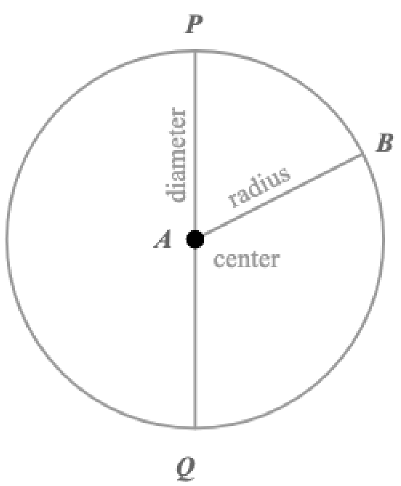


FIGURE 6.55

The perimeter of a circle is called its circumference. The ratio between the circumference and diameter of any circle is π, or "pi," which is a Greek letter (pronounced like 'pie') that stands for an irrational number approximately equal to 3.14.

- Pi is the ratio between the circumference and the diameter: $\frac{C}{d} = \pi$.
- The circumference of a circle is equal to the diameter multiplied by pi: $C = (\pi)(d)$.

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Area of Circle

The formula for the area of a circle can be derived by dissecting a circle into wedges and rearranging them to form a shape that is close to a parallelogram. The parallelogram can then be formed into a shape close to a rectangle.

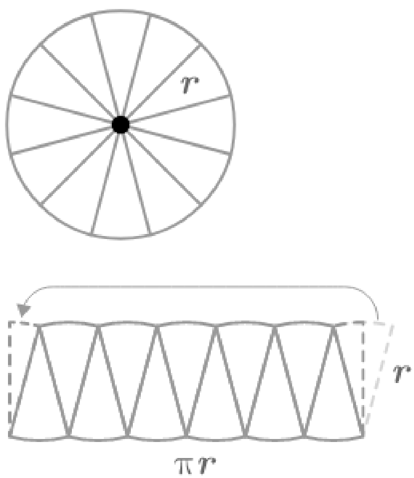

FIGURE 6.56

The lengths of the sides of the "parallelogram" are r and $\frac{2\pi r}{2} = \pi r$. If you imagine cutting the wedges smaller and smaller, the parallelogram will look closer and closer to a rectangle with dimensions πr and r. The rectangle is made by cutting up the circle, the area of the circle is equal to the area of the rectangle.

Circle area
$= \text{Rectangle area}$
$= \text{base} \cdot \text{height}$
$= \pi r \cdot r$
$= \pi r^2$

Circumference in Terms of Radius

What is the circumference of a circle in terms of radius, r?

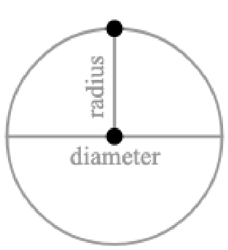

FIGURE 6.57

The circumference of a circle in terms of radius, r is _____.

Arc of a Circle

The arc of a circle is a portion of the circumference of a circle. A minor arc is an arc smaller than a semicircle. The larger of the two arcs is called the major arc. Arcs are named by their endpoints. A minor arc can be written as letters with a curving line above them: $\widehat{AB}$. To indicate the major arc, add an extra point and use three letters in the name, as in $\widehat{AKB}$, which is the long arc from A to B going around the bottom via K.

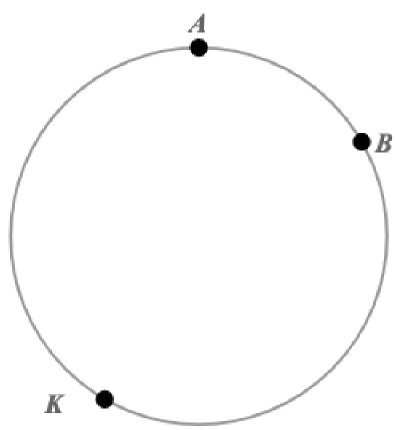

FIGURE 6.58

The formula, $C = \pi d$ yields the arc-length (that is, circumference, or curved line) for the entire circle. Sometimes, you need to work with just a portion of a circle's arc, i.e. fractional part of the circumference.

Consider the following proportion:

$$\frac{\text{arc measure}}{360^\circ} = \frac{\text{arc length}}{\text{circumference}}$$

If we solve the proportion for arc length, and replace "arc measure" with its equivalent "central angle", we can establish the formula:

$$\text{arc length} = \left(\frac{\text{central angle}}{360^\circ}\right) \cdot (\text{circumference})$$

For example, an arc measure of 60°is one-sixth of the circle (360°), so the length of that arc will be one-sixth of the circumference of the circle.

Calculating Arc Length

Play with a circle by dragging one of the dots that define the endpoints of the arc. The arc length is recalculated as you drag.

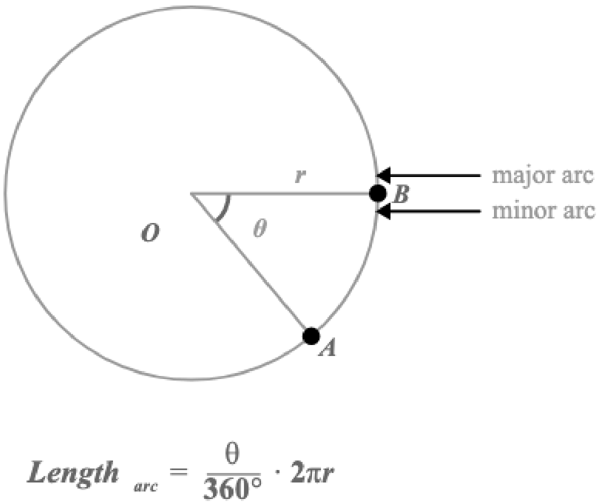

FIGURE 6.59

Sector of a Circle

If we start with a circle with a marked radius, and turn the circle a bit, the area marked off looks something like a wedge of pie or a slice of pizza; this is called a sector of the circle. Hence, a sector of a circle is a region bounded by an arc of the circle and the two radii to the endpoints of the arc, where the smaller area (shaded portion) is known as the minor sector and the larger being the major sector.

When finding the area of a sector, you are actually finding a fractional part of the area of the entire circle. Hence, the area of a sector can be expressed using its central angle or its arc length.

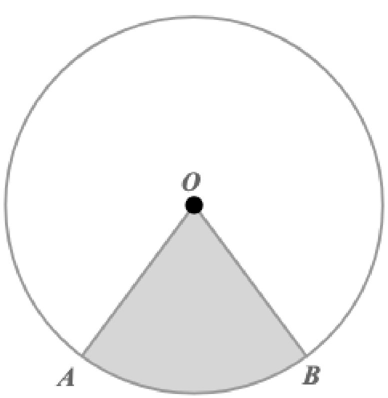

FIGURE 6.60

The following propositions are true regarding the sector:

$$\frac{\text{area of sector}}{\text{area of whole circle}} = \frac{\text{central angle } \theta}{360^\circ} = \frac{\text{arc length}}{\text{circumference}}$$

$$\text{area of sector} = \frac{\text{central angle}}{360^\circ} \cdot \text{area of whole circle}$$

Calculating Sector Area

Play with a circle by dragging one of the dots that define the endpoints of the sector. The sector area is recalculated as you drag.

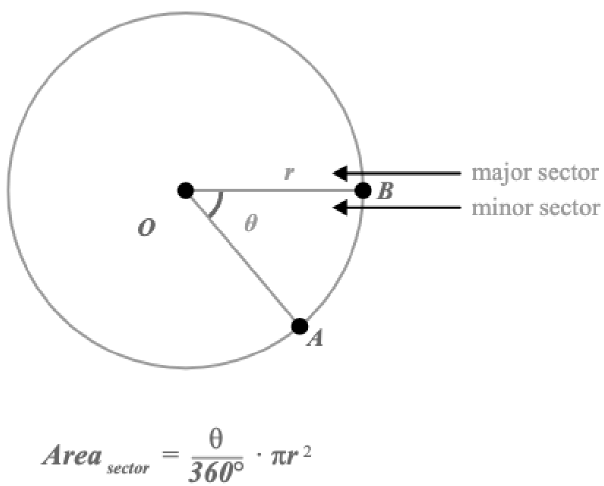

FIGURE 6.61

Examples

1. Find the area and circumference of a circle with radius 5 inches.

$\text{Area} = \pi r^2 = \pi(5)^2 = 25\pi \text{ in}^2$

$\text{Circumference} = \pi d = \pi(2 \cdot 5) = 10\pi \text{ in}$

Leaving your answer with a π symbol in it is called leaving your answer "in terms of π". This is often preferable because it is the exact answer. As soon as you approximate the value of π, your answer is not exact. Keep in mind that π is not a unit. You should still put the appropriate units in your answers.

2. Find the area and circumference of a circle with diameter 16 cm.

If the diameter is 16 cm, then the radius is 8 cm.

$\text{Area} = \pi r^2 = \pi(8)^2 = 64\pi \text{ cm}^2$

$\text{Circumference} = \pi d = 16\pi \text{ cm}$

3. Find the area of a circle with a circumference of 12π cm.

Let r be the radius of the given circle.

$\text{Circumference} = 2\pi r$
$12\pi = 2\pi r$
$r = \frac{12\pi}{2\pi} = 6 \text{ cm}$
$\therefore \text{Area} = \pi r^2$
$\text{Area} = \pi(6)^2 = 36\pi \text{ cm}^2$

CK-12 PLIX: Radius or Diameter of a Circle with Given Area

This PLIX provides interactive examples for practice calculating the dimensions of a circle.

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/radius-or-diameter-of-a-circle-given-area/plix/How-Tall-is-the-Snowman-57586807da2cfe7ec66f760c

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4. The shape below is a portion of a circle called a sector. This sector is $\frac{1}{4}$ of the circle. Find the area and perimeter of the sector.

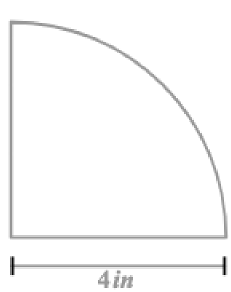


FIGURE 6.62

Since the sector is $\frac{1}{4}$ of the circle, its area will be $\frac{1}{4}$ the area of the circle.

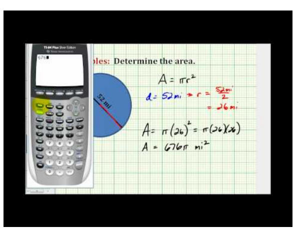
$$
\begin{aligned}
Area_{sector} &= \frac{1}{4} \times Area_{circle}\\
Area_{sector} &= \frac{1}{4}(\pi r^2)\\
Area_{sector} &= \frac{1}{4}\pi(4)^2\\
Area_{sector} &= \frac{1}{4}\pi(16)\\
Area_{sector} &= 4\pi\ in^2
\end{aligned}
$$

FIGURE 6.63

The perimeter of the sector is the sum of the lengths of the two radii and the arc. Each radius is 4 in. The arc is $\frac{1}{4}$ of the circumference of the full circle.

$$
\begin{aligned}
Perimeter_{sector} &= length\ of\ the\ arc + 2 \times radius\\
Perimeter_{sector} &= \frac{2\pi r}{4} + 2r\\
Perimeter_{sector} &= \frac{2\pi(4)}{4} + 2(4)\\
Perimeter_{sector} &= 2\pi + 8\ \ in
\end{aligned}
$$

FIGURE 6.64



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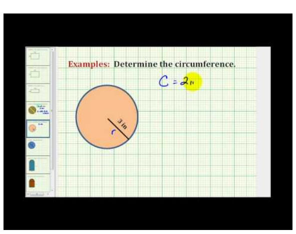

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5. Find the area and perimeter of the sector below. The sector is $\frac{1}{3}$ of the circle.

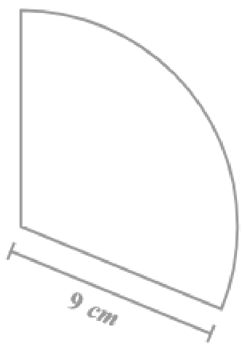


FIGURE 6.65

$$\begin{aligned}
\text{Area}_{\text{sector}} &= \tfrac{1}{3} \cdot \text{Area}_{\text{circle}} \\
\text{Area}_{\text{sector}} &= \tfrac{1}{3} \cdot \pi r^2 \\
\text{Area}_{\text{sector}} &= \tfrac{1}{3} \cdot \pi (9)^2 \\
\text{Area}_{\text{sector}} &= \tfrac{1}{3} \cdot \pi (81) = 27\pi \text{ cm}^2 \\
\text{Perimeter}_{\text{sector}} &= \text{length of the arc} + 2 \cdot \text{radius} \\
\text{Perimeter}_{\text{sector}} &= \tfrac{2\pi r}{3} + 2r \\
\text{Perimeter}_{\text{sector}} &= \tfrac{2\pi(9)}{3} + 2(9) \\
\text{Perimeter}_{\text{sector}} &= 6\pi + 18 \text{ cm} .
\end{aligned}$$

CK-12 PLIX: Area of Sectors and Segments

This PLIX provides examples to use as practice calculating the area of segments and sectors of a circle.

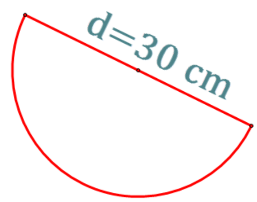
PLIX

Click image to the left or use the URL below.

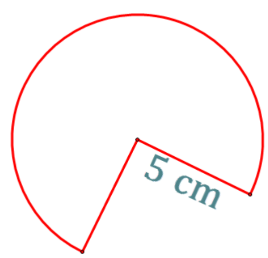
URL: http://www.ck12.org/geometry/area-of-sectors-and-segments/plix/Area-of-Sectors-and-Segments-Ms.-Arcade-Man-53680ee05aa4133b2edfc518

Review

1. Find the area and circumference of a circle with radius 3 *in*.
2. Find the area and circumference of a circle with radius 6 *in*.
3. Find the area and circumference of a circle with diameter 15 *cm*.
4. Find the area and circumference of a circle with diameter 22 *in*.
5. Find the area of a circle with a circumference of 32π *cm*.
6. Find the circumference of a circle with area $32\pi\ cm^2$.
7. The sector below is $\frac{1}{2}$ of a circle. Find the area of the sector.

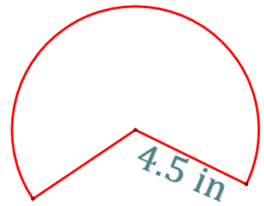


8. Find the perimeter of the sector in #7.
9. The sector below is $\frac{3}{4}$ of a circle. Find the area of the sector.



10. Find the perimeter of the sector in #9.
11. The sector below is $\frac{1}{8}$ of a circle. Find the area of the sector.

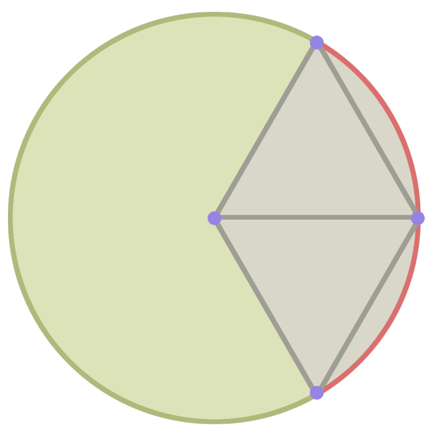
12. Find the perimeter of the sector in #11.
13. The sector below is $\frac{2}{3}$ of a circle. Find the area of the sector.



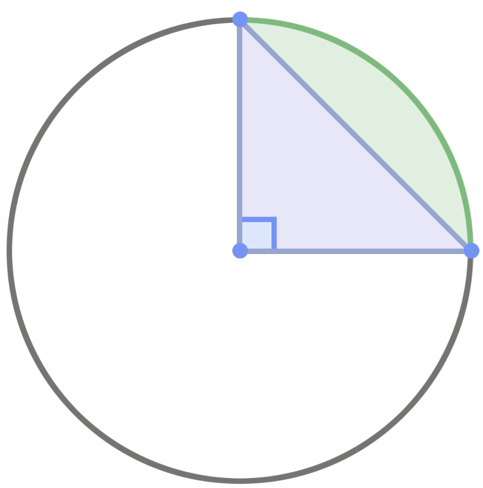
14. Find the perimeter of the sector in #13.

15. Explain the formula $A = \frac{Cr}{2}$ where A is the area of a circle, C is the circumference of the circle, and r is the radius of the circle.

16. The triangles shown are equilateral with a side length of 1. Find the area of the green sector.



17. The right triangle below has a hypotenuse of 5. Find the area of the green region. (This is called a **segment** of the circle.)



18. Why is a complete circle assigned an angle measure of 360° instead of 100° or some other number? Make an argument for using 360°. Make an argument against and suggest an alternative.

19. Given a circle with radius 1. For each of the following **central angles**, find the corresponding arc length. Leave the answers in terms of π.

0°
90°
180°
270°
360°

A new type of angle measurement unit can be derived from the arc length of a circle with radius 1. Since 360° corresponds to an arc length of 2π, the new angle corresponding to a complete circular rotation is 2π **radians**. In addition to measuring an angle in degrees, we can also measure it in **radians.** Do you prefer this new unit, or degrees? Why?

Review (Answers)

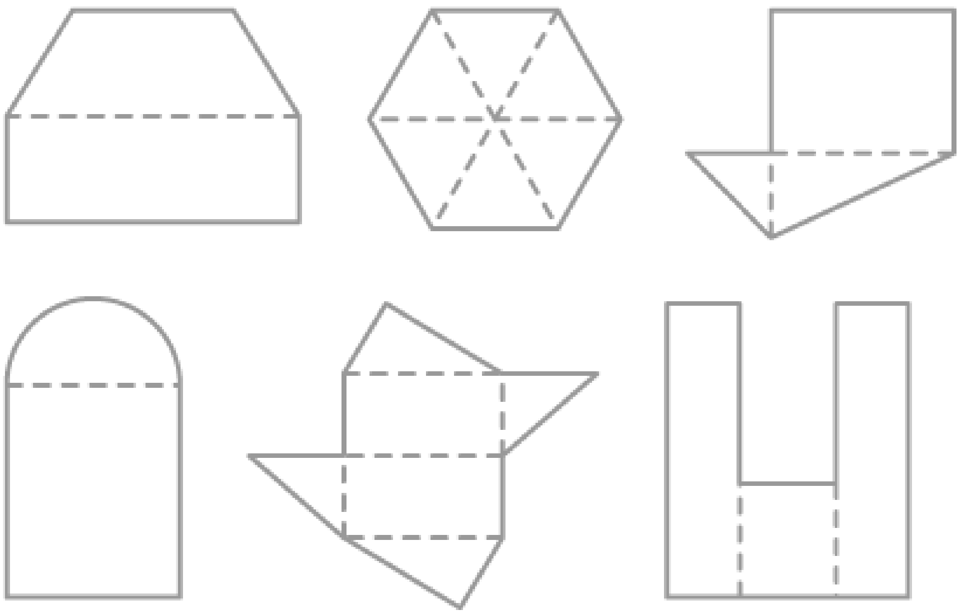
To see the Review answers, open this PDF file and look for section 1.8.

6.8 Composite Shapes

Learning Objectives

Here you will find the area and perimeter of composite shapes.

A composite shape or a composite figure is a two-dimensional figure made up of basic two-dimensional shapes such as triangles, rectangles, circles, semi-circles, etc.



Perimeter of Composite Figures

To find the perimeter of a composite two-dimensional figure, add the lengths of the sides.

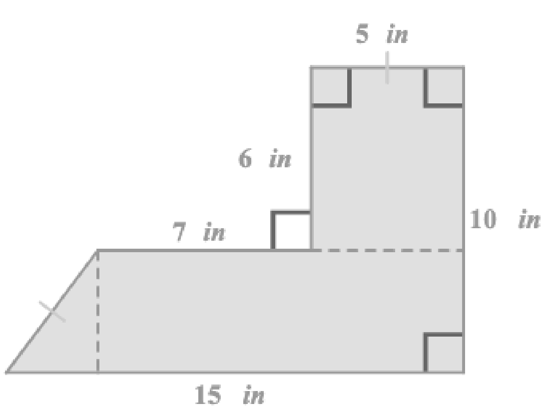


FIGURE 6.66

Area of Composite Figures

There are two general methods for finding the area of a composite shape.

Additive Areas Method

Find the individual areas of each piece of the composite shape. The area of the composite shape will be the sum of the individual areas.

Subtractive Areas Method

Find the area of a shape larger than the composite shape and the areas of the pieces of the larger shape not included in the composite shape. The area of the composite shape will be the difference between the area of the larger shape and the areas of the pieces of the larger shape not included in the composite shape.

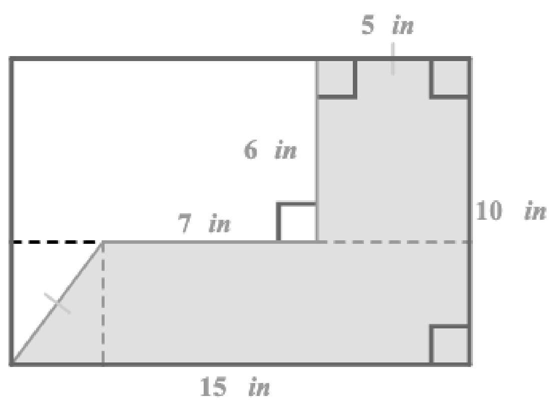


FIGURE 6.67

Regardless of what method you use, you will often have to think carefully in order to find the dimensions necessary for determining the area or perimeter.

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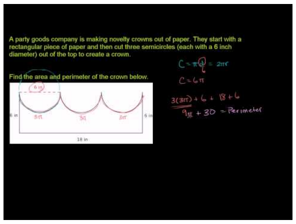

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Finding Area Using the Additive Areas Method

Find the area and perimeter of the composite shape below.

You can find the area, using the additive areas method. The shape has been broken into 1 triangle and 2 rectangles (although it could have been broken up differently).

First, find all of the missing side lengths.

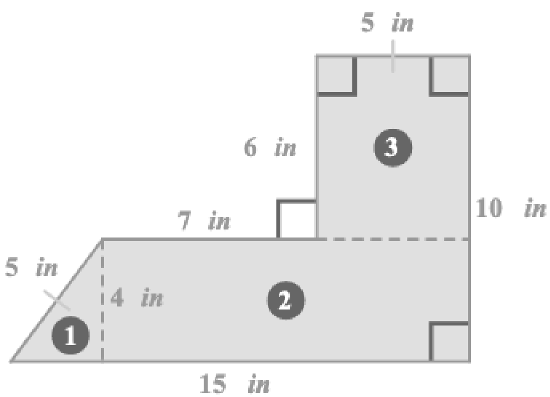

FIGURE 6.68

For the right triangle, you know the hypotenuse is 5 inches because of the tick mark showing it is congruent to the other segment that is 5 inches. The height is 4 inches because the full height of the shape is 10 inches and the height above the triangle is given as 6 inches. Finally, you know the base is 3 inches because of the Pythagorean Theorem (or, because the full base is 15 inches and other portions are 7 inches and 5 inches).

Now you can find the perimeter by finding the sum of all the side lengths.

$P = 5+7+6+5+10+12+3 = 48$ in

Find the area of each of the three pieces and then the total area.

Area of Triangle, $A_1 = \frac{bh}{2} = \frac{3 \cdot 4}{2} = 6 \text{ in}^2$
Area of Rectangle #1, $A_2 = bh = 12 \cdot 4 = 48 \text{ in}^2$
Area of Rectangle #2, $A_3 = bh = 5 \cdot 6 = 30 \text{ in}^2$
Total Area, $A_1 + A_2 + A_3 = 6 + 48 + 30 = 84 \text{ in}^2$

Finding Area Using the Subtractive Area Method.

Find the area of the composite shape:

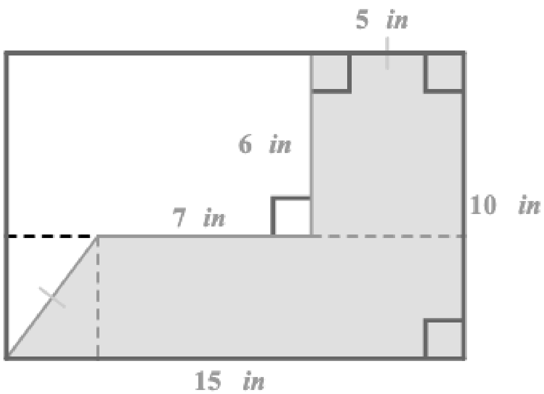

FIGURE 6.69

To find the area using subtractive areas method:

Find the area of the large rectangle, the unshaded triangle, and the unshaded rectangle. Subtract to find the area of the composite shape. Just like with additive areas method, you will first need to find missing side lengths. For the unshaded rectangle, you know the base is 10 inches as the full base is 15 inches and a portion of which is part of the other segment that is 5 inches. The height of the unshaded rectangle is 6 inches. For the triangle, the height is 4 inches because the full height of the shape is 10 inches and the height above the rectangle is given as 6 inches. Finally, you know the base is 3 inches because of the Pythagorean theorem.

$A_{\text{large rectangle}} = bh = 15 \cdot 10 = 150 \text{ in}^2$
$A_{\text{unshaded rectangle}} = bh = 10 \cdot 6 = 60 \text{ in}^2$
$A_{\text{unshaded triangle}} = \frac{bh}{2} = \frac{3 \cdot 4}{2} = 6 \text{ in}^2$
$\text{Total Area} = 150 - 60 - 6 = 84 \text{ in}^2$

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The total area is $12 + 9 = 21 \text{ ft}^2$

CK-12 PLIX: Area and Perimeter of Composite Shapes

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/area-and-perimeter-of-composite-shapes/plix/Area-of-a-Pinwheel-56c4c327da2cfe665725d449

Examples

Example 1

A square is inscribed inside a circle. Find the total area of the shaded regions of the circle below. What method for finding the area makes the most sense in this case? Why?

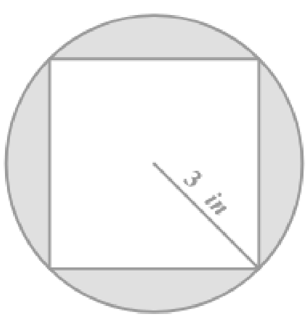


FIGURE 6.70

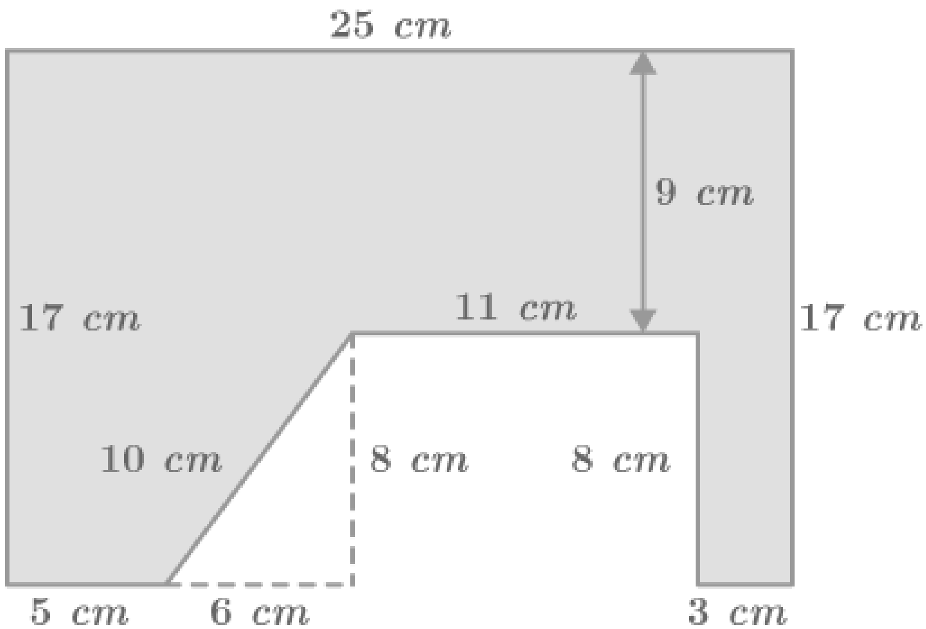
It makes sense to use subtractive areas method to find the area of the shaded region. Finding the area of the whole circle and subtracting the area of the square is much simpler than trying to calculate the area of each of the four shaded pieces directly.

$A_{\text{circle}} = \pi r^2 = \pi(3)^2 = 9\pi \text{ in}^2$
$A_{\text{square}} = \frac{d_1 d_2}{2} = \frac{6 \cdot 6}{2} = \frac{36}{2} = 18 \text{ in}^2$
$A_{\text{shaded region}} = 9\pi - 18 \text{ in}^2$

Example 2

Find the perimeter of the shape.

The figure below is not drawn to scale. Assume all angles that look like right angles are right angles.



First find all of the missing lengths. Note that you will need to use the Pythagorean Theorem to find the side length of 10 cm (the hypotenuse of the right triangle).

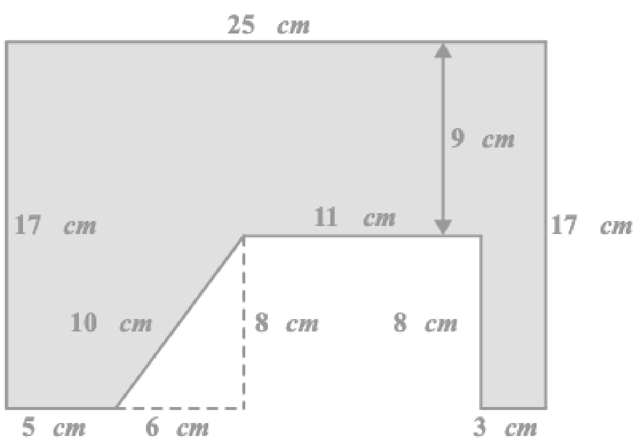


FIGURE 6.71

Now find the perimeter.

$P = 17 + 5 + 10 + 11 + 8 + 3 + 17 + 25 = 96$ cm

Example 3

Choose a method for finding the area of the shape. Justify your method.

You could choose either method. Subtractive areas method might be slightly easier because you will just need to find the area of the big rectangle and subtract the area of the trapezoid.

Example 4

Find the area of the shape using the subtractive areas method.

$$\begin{aligned}
\text{Area} &= \text{Area}_{\text{rectangle}} - \text{Area}_{\text{trapezoid}} \\
&= bh - \frac{(b_1+b_2)h}{2} \\
&= (25 \cdot 17) - \frac{(11+17)(8)}{2} \\
&= (25 \cdot 17) - \frac{(28 \cdot 8)}{2} \\
&= 425 - 112 \\
&= 313 \text{ cm}^2
\end{aligned}$$

CK-12 PLIX: Area and Perimeter of Composite Shapes

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/area-and-perimeter-of-composite-shapes/plix/Magic-Missing-Square-568c1b79da2cfe2623a61661

Review

The figure below is not drawn to scale. Assume all angles that look like right angles are right angles. Point O is the center of the partial circle.

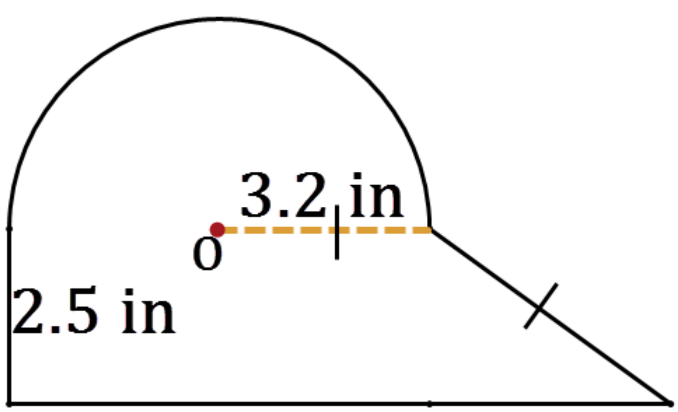


FIGURE 6.72

1. Find all missing side lengths for the shape.
2. Find the perimeter of the shape.

3. Choose a method for finding the area of the shape. Justify your method.

4. Find the area of the shape using the method you chose in #2.

5. Try to find the area using the other method. If not possible, explain why not.

The figure below is not drawn to scale. Assume all angles that look like right angles are right angles.

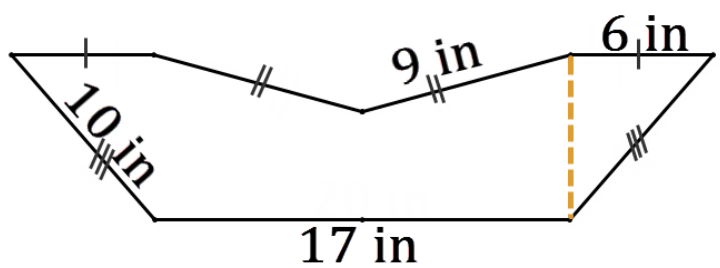


FIGURE 6.73

6. Find all missing side lengths for the shape.

7. Find the perimeter of the shape.

8. Choose a method for finding the area of the shape. Justify your method.

9. Find the area of the shape using the method you chose in #8.

10. Try to find the area using the other method. If not possible, explain why not.

The figure below is not drawn to scale. Assume all angles that look like right angles are right angles.

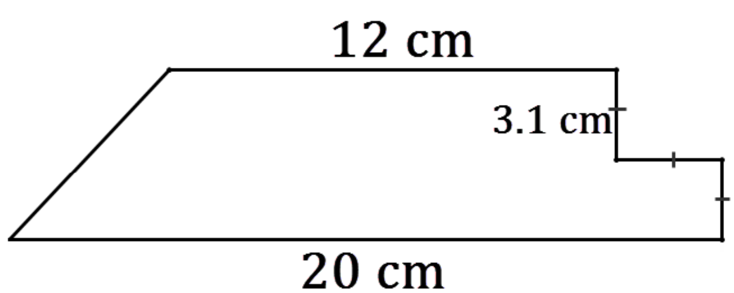


FIGURE 6.74

11. Find all missing side lengths for the shape.

12. Find the perimeter of the shape.

13. Choose a method for finding the area of the shape. Justify your method.

14. Find the area of the shape using the method you chose in #13.

15. Try to find the area using the other method. If not possible, explain why not.

16. Given the rectangle below, find the scalene triangle area using two different methods. (Diagram is not to scale.)

17. In the coordinate plane, sketch a quadrilateral that is not a parallelogram, trapezoid, or kite, such that the vertex coordinates are integers. (Polygons like this are called lattice polygons.) Is it possible to decompose this shape and find its area? If so, find its area. Do you think every quadrilateral whose vertex coordinates are integers can be decomposed to triangles and its area calculated? Experiment with quadrilaterals and decompositions and explain. Extend your exploration to polygons with more sides - it is not necessary to calculate the area, only to decompose the figure in such a way that the area can be calculated. Explain your conclusions.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.9.

6.9 References

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Chapter 7 Trigonometry

Chapter Outline

7.1 Tangent Ratio

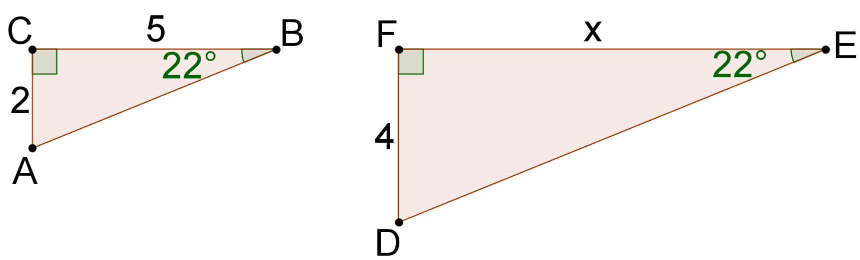
Learning Objectives

Here you will learn how to use the tangent ratio to find missing sides of right triangles.

As the measure of an angle increases between $0°$ and $90°$, how does the tangent ratio of the angle change?

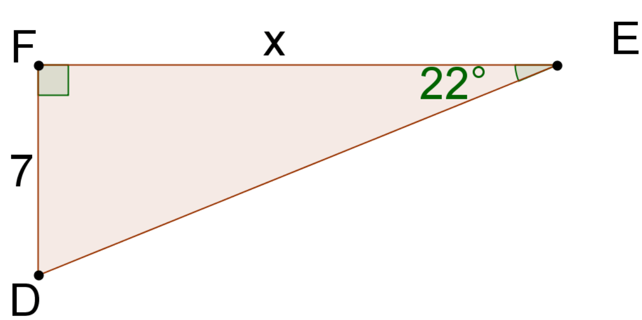
Tangent Ratio

Recall that one way to show that two triangles are similar is to show that they have two pairs of congruent angles. This means that two right triangles will be similar if they have one pair of congruent non-right angles.



The two right triangles above are similar because they have two pairs of congruent angles. This means that their corresponding sides are proportional. $\overline{DF}$ and $\overline{AC}$ are corresponding sides because they are both opposite the $22°$ angle. $\frac{DF}{AC} = \frac{4}{2} = 2$, so the scale factor between the two triangles is 2. This means that $x = 10$, because $\frac{FE}{CB} = \frac{10}{5} = 2$.

The ratio between the two legs of any $22°$ right triangle will always be the same, because all $22°$ right triangles are similar. The ratio of the **length of the leg opposite** the $22°$ angle to the **length of the leg adjacent to** the $22°$ angle will be $\frac{2}{5} = 0.4$. You can use this fact to find a missing side of another $22°$ right triangle.



Because this is a $22°$ right triangle, you know that $\frac{opposite\ leg}{adjacent\ leg} = \frac{2}{5} = 0.4$.

$$\frac{opposite\ leg}{adjacent\ leg} = 0.4$$
$$\frac{7}{x} = 0.4$$
$$0.4x = 7$$
$$x = 17.5$$

The ratio between the opposite leg and the adjacent leg for a given angle in a right triangle is called the **tangent ratio**. Your scientific or graphing calculator has **tangent** programmed into it, so that you can determine the $\frac{opposite\ leg}{adjacent\ leg}$ ratio for any angle within a right triangle. The abbreviation for **tangent** is **tan**.

CK-12 PLIX Interactive

PLIX

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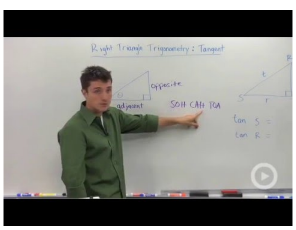

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Calculating Tangent Functions

MEDIA

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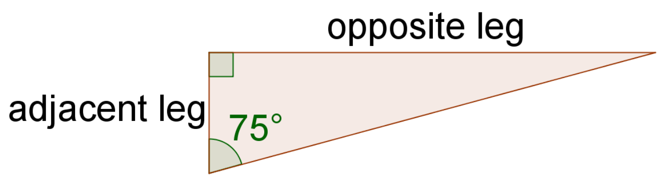
Use your calculator to find the tangent of 75°. What does this value represent?

Make sure your calculator is in degree mode. Then, type "tan(75)".

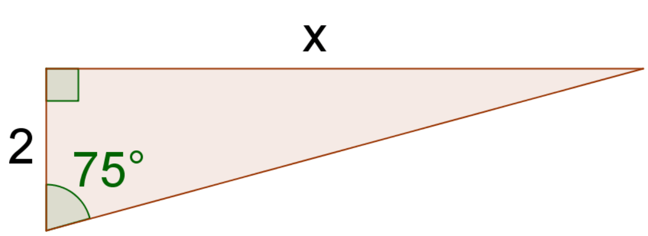
$\tan(75^\circ) \approx 3.732$

This means that the ratio of the ***length of the opposite leg to the length of the adjacent leg*** for a 75° angle within a right triangle will be approximately 3.732.



Solving for Unknown Values

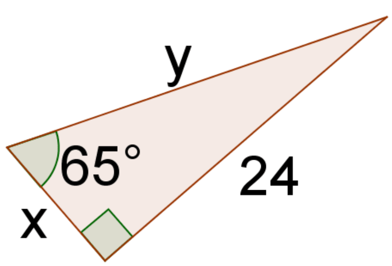
1. Solve for x.



From the previous problem, you know that the ratio $\frac{opposite\ leg}{adjacent\ leg} \approx 3.732$. You can use this to solve for x.

$$\begin{aligned} \frac{opposite\ leg}{adjacent\ leg} &\approx 3.732 \\ \frac{x}{2} &\approx 3.732 \\ x &\approx 7.464 \end{aligned}$$

2. Solve for x and y.



You can use the 65° angle to find the correct ratio between 24 and x.

$$\begin{aligned}
\tan(65^\circ) &= \frac{opposite\ leg}{adjacent\ leg} \\
2.145 &\approx \frac{24}{x} \\
x &\approx \frac{24}{2.145} \\
x &\approx 11.189
\end{aligned}$$

Note that this answer is only approximate because you rounded the value of $\tan 65^\circ$. An exact answer will include "tan". The exact answer is:

$x = \frac{24}{\tan 65^\circ}$

To solve for y, you can use the Pythagorean Theorem because this is a right triangle.

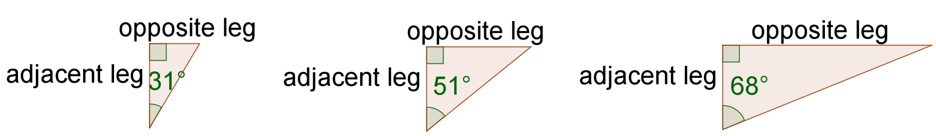
$$\begin{aligned}
11.189^2 + 24^2 &= y^2 \\
701.194 &= y^2 \\
26.48 &= y
\end{aligned}$$

Examples

Example 1

Earlier, you were asked how does the tangent ratio of the angle change.

As the measure of an angle increases between 0° and 90°, how does the tangent ratio of the angle change?



As an angle increases, the length of its opposite leg increases. Therefore, $\frac{opposite\ leg}{adjacent\ leg}$ increases and thus the value of the tangent ratio increases.

Example 2

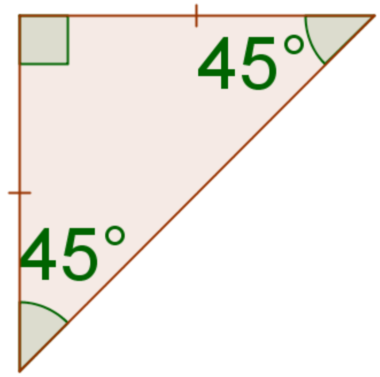
Tangent tells you the ratio of the two legs of a right triangle with a given angle. Why does the tangent ratio not work in the same way for non-right triangles?

Two right triangles with a 32° angle will be similar. Two non-right triangles with a 32° angle will not necessarily be similar. The tangent ratio works for right triangles because all right triangles with a given angle are similar. The tangent ratio doesn't work in the same way for non-right triangles because not all non-right triangles with a given angle are similar. **You can only use the tangent ratio for right triangles.**

Example 3

Use your calculator to find the tangent of 45°. What does this value represent? Why does this value make sense?

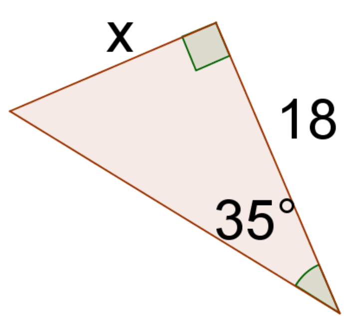
$\tan(45^\circ) = 1$. This means that the ratio of the length of the opposite leg to the length of the adjacent leg is equal to 1 for right triangles with a 45° angle.



This should make sense because right triangles with a 45° angle are isosceles. The legs of an isosceles triangle are congruent, so the ratio between them will be 1.

Example 4

Solve for x.



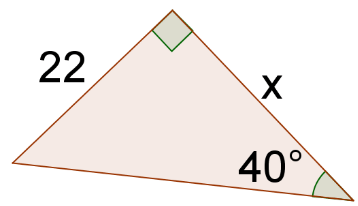
Use the tangent ratio of a 35° angle.

$$\begin{aligned}
\tan(35^\circ) &= \frac{opposite\ leg}{adjacent\ leg} \\
\tan(35^\circ) &= \frac{x}{18} \\
x &= 18\tan(35^\circ) \\
x &\approx 12.604
\end{aligned}$$

Review

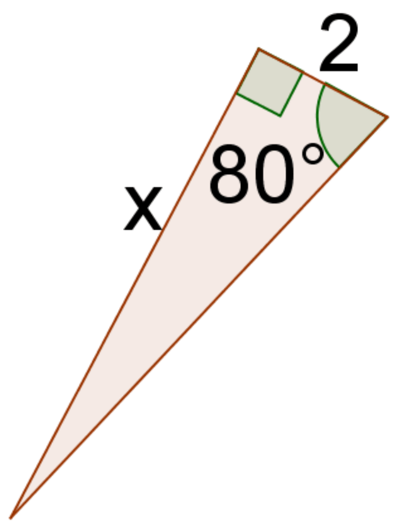
1. Why are all right triangles with a 40° angle similar? What does this have to do with the tangent ratio?
2. Find the tangent of 40°.

3. Solve for x.



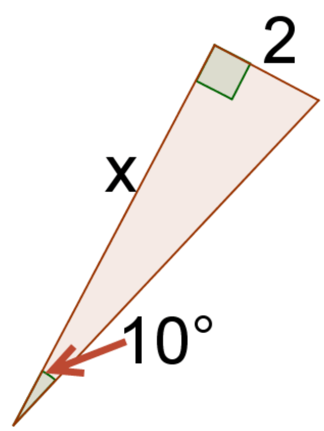
4. Find the tangent of 80°.

5. Solve for x.



6. Find the tangent of 10°.

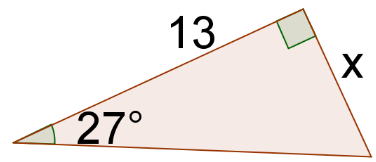
7. Solve for x.



8. Your answer to #5 should be the same as your answer to #7. Why?

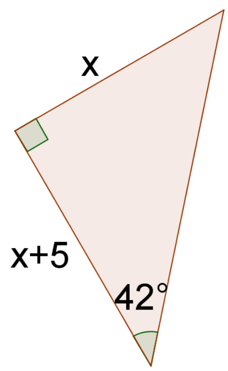
9. Find the tangent of 27°.

10. Solve for x.



11. Find the tangent of 42°.

12. Solve for x.



13. A right triangle has a 42° angle. The base of the triangle, adjacent to the 42° angle, is 5 inches. Find the area of the triangle.

14. Recall that the ratios between the sides of a 30-60-90 triangle are $1:\sqrt{3}:2$. Find the tangent of 30°. Explain how this matches the ratios for a 30-60-90 triangle.

15. Explain why it makes sense that the value of the tangent ratio increases as the angle goes from 0° to 90°.

16. Kaci is building a bike ramp for jumps. She wants the angle of elevation of the ramp with respect to the ground to be 30°. What is the slope of her ramp? What are some reasonable dimensions for the ramp that will have this slope?

17. Theo and Kaci are debating. Theo thinks that the slope of the line below is $\frac{\Delta y}{\Delta x}$ while Kaci thinks it's $\tan\theta$. Who is correct and why?

18. Why is tangent defined as opposite over adjacent instead of the other way around?

19. Theo and Kaci are standing 5 kilometers away from each other, watching a balloon ascending directly between them. The angle of elevation from Kaci to the balloon is 30°. The angle of elevation from Theo to the balloon is 60°. How high is the balloon?

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.1.

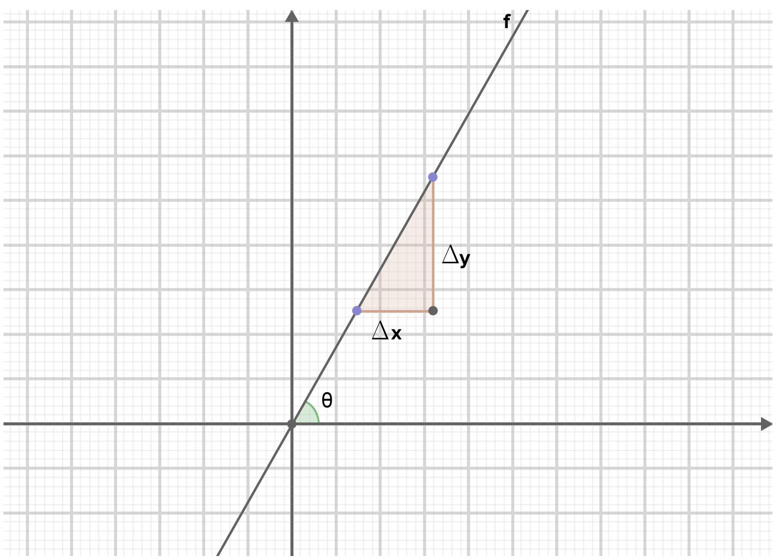


FIGURE 7.1

7.2 Sine and Cosine Ratios

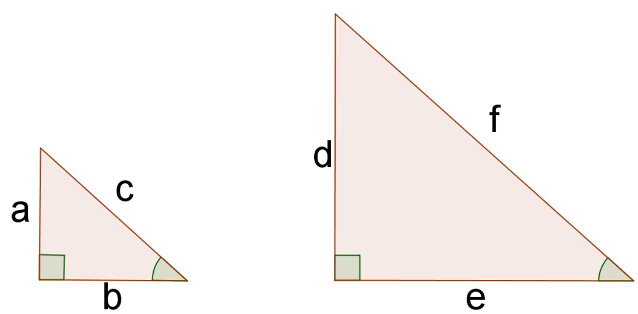
Learning Objectives

Here you will learn how to use sine and cosine ratios to find missing sides of right triangles.

SOH-CAH-TOA, pronounced "sew-cah-toe-ah" is a mnemonic that many people use to remember the difference between sine, cosine, and tangent. How can remembering SOH-CAH-TOA help you?

Sine and Cosine Ratios

Two right triangles with one pair of non-right congruent angles are similar by $AA \sim$. This means the ratio between the side lengths of the first triangle must be congruent to the ratio between the corresponding side lengths of the second triangle.



For example, in the picture above, $\frac{a}{c} = \frac{d}{f}$. Because there are **three *pairs*** of sides for any triangle, there are **three** relevant **ratios** for a given angle.

1. The **tangent** of an angle gives the ratio $\frac{\text{opposite leg}}{\text{adjacent leg}}$. *The abbreviation for tangent is tan.*
2. The **sine** of an angle gives the ratio $\frac{\text{opposite leg}}{\text{hypotenuse}}$. *The abbreviation for sine is sin.*
3. The **cosine** of an angle gives the ratio $\frac{\text{adjacent leg}}{\text{hypotenuse}}$. *The abbreviation for cosine is cos.*

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These are the basic **trigonometric ratios**. **Trigonometry** is the study of triangles. These ratios are called trigonometric ratios because they apply to triangles. Just as your scientific or graphing calculator has **tangent** programmed into it, it also has **sine** and **cosine** programmed into it. This means that you can use your calculator to determine the ratio between the lengths of any pair of sides for any angle within a right triangle.

CK-12 PLIX Interactive

PLIX

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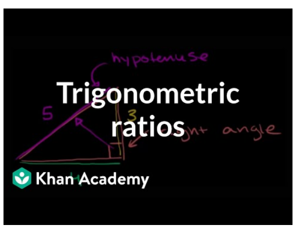
URL: http://www.ck12.org/trigonometry/domain-range-and-signs-of-trigonometric-functions/plix/Range-and-Signs-of-Trigonometric-Functions-Sine-and-Cosine-55b955e45aa4130b42aaf575



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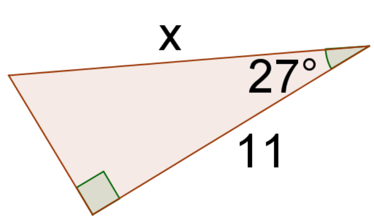
Finding Sine and Cosine Ratios

Use your calculator to find the sine ratio and cosine ratio for a 27° angle.

$\sin(27^\circ) \approx 0.454$ and $\cos(27^\circ) \approx 0.891$.

Solving for Unknown Values

Solve for x.



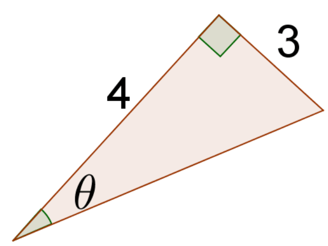
Look to see how the sides that are marked are related to the 27° angle. The side of length 11 is **adjacent** to the angle. The side of length x is the **hypotenuse** of the triangle. When working with the adjacent side and the hypotenuse, you should use the cosine ratio.

$$\begin{aligned}
\cos(27^\circ) &= \frac{\text{adjacent leg}}{\text{hypothenuse}} \\
\cos(27^\circ) &= \frac{11}{x} \\
x &= \frac{11}{\cos(27^\circ)} \\
x &\approx \frac{11}{0.891} \\
x &\approx 12.346
\end{aligned}$$

Note that $\frac{11}{\cos(27^\circ)}$ is the exact answer. 12.346 is an approximate answer because you rounded the value of $\cos(27^\circ)$.

Calculating Sine and Cosine Functions

Find $\sin\theta$ and $\cos\theta$.



Relative to angle θ, 3 is the opposite leg and 4 is the adjacent leg. In order to find the sine and cosine ratios you also need to know the hypotenuse of the triangle. Use the Pythagorean Theorem to find the hypotenuse:

$$\begin{aligned}
3^2 + 4^2 &= h^2 \\
25 &= h^2 \\
h &= 5
\end{aligned}$$

Now, write the sine and cosine ratios:

$$\begin{aligned}
\sin\theta &= \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{3}{5} \\
\cos\theta &= \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{4}{5}
\end{aligned}$$

Examples

Example 1

Earlier, you were asked how can remembering SOH-CAH-TOA can help you.

SOH-CAH-TOA is a mnemonic that many people use to remember the difference between sine, cosine, and tangent. How can remembering SOH-CAH-TOA help you?

SOH-CAH-TOA stands for **S**ine equals **O**pposite over **H**ypotenuse, **C**osine equals **A**djacent over **H**ypotenuse, and **T**angent equals **O**pposite over **A**djacent. This mnemonic helps you remember which trigonometric ratio is which.

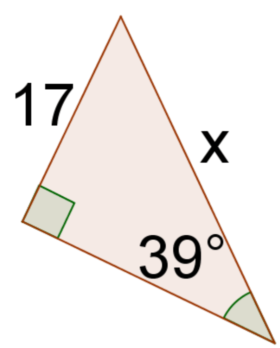
Example 2

Use your calculator to find the sine and cosine ratios for a 39° angle.

$\sin(39^\circ) \approx 0.629$ and $\cos(39^\circ) \approx 0.777$.

Example 3

Solve for x.

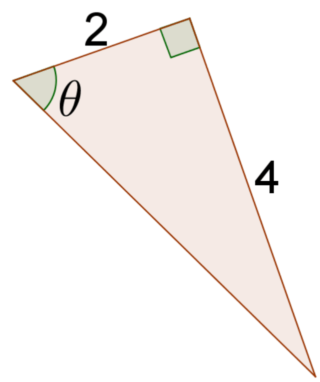


Look to see how the sides that are marked are related to the 39° angle. The side of length 17 is **opposite** from the angle. The side of length x is the **hypotenuse** of the triangle. When working with the opposite side and the hypotenuse, you should use the sine ratio.

$$\begin{aligned}
\sin(39^\circ) &= \frac{\text{opposite leg}}{\text{hypotenuse}} \\
\sin(39^\circ) &= \frac{17}{x} \\
x &= \frac{17}{\sin(39^\circ)} \\
x &\approx \frac{11}{0.629} \\
x &\approx 27.013
\end{aligned}$$

Example 4

Find $\sin\theta$ and $\cos\theta$.



Relative to angle θ, 4 is the opposite leg and 2 is the adjacent leg. In order to find the sine and cosine ratios you also need to know the hypotenuse of the triangle. Use the Pythagorean Theorem to find the hypotenuse:

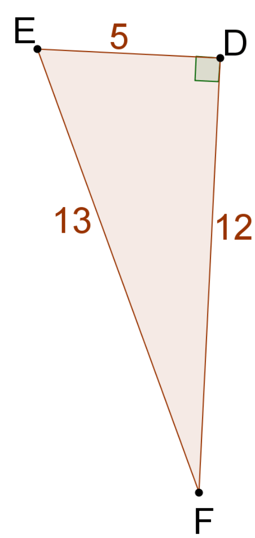
Now, write the sine and cosine ratios:

$$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$
$$\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

Note that in the last step of each calculation the denominator was rationalized. You can choose to rationalize the denominator if you wish.

Review

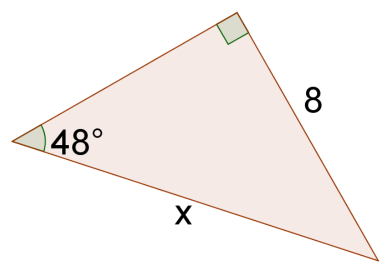
For #1-#6, use the triangle below. Find each exact value.



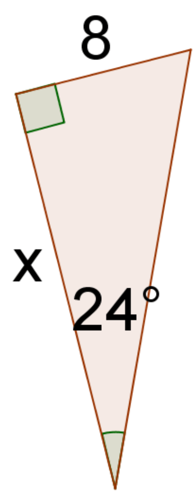
1. $\sin E$
2. $\cos E$
3. $\tan E$
4. $\sin F$
5. $\cos F$
6. $\tan F$

Identify whether the sine, cosine, or tangent ratio is most useful for helping to solve the problem. Then, solve for x.

7.



8.



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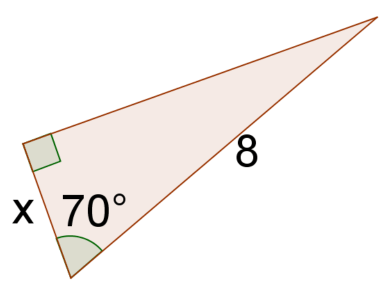

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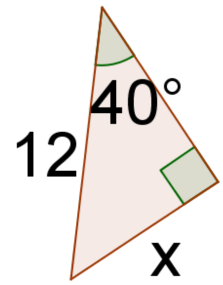

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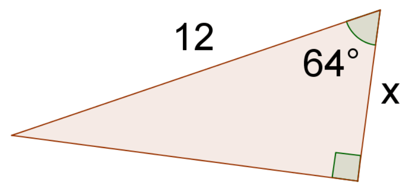

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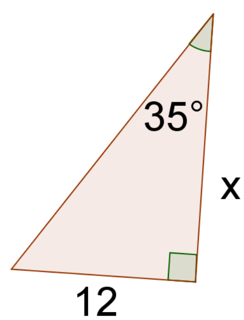
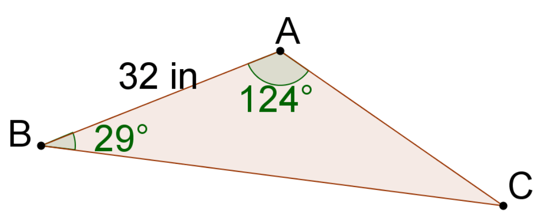


Use the triangle below for #13-#15.



13. Find $m\angle C$.

14. Draw an altitude from $\angle B$ to divide the triangle into two right triangles. Use trigonometry to find the lengths of the sides of each of these right triangles.

15. Find the perimeter of ΔABC.

16. True or false: You need to know the length of at least one side of a triangle to find the lengths of the other sides.

17. What are the trigonometric ratios?

18. What do the trigonometric ratios have to do with similar triangles?

19. The curvature of the earth makes it impossible to see beyond the horizon, but as an observer ascends, the distance to the horizon increases. The radius of the earth is 3,959 miles. How high must an individual ascend in order to see a horizon line corresponding to a 1° rotation around the center of the earth? (Sketch is not to scale.)

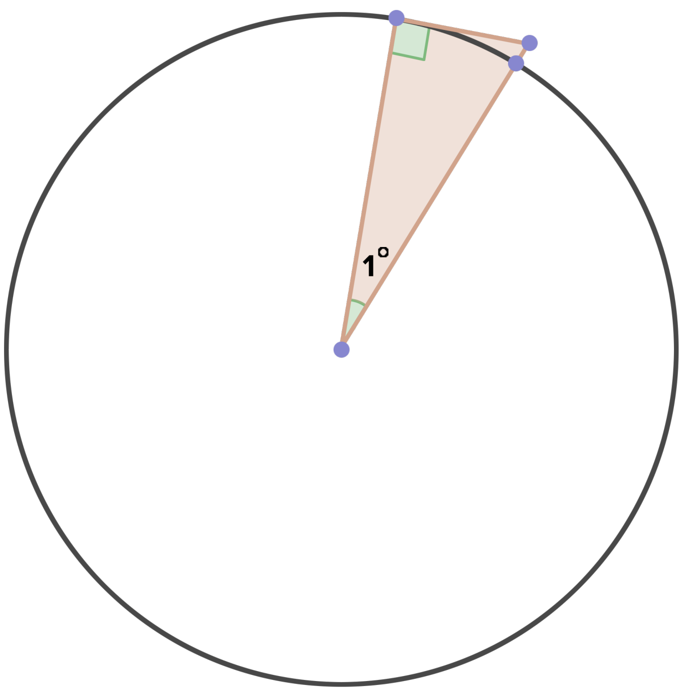


FIGURE 7.2

20. Use the interactive in this lesson to help you, or sketch by hand. What is the range of possible values for an acute angle of a right triangle? If one acute angle of a right triangle is very small, how does its side opposite compare to the other sides in the triangle? How does this impact the sine of the angle? If one acute angle of a right triangle is very large, how does its side opposite compare to the other sides of the triangle? How does this impact the sine of the angle? What is the range of values for the sine of an acute angle? What is the range of values for the cosine of an acute angle?

21. Sine is defined as opposite over hypotenuse and cosine is defined as adjacent over hypotenuse. Why not hypotenuse over opposite and hypotenuse over adjacent?

22. Use a calculator to test whether or not this is true: $(\sin\theta)^2 + (\cos\theta)^2 = 1$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.2.

7.3 Sine and Cosine of Complementary Angles

Learning Objectives

Here you will explore how the sine and cosine of complementary angles are related.

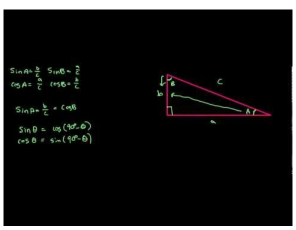
ΔABC is a right triangle with $m\angle C = 90^\circ$ and $\sin A = k$. What is $\cos B$?

Sine and Cosine of Complementary Angles

Recall that the **sine** and **cosine** of angles are ratios of pairs of sides in right triangles.

- The **sine** of an angle in a right triangle is the ratio of the side *opposite* the angle to the *hypotenuse*.
- The **cosine** of an angle in a right triangle is the ratio of the side *adjacent* to the angle to the *hypotenuse*.

In the problems below, you will explore how the sine and cosine of the angles in a right triangle are related.



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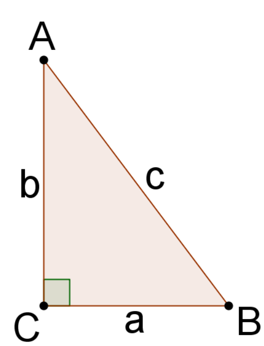
Let's take a look at a few example problems.

1. Consider the right triangle below. Find the sine and cosine of angles A and B in terms of a, b, and c. What do you notice?



$\sin A = \frac{a}{c}, \sin B = \frac{b}{c}, \cos A = \frac{b}{c}, \cos B = \frac{a}{c}$ Note that $\sin A = \cos B$ and $\sin B = \cos A$.

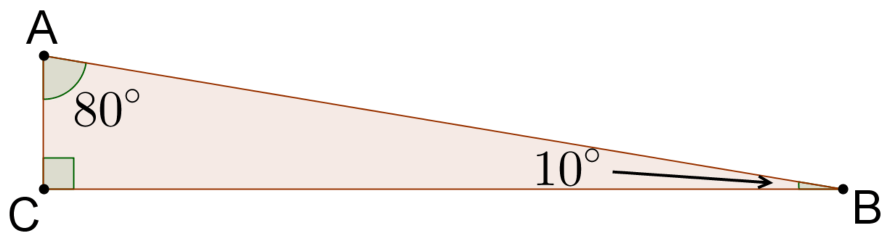
2. Consider the triangle from the previous problem. How is $\angle A$ related to $\angle B$?

The sum of the measures of the three angles in a triangle is 180°. This means that $m\angle A + m\angle B + m\angle C = 180^\circ$. $\angle C$ is a right angle so $m\angle C = 90^\circ$. Therefore, $m\angle A + m\angle B = 90^\circ$. Angles A and B are complementary angles because their sum is 90°.

In #1 you saw that $\sin A = \cos B$ and $\sin B = \cos A$. This means that the sine and cosine of *complementary angles* are equal.

3. Find 80° and $\cos 10^\circ$. Explain the result.

$\sin 80^\circ \approx 0.985$ and $\cos 10^\circ \approx 0.985$. $\sin 80^\circ = \cos 10^\circ$ because 80° and 10° are complementary angle measures. $\sin 80^\circ$ and $\cos 10^\circ$ are the ratios of the same sides of a right triangle, as shown below.



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Examples

Example 1

Earlier, you were asked what is $\cos B$.

ΔABC is a right triangle with $m\angle C = 90^\circ$ and $\sin A = k$. What is $\cos B$?

$\angle A$ and $\angle B$ are complementary because they are the two non-right angles of a right triangle. This means that $\sin A = \cos B$ and $\sin B = \cos A$. If $\sin A = k$, then $\cos B = k$ as well.

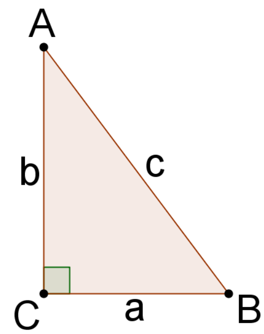
Example 2

If $\sin 30^\circ = \frac{1}{2}, \cos ? = \frac{1}{2}$.

The sine and cosine of **complementary** angles are equal. $90^\circ - 30^\circ = 60^\circ$ is complementary to 30°. Therefore, $\cos 60^\circ = \frac{1}{2}$.

Example 3

Consider the right triangle below. Find $\tan A$ and $\tan B$.



$\tan A = \frac{a}{b}$ and $\tan B = \frac{b}{a}$.

Example 4

In general, what is the relationship between the tangents of complementary angles?

In general, the tangents of complementary angles are reciprocals.

CK-12 PLIX Interactive

PLIX

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URL: http://www.ck12.org/trigonometry/sine-and-cosine-of-complementary-angles/plix/Find-the-Sine-Cosine-and-Tangent-56b91020da2cfe61dcfc62be

1. How are the two non-right angles in a right triangle related? Explain.

2. How are the sine and cosine of complementary angles related? Explain.

3. How are the tangents of complementary angles related? Explain.

Let A and B be the two non-right angles in a right triangle.

4. If $\tan A = \frac{1}{2}$, what is $\tan B$?

5. If $\sin A = \frac{7}{10}$, what is $\cos B$?

6. If $\cos A = \frac{1}{4}$ what is $\sin B$?

7. If $\sin A = \frac{3}{5}, \cos ? = \frac{3}{5}$?

8. Simplify $\frac{\sin A + \cos B}{2}$.

9. If $\tan A = \frac{2}{3}$ what is $\tan B$?

10. If $\tan B = \frac{1}{5}$, what is $\tan A$? Which angle is bigger, $\angle A$ or $\angle B$?

Solve for θ.

11. $\cos 30^\circ = \sin\theta$

12. $\sin 75^\circ = \cos\theta$

13. $\cos 52^\circ = \sin\theta$

14. $\sin 18^\circ = \cos\theta$

15. $\cos 49^\circ = \sin\theta$

16. What is the slope of an increasing line that intersects the x-axis at an acute angle θ? What is the slope of any line parallel to this line? What is the slope of any line perpendicular to this line? Why?

17. Sine is opposite over hypotenuse, and cosine is adjacent over hypotenuse. What is the sine of angle divided by the cosine of the same angle? What do you observe?

18. In this lesson you explored the sine, cosine and tangent of complements. Use a calculator to explore the sine, cosine and tangent of complements. What do you observe?

19. Do the patterns you observed for sine, cosine and tangent of complements hold true for 0° and 90°? Use a calculator to explore further. Why or why not?

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.3.

7.4 Inverse Trigonometric Ratios

Learning Objectives

Here you will learn how to use the inverses of the trigonometric ratios to find the measures of angles within right triangles.

The maximum slope of a wheelchair ramp is 1:12. For a wheelchair ramp made with these specifications, what angle does the ramp make with the flat ground?

Inverse Trigonometric Ratios

Recall that the **sine, cosine,** and **tangent** of angles are ratios of pairs of sides in right triangles.

- The **sine** of an angle in a right triangle is the ratio of the side *opposite* the angle to the *hypotenuse.*
- The **cosine** of an angle in a right triangle is the ratio of the side *adjacent* to the angle to the *hypotenuse*.
- The **tangent** of an angle in a right triangle is the ratio of the side *opposite* the angle to the side *adjacent* to the angle.

You can use the trigonometric ratios to find missing sides of right triangles when given at least one side length and one angle measure. You can use the ***inverse*** **trigonometric ratios** to find a missing angle in a right triangle when given two sides.

- The **inverse sine** of a *ratio* gives the *angle* in a right triangle whose sine is the given ratio. Inverse sine is also called **arcsine** and is labeled $\sin^{-1}$ or **arcsin**.
- The **inverse cosine** of a *ratio* gives the *angle* in a right triangle whose cosine is the given ratio. Inverse cosine is also called **arccosine** and is labeled $\cos^{-1}$ or **arccos**.
- The **inverse tangent** of a *ratio* gives the *angle* in a right triangle whose tangent is the given ratio. Inverse tangent is also called **arctangent** and is labeled $\tan^{-1}$ or **arctan**.

Note that in each case the "-1" is to indicate inverse, and is not an exponent.

To find the measure of an angle using an inverse trigonometric ratio, you will need to use your calculator. Most scientific and graphing calculators have buttons that look like $[\sin^{-1}], [\cos^{-1}]$, and $[\tan^{-1}]$. You will want to make sure that your calculator is in degree mode so that the angle measure that the calculator produces is in degrees.

CK-12 PLIX Interactive

PLIX

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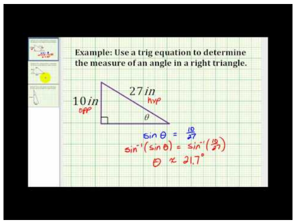
URL: http://www.ck12.org/trigonometry/inverse-trig-functions/plix/Basic-Inverse-Trig-Functions-Ladder-54c2e696da2cfe7bd16cc0cd



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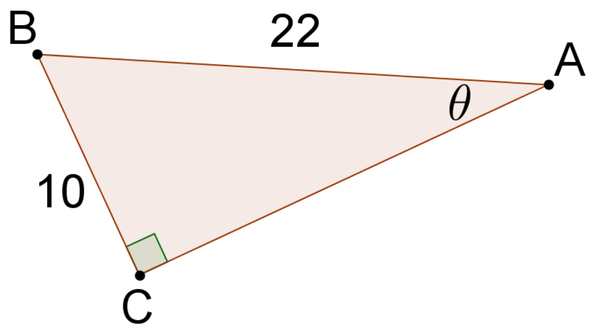
Let's take a look at some example problems.

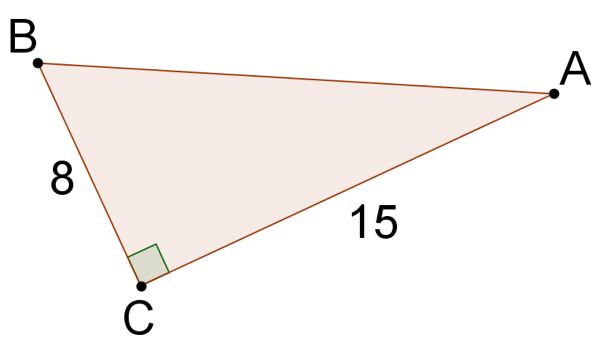
1. Solve for θ.



The side **opposite** the given angle is length 10 and the **hypotenuse** is length 22. This is a sine relationship.

$$\sin\theta = \frac{10}{22}$$
$$\theta = \sin^{-1}\left(\frac{10}{22}\right)$$
$$\theta \approx 27.04^\circ$$

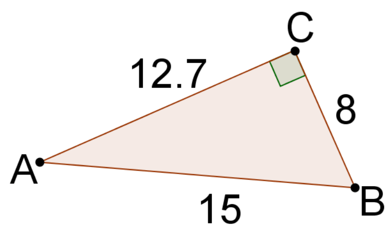
2. Find $m\angle B$.



The side **opposite** $\angle B$ is length 15 and the side **adjacent** to $\angle B$ is length 8. This is a tangent relationship.

$$\tan B = \frac{15}{8}$$
$$m\angle B = \tan^{-1}\left(\frac{15}{8}\right)$$
$$m\angle B \approx 61.93^\circ$$

3. Find $m\angle A$.



You only need to know two sides of the triangle in order to find the measure of one of the angles. Since you are given all three sides, you can choose which two sides you want to use.

The side **adjacent** to $\angle A$ is length 12.7 and the **hypotenuse** is length 15. These two sides are a cosine relationship.

$$\cos A = \frac{12.7}{15}$$
$$m\angle A = \cos^{-1}\left(\frac{12.7}{15}\right)$$
$$m\angle A \approx 32.15^\circ$$

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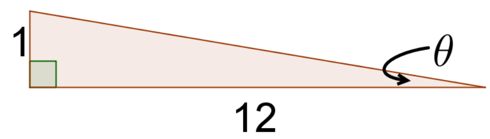
Examples

Example 1

Earlier, you were asked what angle a ramp makes with the flat ground.

The maximum slope of a wheelchair ramp is 1:12. For a wheelchair ramp made with these specifications, what angle does the ramp make with the flat ground?

Draw a picture to represent this situation.

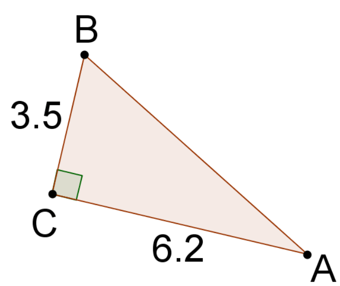


The side opposite the given angle is length 1 and the side adjacent to the given angle is length 12. This is a tangent relationship.

$$\begin{aligned}\tan\theta &= \frac{1}{12}\\ \theta &= \tan^{-1}\frac{1}{12}\\ \theta &\approx 4.76^\circ\end{aligned}$$

The wheelchair ramp makes approximately a 4.76° angle with the ground.

Use the triangle below for #2-#4.



Example 2

Find $m\angle A$.

The given sides are opposite and adjacent to $\angle A$. This is a tangent relationship.

$$\begin{aligned}\tan A &= \frac{3.5}{6.2}\\ \angle A &= \tan^{-1}\frac{3.5}{6.2}\\ \angle A &\approx 29.45^\circ\end{aligned}$$

Example 3

Find $m\angle B$.

$\angle A$ and $\angle B$ are complementary:

$$\begin{aligned} m\angle B &= 90^\circ - m\angle A \\ &= 90^\circ - 29.45^\circ \\ &= 60.55^\circ \end{aligned}$$

You could also use inverse tangent to find $m\angle B$.

Example 4

Find AB.

With all angle measures and two sides, you could use sine, cosine, or the Pythagorean Theorem to find AB. Using the Pythagorean Theorem:

$$\begin{aligned} AB^2 &= 3.5^2 + 6.2^2 \\ AB^2 &= 50.69 \\ AB &\approx 7.12 \end{aligned}$$

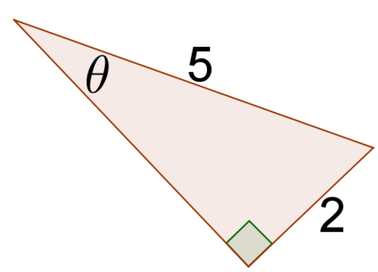
Review

1. What is the difference between $\sin^{-1}$ and sin?

2. When do you use regular trigonometric ratios and when do you use inverse trigonometric ratios?

Solve for θ.

3.



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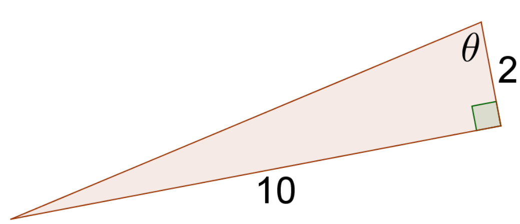

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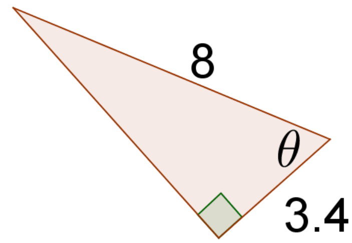

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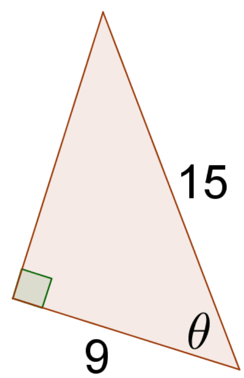

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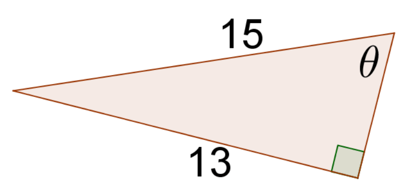

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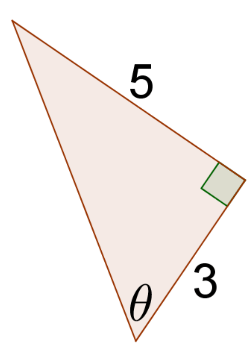

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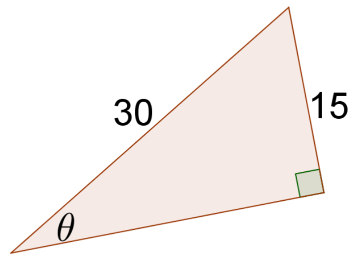
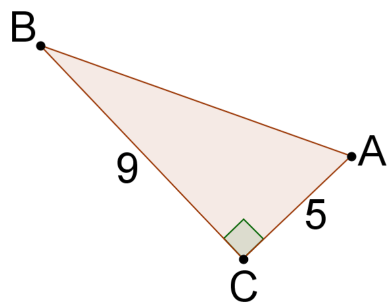


10. How could you have found θ in #9 without using an inverse trigonometric ratio?

Find all missing information (sides and angles) for each triangle.

11.



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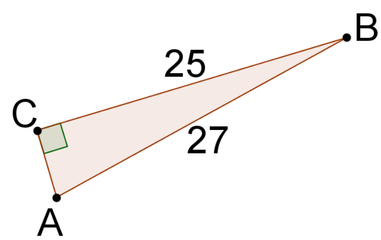

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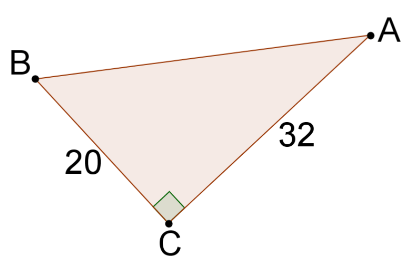


14. How are inverse cosine and inverse sine of the same ratio related?

15. How are inverse tangents of a and $\frac{1}{a}$ related?

16. Given the following graph of a line, find the angles shown. Are the two angles the same measure? How do you know?

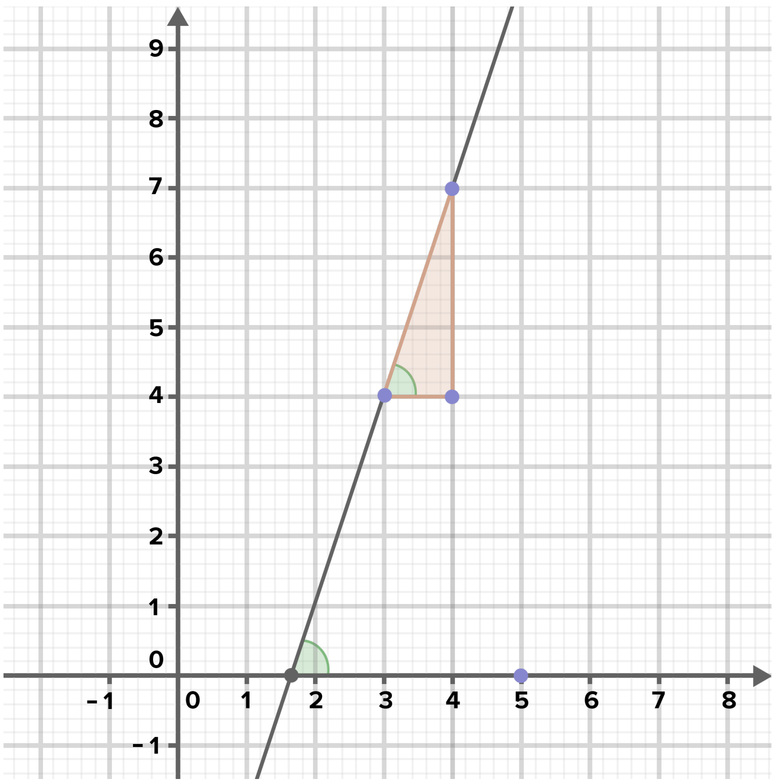


FIGURE 7.3

17. Consider the graphs of the following lines. How does the slope change from line to line? How does this impact the graphs of the lines? Find the angle of intersection of each line with the x-axis. Is it true that if we double the slope, the angle doubles? Why or why not?

$y = \frac{1}{4}x$

$y = \frac{1}{2}x$

$y = x$

$y = 2x$

$y = 4x$

$y = 8x$

18. Consider a line with a slope of 100. What is the angle of intersection of this line with the x-axis? Is there any limit to how steep a line can get? Describe the steepness in terms of slope and in terms of the angle with the x-axis.

19. Use a calculator to compute $\tan 90^\circ$. What do you observe? Explain this result.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.4.

7.5 Sine to Find the Area of a Triangle

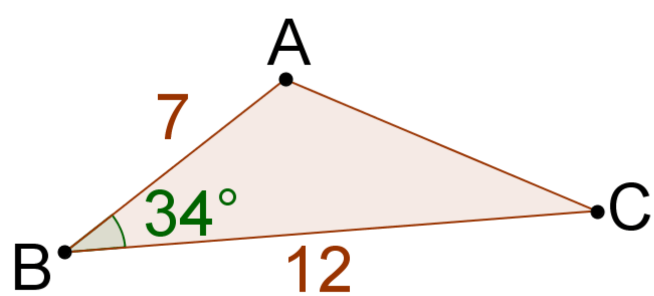
Learning Objectives

Here you will learn how to find the area of a triangle using the sine ratio.

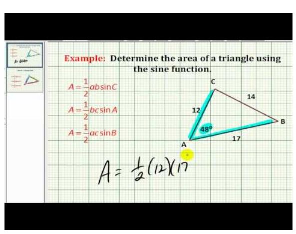
You have previously learned that the area of a triangle is $A = \frac{1}{2}bh$ where b is a base of the triangle and h is the corresponding height. Why is it helpful to have another formula for calculating the area of a triangle?

Finding the Area of a Triangle by using Sine

One way to find the area of a triangle is by calculating $\frac{1}{2}bh$. This formula works when you know or can determine the base and the height of a triangle. What if you wanted to find the area of the following triangle:



The two given sides are *not* the base and the height. In the examples you will derive a formula for calculating the area of a triangle given this type of information.



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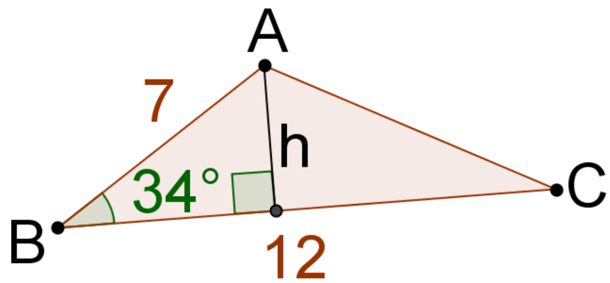
Below, the altitude of ΔABC from A has been drawn. Write an equation that relates $h, 34^\circ$ and 7.



Notice that two right triangles have been formed. This means you can use the trigonometric ratios to relate the sides. h is opposite the 34° angle and 7 is the length of the hypotenuse of the right triangle. This is the sine relationship.

$\sin 34^\circ = \frac{h}{7}$

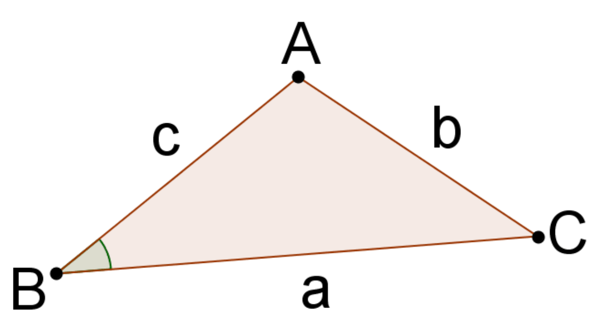
Solving for Unknown Values

Solve the equation for h. Then, use $A = \frac{1}{2}bh$ to find the area of the triangle using your value for h. Can you generalize the formula?

If $\sin 34^\circ = \frac{h}{7}$ then $h = 7\sin 34^\circ$. For the height that has been drawn, $\overline{BC}$ is the base.

$$
\begin{aligned}
A &= \frac{1}{2}bh \\
A &= \frac{1}{2}(12)(7\sin 34^\circ) \\
A &= \frac{1}{2}(12)(7)(\sin 34^\circ) \\
A &\approx 23.45\ un^2
\end{aligned}
$$

Notice the calculation: $A = \frac{1}{2}(12)(7)(\sin 34^\circ)$. You found half the product of the two sides and the sine of the included angle. Consider the general triangle below:



If you know $\angle B$, the area will be $Area = \frac{1}{2}(a)(c)\sin B$. If you know $\angle C$, the area will be $Area = \frac{1}{2}(a)(b)\sin C$.

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PLIX

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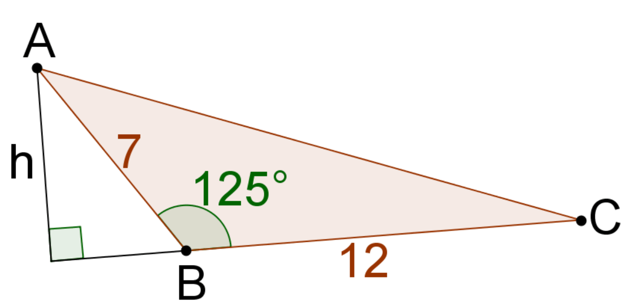
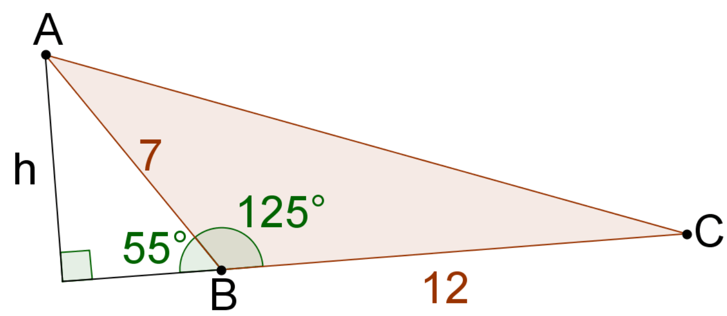
Finding the Area Given Angle Measurements

Previously, you have only used the sine ratio on acute angles. Can you find the area of a triangle when given an obtuse angle? Consider the triangle below. Find an equation to determine the value of h. Then, find the area of the triangle.



First find the measure of the exterior angle of the triangle at vertex B. Remember that the interior and exterior angles at vertex B must sum to 180°.



Now, consider the right triangle with $\overline{AB}$ as the hypotenuse. h is the side opposite the 55° angle and 7 is the hypotenuse. Once again, this is the sine relationship.

$$\sin 55^\circ = \frac{h}{7}$$
$$\rightarrow h = 7\sin 55^\circ$$

For the height that has been drawn, $\overline{BC}$ is the base.

$$A = \frac{1}{2}bh$$
$$A = \frac{1}{2}(12)(7\sin 55^\circ)$$
$$A = \frac{1}{2}(12)(7)(\sin 55^\circ)$$
$$A \approx 34.4\ un^2$$

In general, if your given angle is obtuse, you can use the angle supplementary to your given angle in your area calculation.

Examples

Example 1

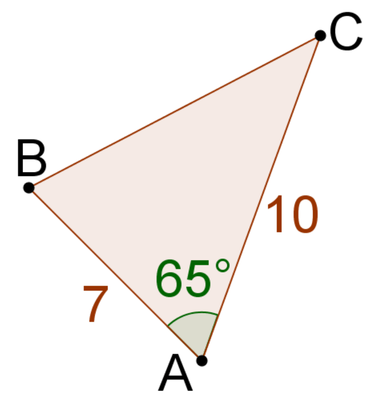
Earlier, you were asked why is it helpful to have another formula for calculating the area of a triangle.

You have previously learned that the area of a triangle is $A = \frac{1}{2}bh$ where b is a base of the triangle and h is the corresponding height. Why is it helpful to have another formula for calculating the area of a triangle?

Often you only know the sides and angles of a triangle and not the height. $A = \frac{1}{2}(a)(b)\sin C$ allows you to quickly find the area of acute or obtuse triangles when given two sides and an included angle.

Example 2

Find the area of the triangle.

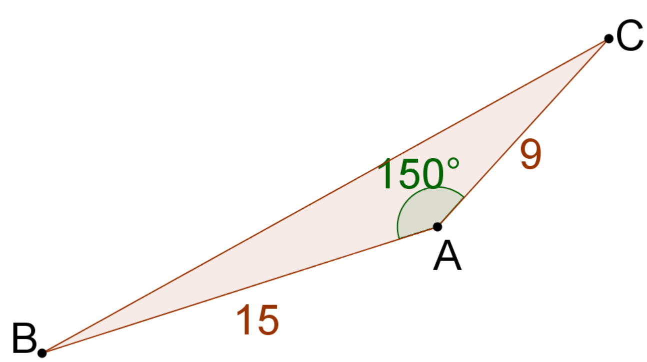


The 65° angle is the included angle to the two given sides of length 7 and 10. Use the new area formula.

$$A = \frac{1}{2}(7)(10)\sin 65^\circ$$
$$A \approx 31.72\ un^2$$

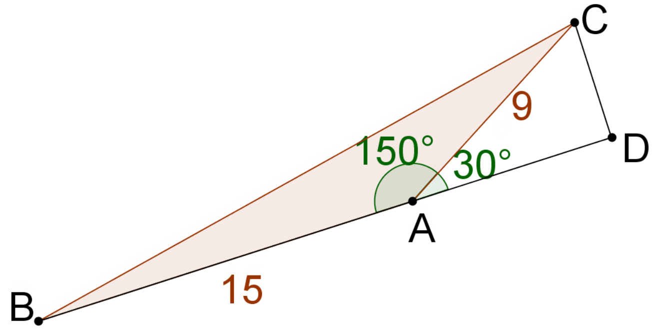
Example 3

Find the area of the triangle.



The **sine ratio** is actually a **function** that takes <u>*any*</u> angle measure as an input (not just angles between 0° and 90°). One property of the sine function is that the sine of supplementary angles will always be equal. Therefore, when finding the area of a triangle given an obtuse angle, you can use the obtuse angle in your calculation instead of the acute supplementary angle and your answer will be the same. *Note: You will study the sine function in much more detail in future courses!*

Notice that the given angle is obtuse. Draw the altitude from vertex C so that it intersects the extension of $\overline{AB}$ at point D. The exterior angle at $\angle A$ is 30°.



Use the new area formula with the 30° angle.

$$A = \frac{1}{2}(9)(15)\sin 30^\circ$$
$$A = 33.75\ un^2$$

Example 4

Find the area of the triangle from #2 using the obtuse angle in your calculations. What do you notice?

Use the area formula as in #3. Instead of using 30°, use 150°.

$$A = \frac{1}{2}(9)(15)\sin 150^\circ$$
$$A = 33.75\ un^2$$

The result is the same. $\sin 30^\circ = 0.5$ and $\sin 150^\circ = 0.5$.

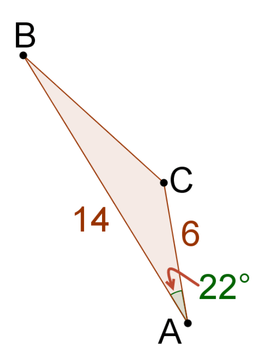
This means that regardless of whether the given angle is acute or obtuse, you can always find the area of a triangle by finding the half the product of two sides and the sine of their included angle:

For all triangles: $A = \frac{1}{2}(a)(b)\sin C$

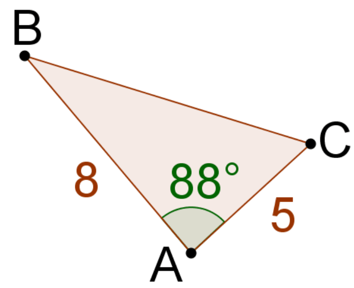
Review

Find the area of each triangle.

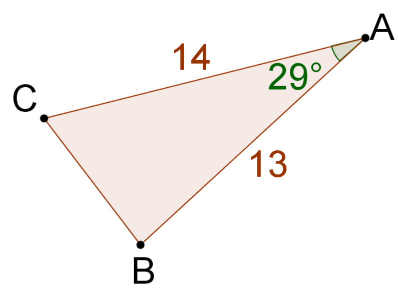
1.



2.



3.



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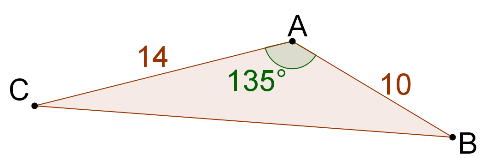
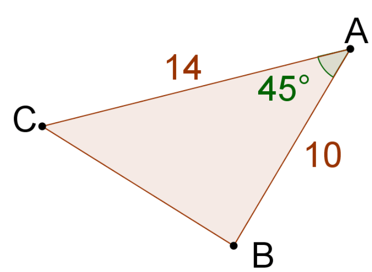
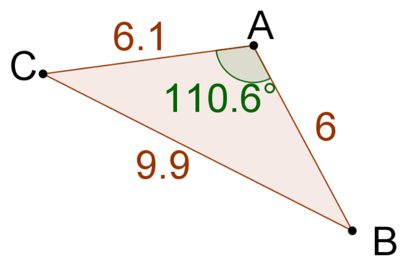


5. Explain why the following triangle has the same area as the triangle in #4.



Find the area of each triangle.

6.



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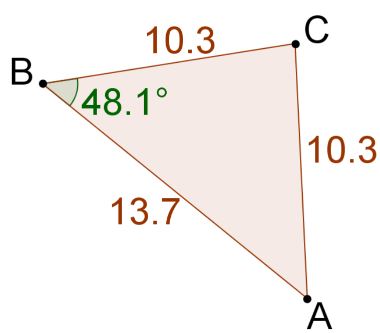

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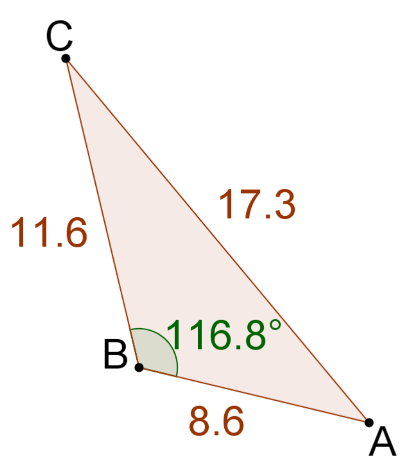
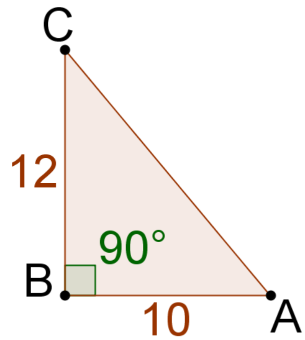
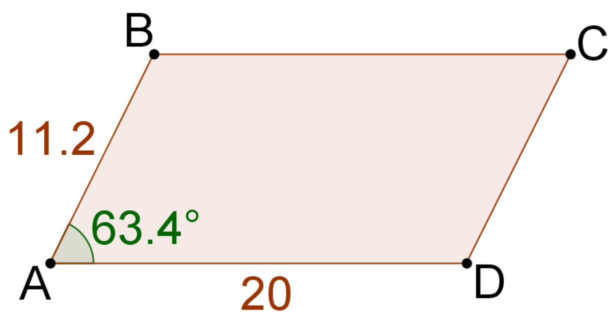


9. Use your calculator to find $\sin 90^\circ$.

10. Find the area of the triangle below in two ways. First, use the formula $A = \frac{1}{2}bh$. Then, use the new formula from this concept. What do you notice?

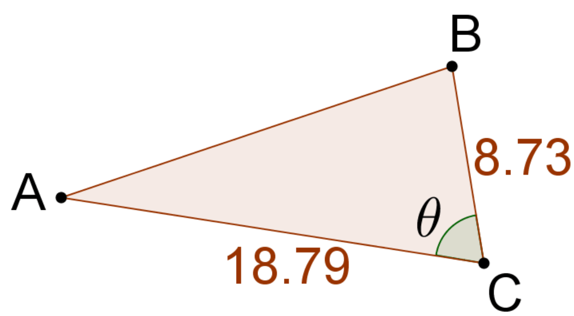


11. Find the area of the parallelogram:

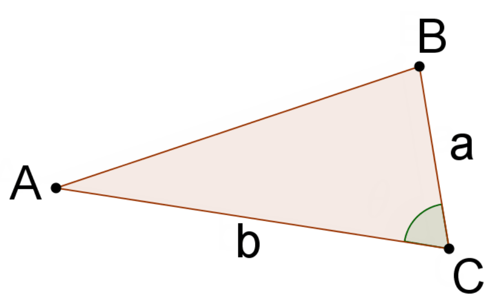


12. Use your work from #11 to help you to describe a general method for calculating the area of a parallelogram given its sides and angles.

13. The area of the triangle below is 78 un^2. Find the measure of θ rounded to the nearest degree.



14. Use the triangle below to explain where the area formula $A = \frac{1}{2}ab\sin C$ comes from.



15. Why does the area formula $A = \frac{1}{2}ab\sin C$ work even if $\angle C$ is obtuse?

16. Consider a right triangle with legs of length 3 and 4. Find the area using two different formulas. Is the result the same? Why or why not?

17. Given the triangle shown, use the formula learned in this lesson to write the area two different ways. Are these two results equal? Set them equal to each other and simplify as much as possible, so that the result shows two equivalent fractions. Create a problem that can be solved with this new formula.

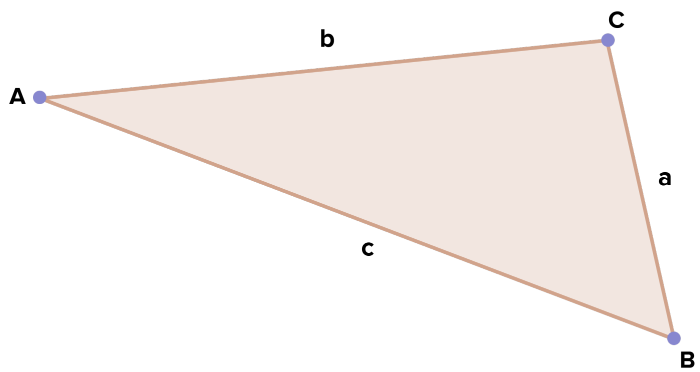


FIGURE 7.4

18. For this problem, keep in mind that the sine of the supplement of an acute angle θ is the **same** as the sine of θ. Create two **different** triangles for which two sides are the **same**, such that the **areas** are the same.

Review (Answers)

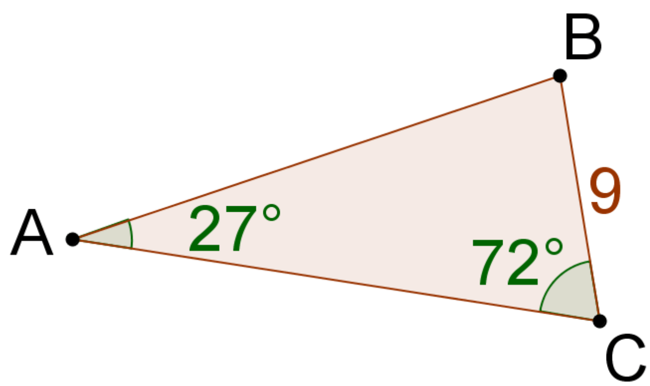
To see the Review answers, open this PDF file and look for section 7.5.

7.6 Law of Sines

Learning Objectives

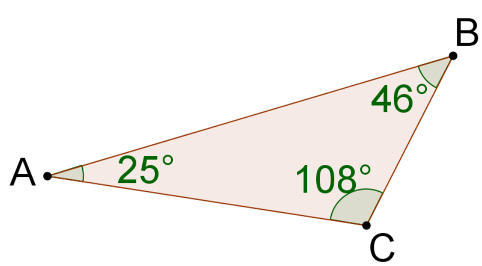
Here you will prove the Law of Sines and learn how to use it.

You know how to use trigonometric ratios to find missing sides in right triangles, but what about non-right triangles? For the triangle below, can you find AB?



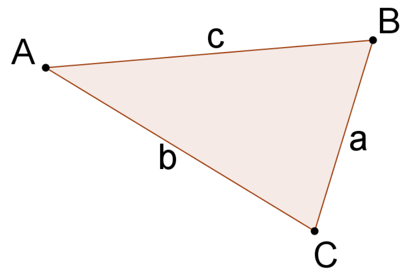
Law of Sines

Look at the triangle below. Based on the angles, can you tell which side is the shortest? Which side is the longest?

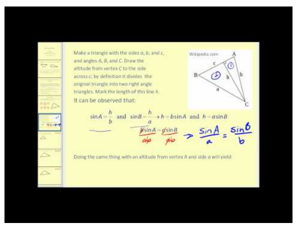


The smallest angle is $\angle A$. It opens up to create the shortest side, $\overline{BC}$. The largest angle is $\angle C$. It opens up to create the longest side, $\overline{AB}$. Clearly, angles and opposite sides within triangles are connected. In fact, the Law of Sines states that the ratio between the sine of an angle and the side opposite the angle is **constant** for each of the three angle/side pairs within a triangle.

Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



In the past, you used the sine ratio to find missing sides and angles in *right* triangles. With the Law of Sines, you can find missing sides and angles in *any type* of triangle. In the following problems you will learn how to prove the Law of Sines.



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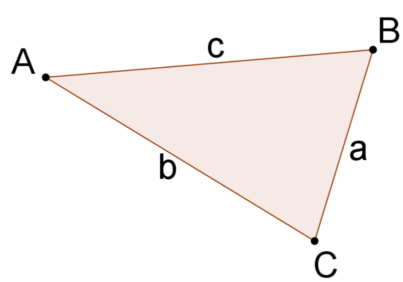
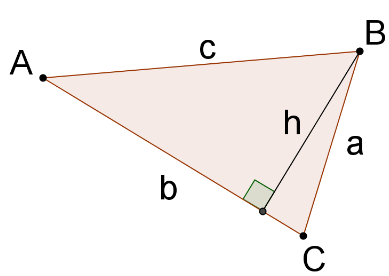
1. Consider ΔABC below. Draw an altitude from vertex B that intersects $\overline{AC}$ to divide ΔABC into two right triangles. Find two equations for the length of the altitude.



Below, the altitude has been drawn and labeled h.



Consider the right triangle with hypotenuse of length c. h is opposite $\angle A$ in this triangle. With an opposite side and hypotenuse you can use the sine ratio.

$$\sin A = \frac{h}{c}$$
$$h = c \sin A$$

Now consider the right triangle with hypotenuse of length a. h is opposite $\angle C$ in this triangle. With an opposite side and hypotenuse you can use the sine ratio.

$$\sin C = \frac{h}{a}$$
$$h = a \sin C$$

You have now found two equations for h.

2. From first problem, $h = c \sin A$ and $h = a \sin C$. This means that $c \sin A = a \sin C$. Divide both sides by ac and you will see the Law of Sines.

$$c \sin A = a \sin C$$
$$\frac{c \sin A}{ac} = \frac{a \sin C}{ac}$$
$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

A and C were two random angles in the original triangle. This proves that in general, within any triangle the ratio of the sine of an angle to its opposite side is constant.

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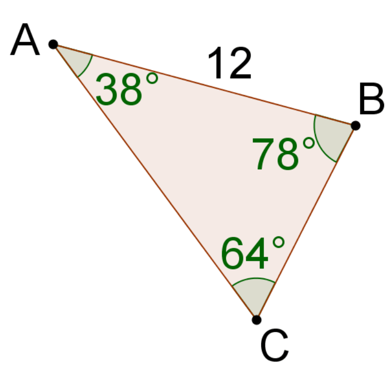
Now, let's do a problem using the Law of Sines.

Use the Law of Sines to find BC and AC.



Look for an angle and side pair whose measurements are both given. $m\angle C = 64^\circ$, and its opposite side is $AB = 12$. Next, set up an equation using the two known measurements as one of the ratios in the Law of Sines. Make sure to match angles with opposite sides.

Solve for BC:

$$\frac{\sin 64^\circ}{12} = \frac{\sin 38^\circ}{BC}$$
$$BC = \frac{12 \sin 38^\circ}{\sin 64^\circ}$$
$$BC \approx 8.22$$

Solve for AC:

$$\frac{\sin 64^\circ}{12} = \frac{\sin 78^\circ}{AC}$$
$$AC = \frac{12 \sin 78^\circ}{\sin 64^\circ}$$
$$AC \approx 13.06$$

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CK-12 PLIX Interactive

PLIX

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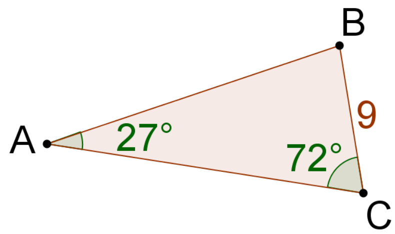
Example 1

Earlier, you were asked to find the side AB.

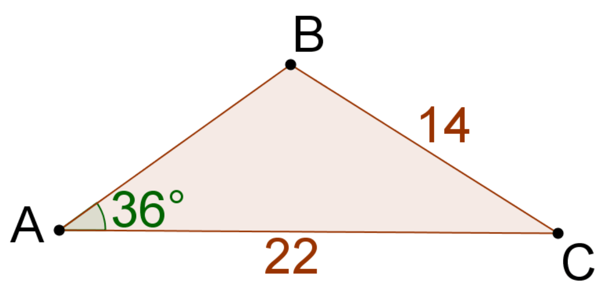


To find AB, you can use the Law of Sines. Match angles with opposite sides.

$$\frac{\sin 27^\circ}{9} = \frac{\sin 72^\circ}{AB}$$
$$AB = \frac{9 \sin 72^\circ}{\sin 27^\circ}$$
$$AB \approx 18.85$$

Example 2

The triangle below is drawn to scale. Use the Law of Sines to solve for the measure of angle B. Does your answer seem correct?



From looking at the triangle, $\angle B$ looks to be an obtuse angle. Match angles with opposite sides.

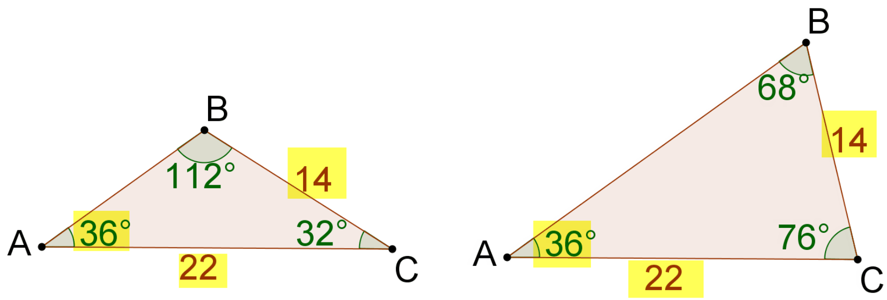
$$\frac{\sin 36^\circ}{14} = \frac{\sin B}{22}$$
$$\sin B = \frac{22 \sin 36^\circ}{14}$$
$$m\angle B = \sin^{-1}\left(\frac{22 \sin 36^\circ}{14}\right)$$
$$m\angle B \approx 67.5^\circ$$

This answer doesn't seem correct because the angle appears to be obtuse (greater than 90°).

Example 3

Recall that SSA was not a criterion for triangle congruence. This was because two non-congruent triangles could have the same "side-side-angle" pattern. What does this have to do with the seemingly wrong answer to #2?

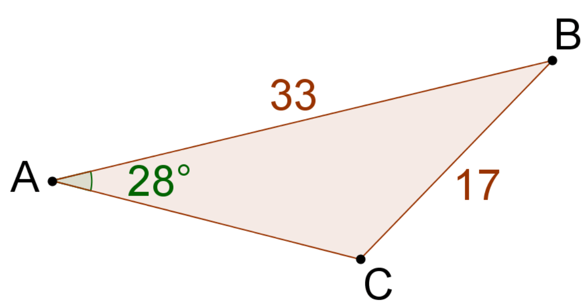
There are two triangles that fit the criteria given in #2 (angle measures have been slightly rounded).



The two possible measures of $\angle B$ are supplementary $(112^\circ + 68^\circ = 180^\circ)$. Also note that $\sin 112^\circ = 0.927 = \sin 68^\circ$. The inverse sine function on your calculator will only produce angles between 0° and 90°. In a sense, the calculator was imagining the triangle on the right while you were imagining the triangle on the left. **This is a problem when using the Law of Sines to solve for missing angles**. If the information you are given fits SSA, it is possible that there are two answers. If your picture is drawn to scale, you can determine whether your answer should be the acute angle or the obtuse angle by looking at the picture.

Example 4

The triangle below is drawn to scale. Use the Law of Sines to solve for the measure of angle C.



Angle C appears to be obtuse. Match angles with opposite sides.

$$
\begin{aligned}
\frac{\sin 28^\circ}{17} &= \frac{\sin C}{33} \\
\sin C &= \frac{33 \sin 28^\circ}{17} \\
m\angle C &= \sin^{-1}\left(\frac{33 \sin 28^\circ}{17}\right) \\
m\angle C &\approx 65.7^\circ
\end{aligned}
$$

This is the acute version of the answer. You know the angle should be obtuse, so find the angle supplementary to 65.7°:

$180^\circ - 65.7^\circ = 114.3^\circ$

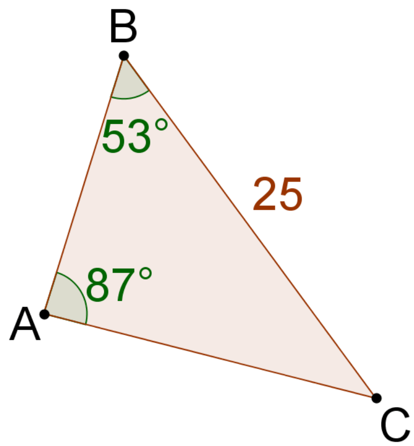
Note that $\sin 65.7^\circ = \sin 114.3$

The final answer is: $m\angle C = 114.3^\circ$**.**

Review

For each triangle, find the measure of all missing sides and angles.

1.



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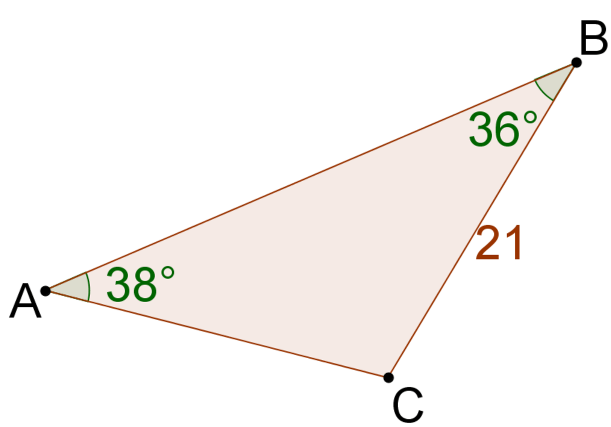

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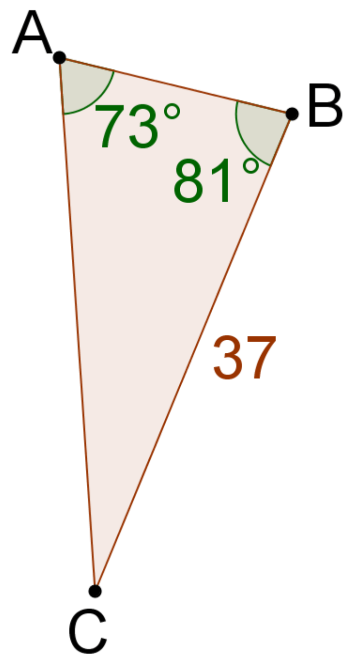

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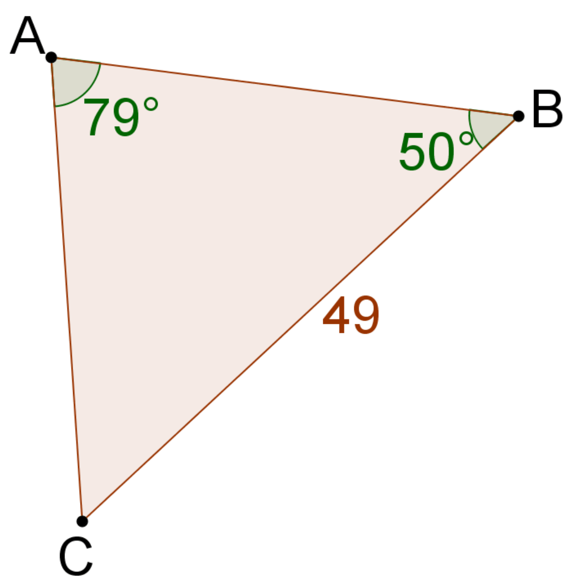

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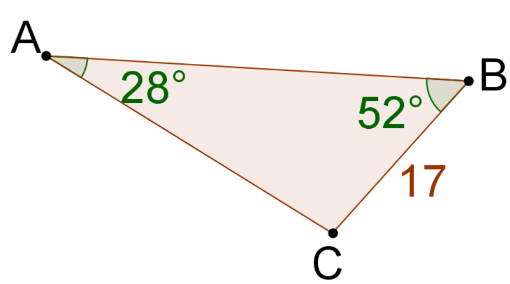

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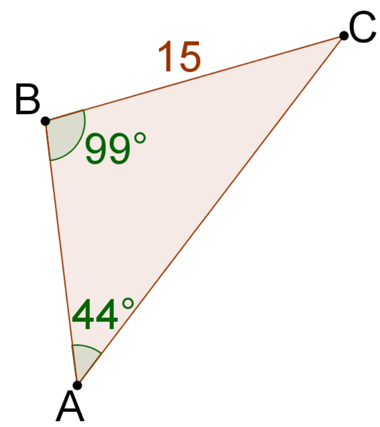

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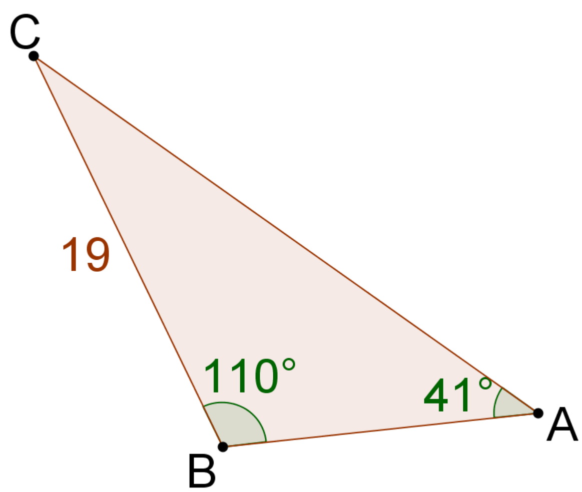

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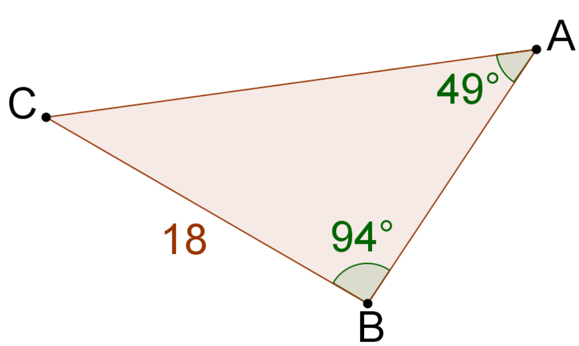
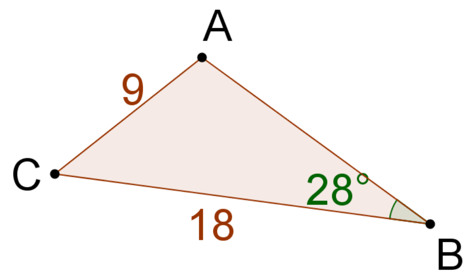
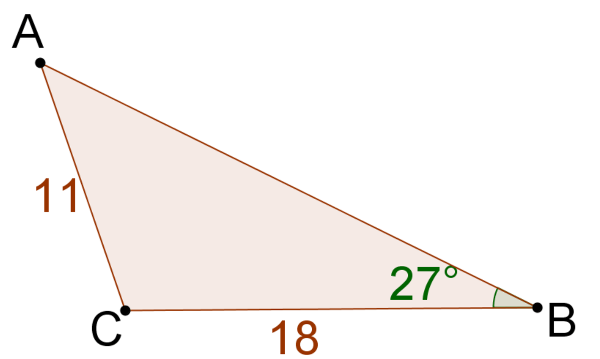


9. What does SSA have to do with the Law of Sines? What type of problems do you have to think extra carefully about to make sure you have the correct answer?

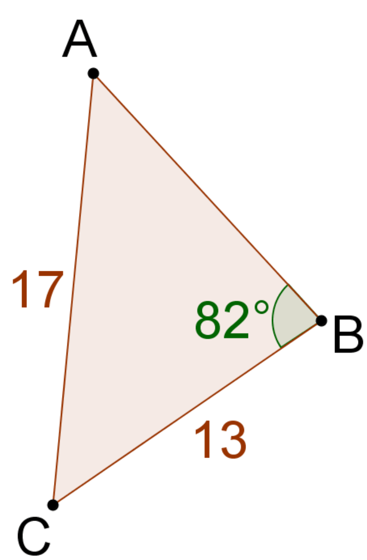
10. The triangle below is drawn to scale. Determine the measure of each of the missing angles.



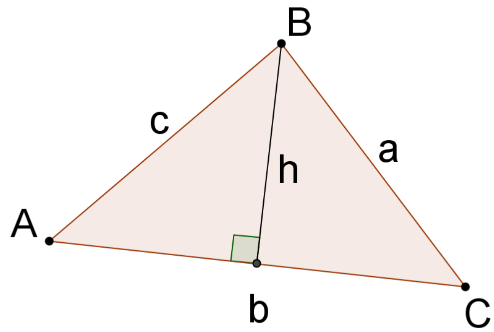
11. The triangle below is drawn to scale. Determine the measure of each of the missing angles.



12. The triangle below is drawn to scale. Determine the measure of each of the missing angles.

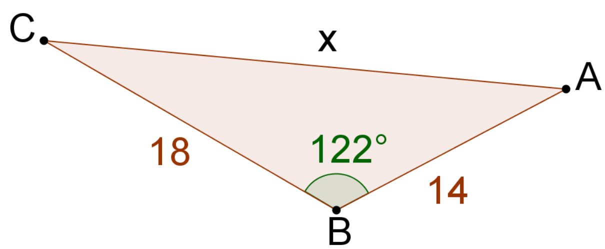


13. Use the picture below to derive the Law of Sines.



14. What type of problems can you solve with the Law of Sines?

15. Explain why you *cannot* use the Law of Sines to solve for x in the triangle below.



16. A park ranger at a watch tower at point A spots a fire at point C. She measures the angle between the trail and the fire at 40 degrees. She hikes along a straight trail for 1 mile, observing the fire as she goes. She stops at point B and measures the angle between the fire and the trail at 30 degrees. How far is she from the fire?

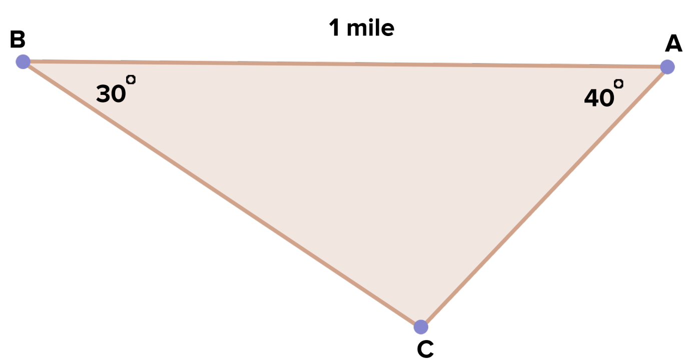


FIGURE 7.5

17. A hiker is walking through a flat meadow, observing a mountain in the distance. She measures the angle of elevation between the ground and the mountaintop to be 25 degrees. She walks 5 miles, and then measures the angle of elevation again—now it's 40 degrees. How tall is the mountain?

Review (Answers)

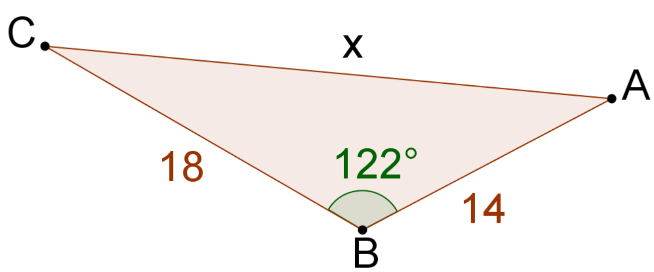
To see the Review answers, open this PDF file and look for section 7.6.

7.7 Law of Cosines

Learning Objectives

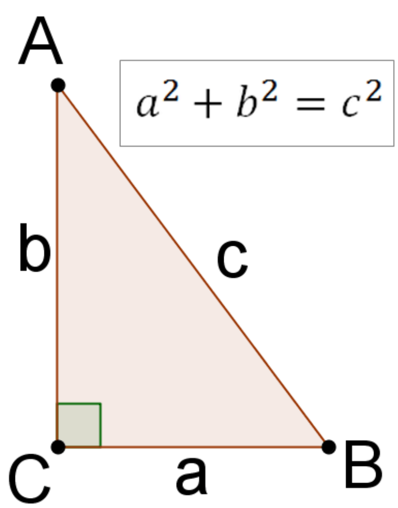
Here you will prove the Law of Cosines and learn how to use it.

You know how to use trigonometric ratios to find missing sides in right triangles. You also know how to use the Law of Sines to find missing sides in non-right triangles given certain information. What about the triangle below? Can you solve for x?

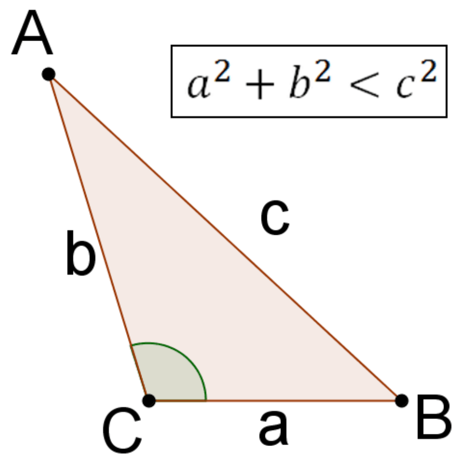


Law of Cosines

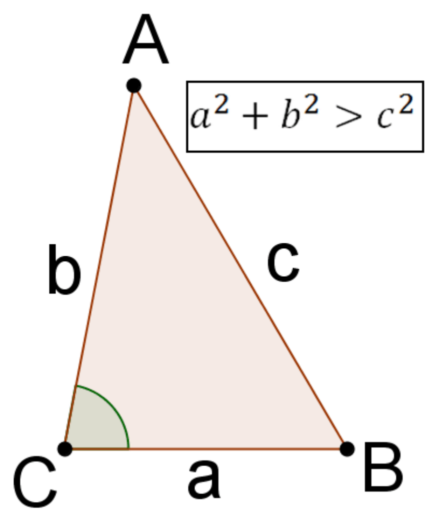
The Pythagorean Theorem relates the three sides of a right triangle.



When you increase the measure of $\angle C$, the length of the opposite side increases. There is a new relationship between the sides of the triangle.



Similarly, when you decrease the measure of $\angle C$, the length of the opposite side decreases. There is a new relationship between the sides of the triangle.



The Law of Cosines takes these relationships one step further. It uses the measure of angle C to provide an equation that relates the three sides of the triangle with angle C.

Law of Cosines: $a^2 + b^2 - 2ab\cos C = c^2$

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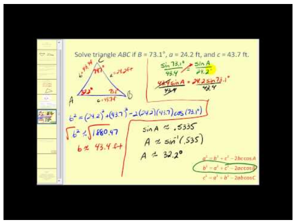
In the past, you used the Pythagorean Theorem to find missing sides in *right* triangles. With the Law of Cosines, you can find missing sides and angles in *any type* of triangle. In the following problems you will learn how to prove the Law of Cosines.



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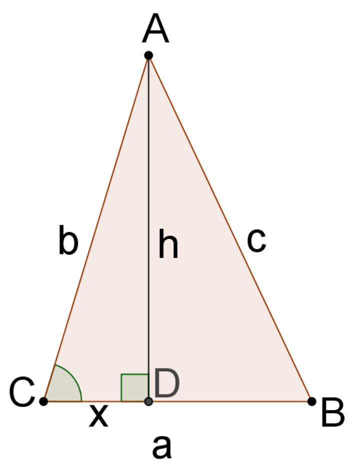
1. Given ΔABC with lengths a, b and c, find DB in terms of x and a. Then, use the Pythagorean Theorem twice to come up with two equations that relate h with the sides of ΔABC.



$CB = a$ and $CD = x$, so $DB = a - x$. You can use the Pythagorean Theorem with both ΔADC and ΔADB.

For $\Delta ADC : x^2 + h^2 = b^2$

For $\Delta ADB : (a-x)^2 + h^2 = c^2$

2. Use the triangle and equations from #1 to derive the Law of Cosines.

From the first problem, you have two equations that you can solve for h^2.

$$x^2 + h^2 = b^2 \rightarrow h^2 = b^2 - x^2$$
$$(a-x)^2 + h^2 = c^2 \rightarrow h^2 = c^2 - (a-x)^2$$

Set the new equations equal to each other, expand, and simplify:

$$\begin{aligned} b^2 - x^2 &= c^2 - (a-x)^2 \\ b^2 - x^2 + (a-x)^2 &= c^2 \\ b^2 - x^2 + a^2 - 2ax + x^2 &= c^2 \\ a^2 + b^2 - 2ax &= c^2 \end{aligned}$$

You can now consider x in terms of $\angle C$. Within ΔADC, x is adjacent to $\angle C$. b is the hypotenuse. This is a cosine relationship.

$$\cos C = \frac{x}{b}$$
$$x = b \cos C$$

Substitute this equation for x into the earlier equation:

$$a^2 + b^2 - 2ax = c^2$$
$$a^2 + b^2 - 2a(b \cos C) = c^2$$
$$a^2 + b^2 - 2ab \cos C = c^2$$

Here you have derived the Law of Cosines for acute angles C. Just like the sine ratio, the **cosine ratio** is actually a **function** that takes *any* angle measure as an input (not just angles between 0° and 90°). One property of the cosine function is that the cosine of supplementary angles will always be opposites. In other words, $\cos\theta = -\cos(180 - \theta)$. In the practice exercises, you will use this fact to show that the Law of Cosines works even if C is a right or obtuse angle. *Note: You will study the cosine function in much more detail in future courses!*

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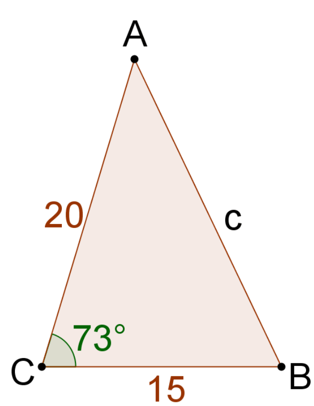
Now let's do a problem using the Law of Cosines.

Use the Law of Cosines to solve for c.



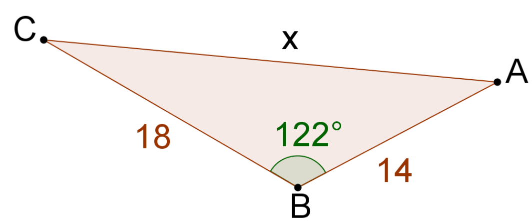
15 and 20 are the values for a and $b, m\angle C = 73^\circ$, and c is the unknown.

$$
\begin{aligned}
a^2+b^2-2ab\cos C &= c^2 \\
15^2+20^2-2(15)(20)(\cos 73^\circ) &= c^2 \\
225+400-175.42 &= c^2 \\
449.58 &= c^2 \\
c &\approx 21.2
\end{aligned}
$$

Examples

Example 1

Earlier, you were asked can you solve for x?



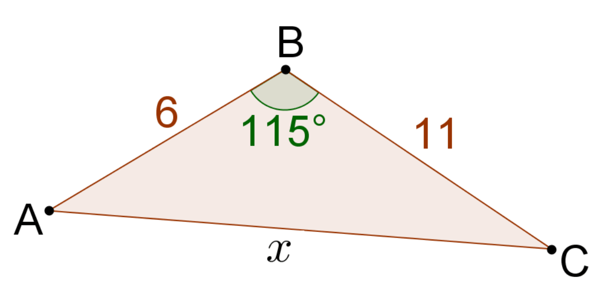
Notice that in this triangle, the included angle is obtuse. As was explained earlier, the Law of Cosines works for acute, right, and obtuse angles. The names of the sides do not match a, b, and c from the Law of Cosines exactly. What is important is that the angle you choose for $\angle C$ is the **included angle** of the two sides you choose for a and b. Side c is opposite $\angle C$.

In this triangle, the sides 18 and 14 have an included angle of 122°. x is opposite the 122° angle. Use the Law of Cosines to relate these 4 values.

$$
\begin{aligned}
18^2+14^2-2(18)(14)\cos 122^\circ &= x^2 \\
324+196+267.08 &= x^2 \\
787.08 &= x^2 \\
x &\approx 28.05
\end{aligned}
$$

Example 2

Solve for x or θ.



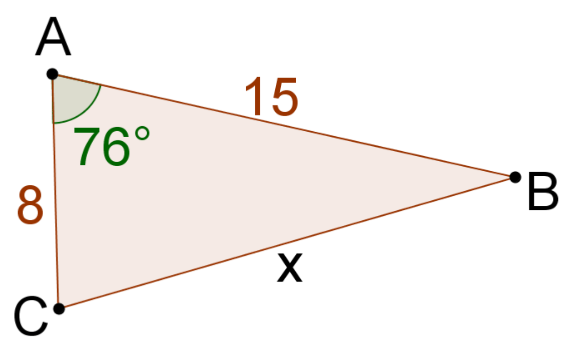
The Law of Cosines states that $a^2 + b^2 - 2ab\cos C = c^2$. Remember that the names of your sides might not match a, b, and c exactly. What is important is that $\angle C$ is the **included angle** of sides a and b. Side c is opposite $\angle C$.

In this triangle, the sides 11 and 6 have an included angle of 115°. x is opposite the 115° angle. Use the Law of Cosines to relate these 4 values.

$$\begin{aligned} 11^2 + 6^2 - 2(11)(6)\cos 115^\circ &= x^2 \\ 121 + 36 + 55.8 &= x^2 \\ 212.8 &= x^2 \\ x &\approx 14.6 \end{aligned}$$

Example 3

Solve for x or θ.

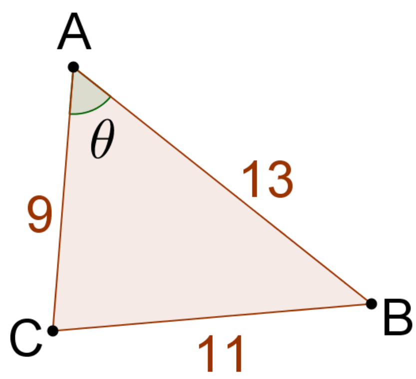


In this triangle, the sides 8 and 15 have an included angle of 76°. x is opposite the 76° angle. Use the Law of Cosines to relate these 4 values.

$$\begin{aligned} 8^2 + 15^2 - 2(8)(15)\cos 76^\circ &= x^2 \\ 64 + 225 - 58.06 &= x^2 \\ 230.94 &= x^2 \\ x &\approx 15.2 \end{aligned}$$

Example 4

Solve for x or θ.



In this triangle, the sides 9 and 13 have an included angle of θ. 11 is opposite the angle θ. Use the Law of Cosines to relate these 4 values.

$$\begin{aligned}
9^2+13^2-2(9)(13)\cos\theta &= 11^2\\
81+169-234\cos\theta &= 121\\
129 &= 234\cos\theta\\
0.5513 &= \cos\theta\\
\theta &= \cos^{-1}(0.5513)\\
\theta &= 56.5^\circ
\end{aligned}$$

Note that you can use the Law of Cosines to solve for missing angles as well as missing sides.

CK-12 PLIX Interactive

PLIX

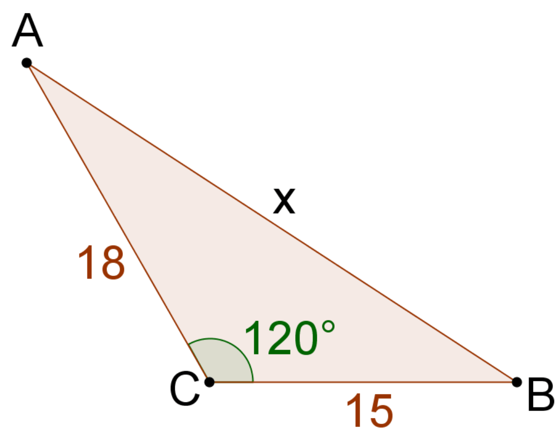
Click image to the left or use the URL below.

URL: http://www.ck12.org/trigonometry/applications-of-the-law-of-cosines/plix/Building-a-Zip-Line-54555033da2cfe01d13ef494

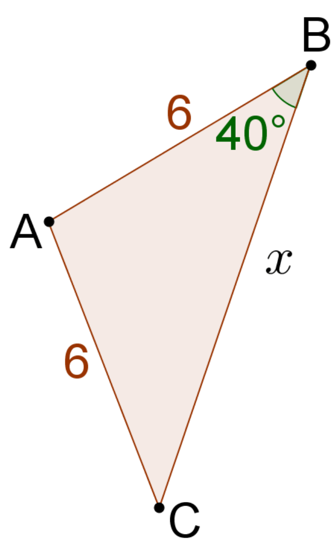
1. What is the Law of Cosines?
2. When using the Law of Cosines, how do you decide on the values for a, b, c, and C?
3. Show that the Law of Cosines is identical to the Pythagorean Theorem when $\angle C$ is a right angle.

Use the Law of Cosines to solve for x or θ.

4.



5. The following triangle is isosceles. Solve for x by first figuring out the measure of $\angle A$.



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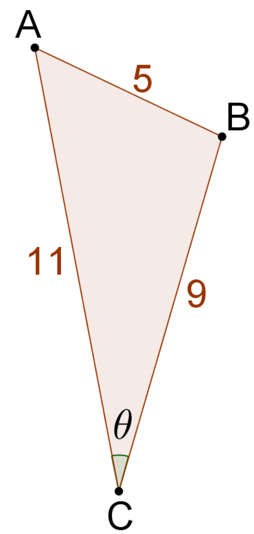

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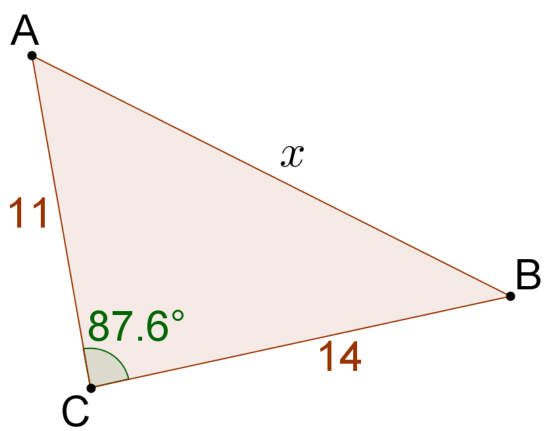
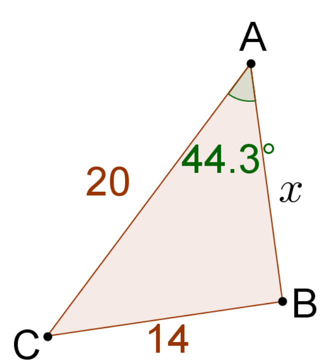


8. Attempt to solve for x. Explain why you get two solutions.



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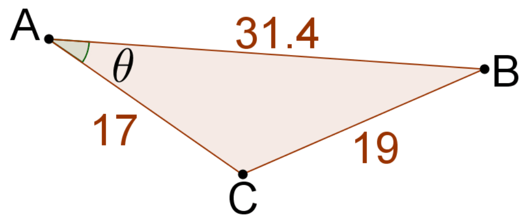


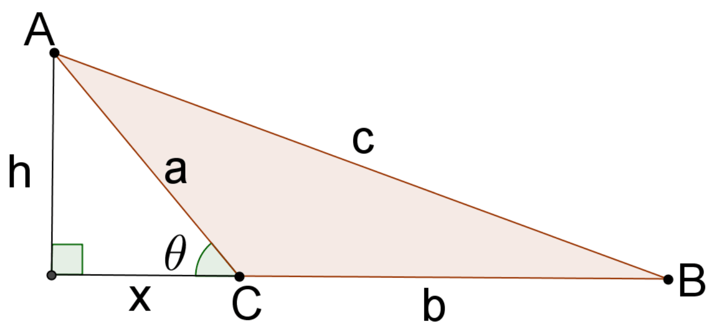
In problems 10-12, you will investigate the relationship between the cosine of supplementary angles.

10. Find $\cos 30^\circ$ and $\cos 150^\circ$. What do you notice?

11. Find $\cos 80^\circ$ and $\cos 100^\circ$. What do you notice?

12. Make a conjecture based on the last two problems. How are the cosine of supplementary angles related? *Note: You will prove this conjecture in future courses!*

In problems 13-16, you will prove that the Law of Cosines works even if $\angle C$ is an obtuse angle.



13. Use the Pythagorean Theorem to write an equation relating h, a, and x. Then, use the Pythagorean Theorem again to write an equation relating h, x, b, and c.

14. Use algebra and your work from #13 to show that the following equation is true: $a^2 + b^2 + 2bx = c^2$.

15. Find an equation that relates and θ, x and a. Show that your equation is equivalent to $\cos(180 - C) = \frac{x}{a}$. Solve for x and substitute this quantity into the equation from #14.

16. The cosine of supplementary angles are opposites. Substitute $-\cos C$ for $\cos(180 - C)$. Show that the result is the Law of Cosines.

17. This chapter has explored many different tools that can be used to find sides or angles in triangles. List the tools and the conditions under which they are best applied. (For example, the Pythagorean Theorem is particularly useful when there is a right triangle and two known sides, and the third side is unknown.)

18. A trail surrounds a lake in the shape of a triangle. The angle between two sections of trail is 50 degrees. These sections are 4 and 6 miles in length. How long is the last section of trail?

19. The law of sines showed us that we can take the sine of an obtuse angle, and the law of cosines showed us that we can take the cosine of an obtuse angle. How are the results different? Justify the difference in terms of the two laws.

20. If the sign of the sine of an obtuse angle is positive, and the sign of the cosine of an obtuse angle is negative, what is the sign of the tangent of an obtuse angle? Why?

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.7.

7.8 Triangles in Applied Problems

Here you will apply your knowledge of trigonometry to solve problems related to triangles.

A community garden is being built in your neighborhood. The garden will be triangular in shape and a fence will surround the garden. Two sides of the garden will be 33 feet and 24 feet. The angle between those two sides will measure 62°. Find the area and perimeter of the garden.

Triangle Summary

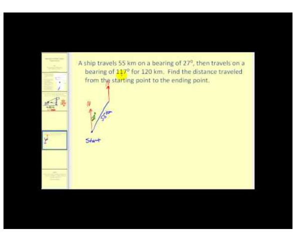
There are many different problems you can solve with your knowledge of triangles and trigonometry. Here is a summary of all the key facts and formulas that will be helpful.

TRIANGLE SUMMARY:

- The sum of the measures of the three angles in a triangle is 180°
- In 30-60-90 right triangles the sides are in the ratio $1 : \sqrt{3} : 2$
- In 45-45-90 right triangles the sides are in the ratio $1 : 1 : \sqrt{2}$
- The Pythagorean Theorem states that for a right triangle with legs a and b and hypotenuse c, $a^2 + b^2 = c^2$
- SOH CAH TOA is a mnemonic device to help you remember the three trigonometric ratios:

$$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}} \quad \cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}} \quad \tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}}$$

- The Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ (watch out for the SSA case)
- The Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$
- The area of a triangle is $\frac{1}{2}bh$ or $\frac{1}{2}ab\sin C$



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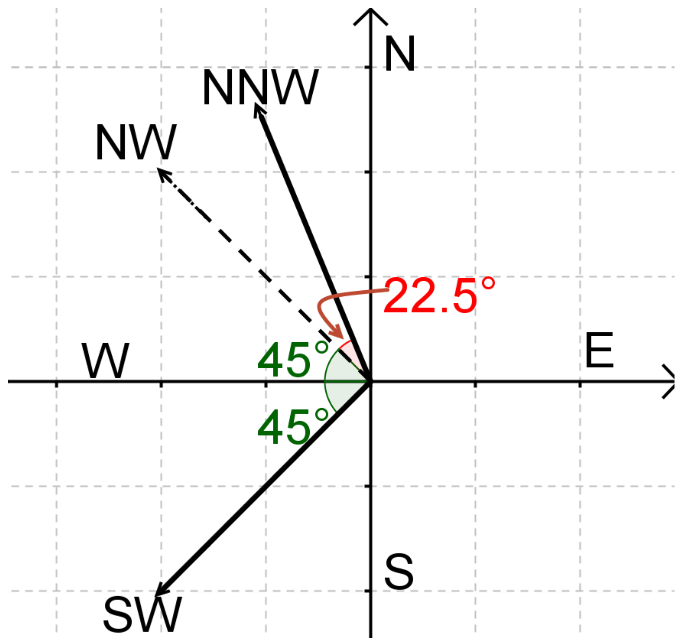
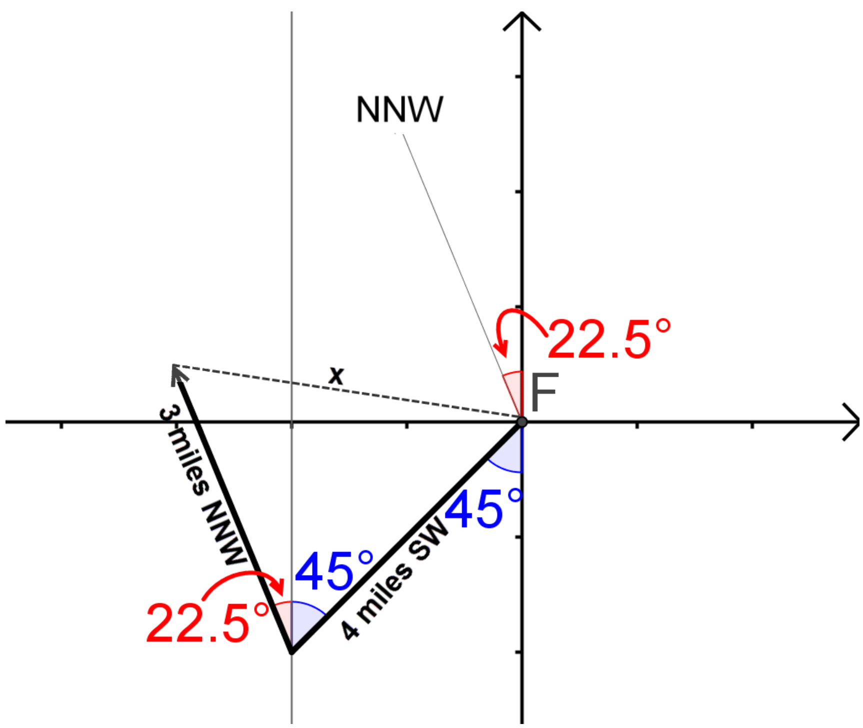
If a boat travels 4 miles SW (southwest) and then 3 miles NNW (north-northwest), how far away is the boat from its starting point?

First, think about what southwest and north-northwest mean. The picture below shows the four basic directions of north, south, east, and west. It also shows southwest and northwest. North-northwest is directly in between northwest and north. Make sure you understand where the angles in the picture came from.



Next, draw a picture of the situation.



Note that the angle between the SW direction and the NNW direction is $22.5^\circ + 45^\circ = 67.5^\circ$.

Now make a plan for how you will solve. Look to see what is given and what you are looking for and think about what method or technique would be helpful. In this situation, you have two sides and an included angle and are looking for the third side. You can use the Law of Cosines.

Now that you have a plan, you can solve the problem. For the Law of Cosines, the 3 and 4 are the values for a and b and 67.5° is the value for $\angle C$. x is the side across from the angle, so it is c.

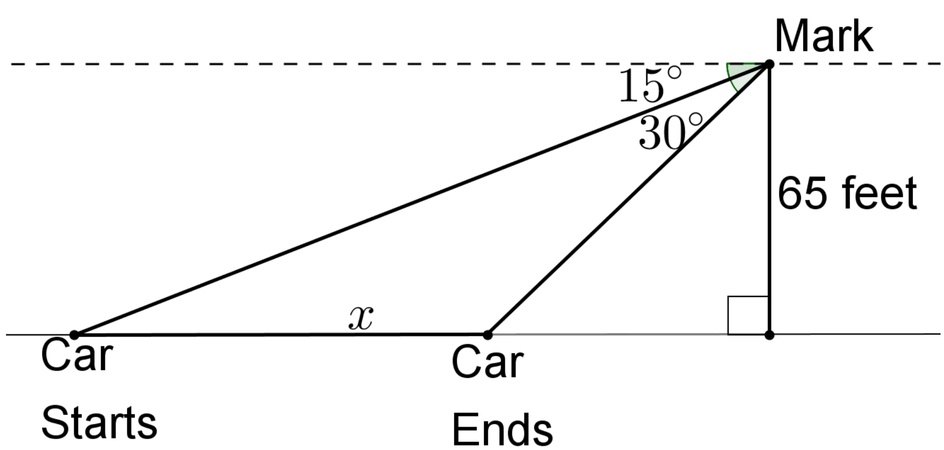
$$\begin{aligned}
a^2+b^2-2ab\cos C &= c^2 \\
3^2+4^2-2(3)(4)(\cos 67.5) &= x^2 \\
9+16-9.18 &= x^2 \\
15.82 &= x^2 \\
x &\approx 3.98
\end{aligned}$$

The boat is 3.98 miles from its starting place.

Real-World Application: Observation Points

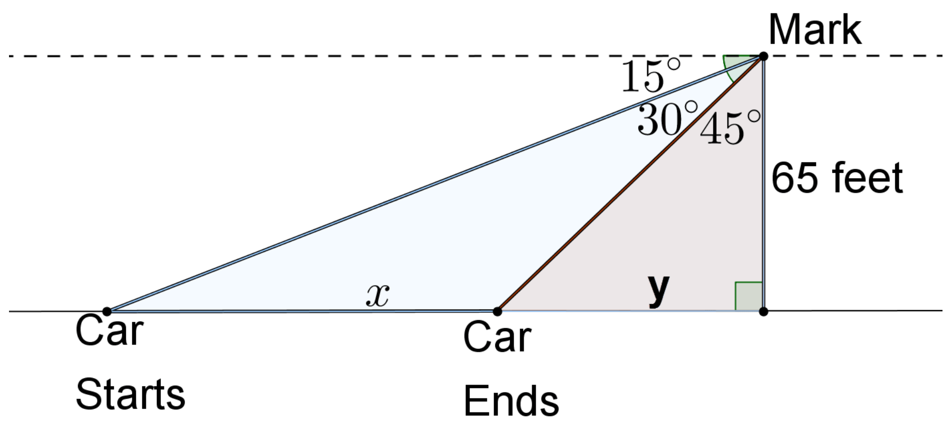
1. From the fourth story of a building (65 feet) Mark observes a car moving towards the building driving on the street below. If the angle of depression of the car changes from 15° to 45° while he watches, how far did the car travel?

The **angle of depression** is the angle at which you view an object below the horizon. Start by making a detailed picture of the situation and labeling what you know.



Note that $15^\circ + 30^\circ = 45^\circ$, the angle of depression for the ending location of the car.

Now make a plan for how you will solve. Look to see what is given and what you are looking for and think about what method or technique would be helpful. In this situation, you have two right triangles (the smaller brown triangle and the larger blue triangle). In each case, you know a side and an angle. You are looking for a portion of one of the sides of these triangles (x).



You can use trigonometric ratios to find the missing sides of these triangles to help you to find the length of x.

First look at the small brown triangle. y is opposite the 45° angle and 65 is adjacent to the 45° angle. This is a tangent relationship (or you could use 45-45-90 triangle ratios):

$$\begin{aligned}\tan 45^\circ &= \frac{y}{65}\\ y &= 65 \tan 45^\circ\\ y &= 65\ ft\end{aligned}$$

Now look at the larger right triangle. $x+y$ is opposite the 75° angle $(30^\circ + 45^\circ = 75^\circ)$ and 65 is adjacent to the 75° angle. Again you can use tangent.

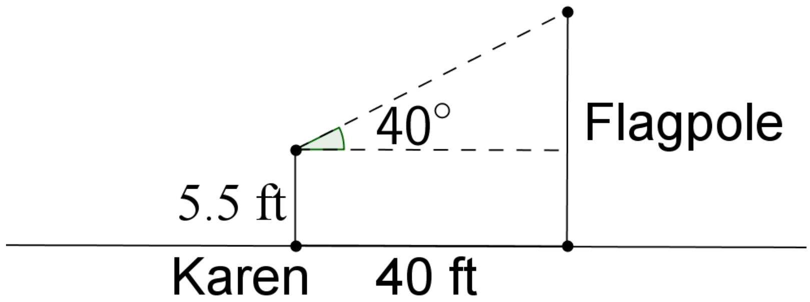
$$\begin{aligned}\tan 75^\circ &= \frac{x+y}{65}\\ 65 \tan 75^\circ &= x+y\\ x+y &\approx 242.58\ ft\end{aligned}$$

Since $y = 65\ ft$, x, must equal $242.58 - 65 = 177.58\ ft$.

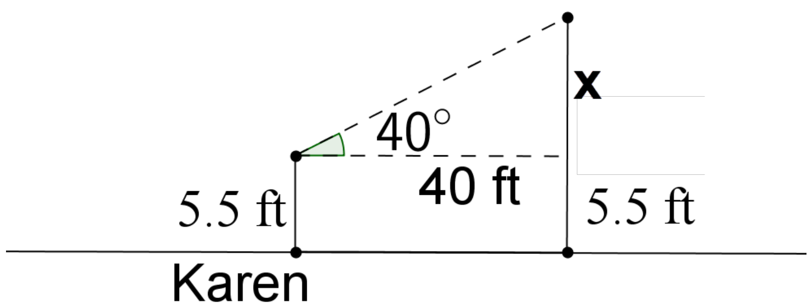
The car traveled 177.58 feet.

2. Karen is 5.5 feet tall and looks up at a 40° angle to see the top of the flagpole in front of a building. She is standing 40 feet from the flagpole. How tall is the flagpole?

Again, start by making a detailed picture of the situation and labeling what you know.



Now, make a plan for how you will solve. Look to see what is given and what you are looking for and think about what method or technique would be helpful. In this situation, you have a right triangle. You know an angle and a side within the right triangle (40° and 40 ft). You are looking for another side of the triangle (x).



Think of the flagpole as being made up of two pieces. The first piece is Karen's height of 5.5 feet. The next piece is the rest of the flagpole that is taller than Karen (x in the picture above). x is opposite the 40° angle and 40 ft is adjacent to the 40° angle. This is a tangent relationship.

$$\begin{aligned}\tan 40^\circ &= \frac{x}{40}\\ x &= 40\tan 40^\circ\\ x &\approx 33.56\ ft\end{aligned}$$

Therefore, the complete height of the flagpole is approximately $33.56 + 5.5 = 39.06\ feet$.

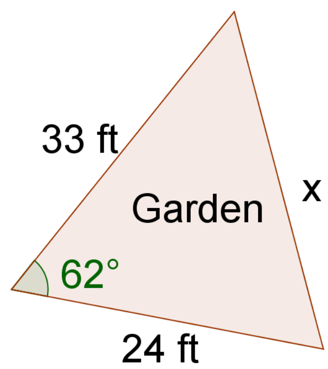
Examples

Example 1

Earlier, you were asked to find the area and perimeter of the garden.

A community garden is being built in your neighborhood. The garden will be triangular in shape and a fence will surround the garden. Two sides of the garden will be 33 feet and 24 feet. The angle between those two sides will measure 62°. Find the area and perimeter of the garden.

Start by drawing a picture and carefully labeling everything that you know.



To find the area of the garden you can use the sine area formula.

$$A = \frac{1}{2}(33)(24)\sin 62^\circ$$
$$A \approx 349.6\ ft^2$$

In order to find the perimeter of the garden you need to know the length of the third side. In this situation you know two sides and an included angle, so you can use the Law of Cosines to find the length of the third side.

$$\begin{aligned} 33^2 + 24^2 - 2(33)(24)\cos 62^\circ &= x^2 \\ 1089 + 576 - 743.64 &= x^2 \\ 921.36 &= x^2 \\ x &\approx 30.35\ ft \end{aligned}$$

Now that you know the length of the third side, you can find the perimeter of the garden.

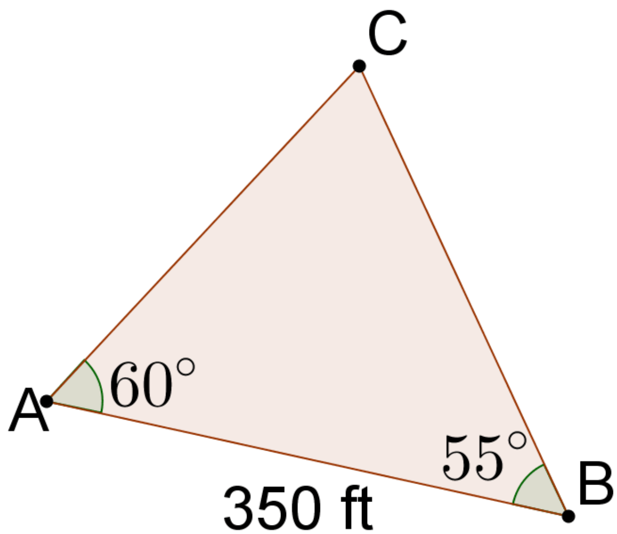
$$P = 33 + 24 + 30.35 = 87.35\ ft$$

A surveyor wants to find the distance from points A and B to an inaccessible point C. These three points form a triangle. Standing at point A, he finds $m\angle A$ in the triangle is equal to 60°. Standing at point B, he finds $m\angle B$ in the triangle is equal to 55°. He measures the distance from point A to point B and finds it to be 350 feet. Find the distance from points A and B to point C.

Example 2

Draw a picture of this situation.

Here is a picture:



Example 3

Make a plan: what method(s) or technique(s) can you use to solve this problem?

You can find the measure of angle C using the fact that the sum of the measures of the three angles in a triangle is 180°. Then, you will know all the angles and one side. You could then use the Law of Sines to set up equations to find AC and BC.

Example 4

Solve the problem.

$m\angle C = 65^\circ$. Now, use the Law of Sines twice:

To find AC:

$$\begin{aligned}
\frac{\sin 65^\circ}{350} &= \frac{\sin 55^\circ}{AC} \\
AC &= \frac{350 \sin 55^\circ}{\sin 65^\circ} \\
AC &\approx 316.34 \ ft
\end{aligned}$$

To find BC:

$$\begin{aligned}
\frac{\sin 65^\circ}{350} &= \frac{\sin 60^\circ}{BC} \\
AC &= \frac{350 \sin 60^\circ}{\sin 65^\circ} \\
AC &\approx 334.44 \ ft
\end{aligned}$$

The distance from A to C is approximately 316 feet and the distance from B to C is approximately 334 feet.

Review

The angle of depression of a boat in the distance from the top of a lighthouse is 25°. The lighthouse is 200 feet tall. Find the distance from the base of the lighthouse to the boat.

1. Draw a picture of this situation.
2. Make a plan: what method(s) or technique(s) can you use to solve this problem?
3. Solve the problem.

A pilot is flying due west and gets word that a major storm is in her path. She turns the plane 40° to the left of her intended course and continues the flying. After passing the storm, she turns 50° to the right and flies until she has returned to her original flight path. At this point she is 75 miles from where she left her original path when she first made a turn. How much further did the pilot fly as a result of the detour?

4. Draw a picture of this situation.
5. Make a plan: what method(s) or technique(s) can you use to solve this problem?
6. Solve the problem.

A new bridge is being built across a river in your town. You want to figure out how long the bridge will be. You find two points on one side of the river that are 30 feet apart. These two points with the point at the end of the bridge on the other side of the river form a triangle. You stand at each of the two points on your side of the river and measure the angles of the triangle. You find the two angles are 45° and 70°. How far are each of the two points on your side of the river from the end of the bridge on the other side of the river? How long will the bridge be?

7. Draw a picture of this situation.
8. Make a plan: what method(s) or technique(s) can you use to solve this problem?
9. Solve the problem.

ΔABC has two sides of length 12 and a non-included angle that measures 60°.

10. Draw a possible picture of this situation.
11. Find the measure of all sides and angles of ΔABC.
12. Find the area of ΔABC.

Lily starts at point A and walks straight for 100 feet. Then, she turns right at an 80° angle and continues walking for another 150 feet. In order to go straight back to her starting place, how far will she need to walk? At want angle should she turn right?

13. Draw a picture of this situation.
14. Make a plan: what method(s) or technique(s) can you use to solve this problem?
15. Solve the problem.
16. Create a triangle such that the two sides and a non-included angle are known. Find the possible measures of the included angle. How many solutions resulted? Explain why. Recreate the triangle from scratch so that there are a number of different solutions. Explain the results.
17. Create a scenario which can be solved with the law of sines. Create another scenario that can be solved with the law of cosines. Solve both.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.8.

7.9 References

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CHAPTER 8 Circles

Chapter Outline

8.1 Circles and Similarity

Learning Objectives

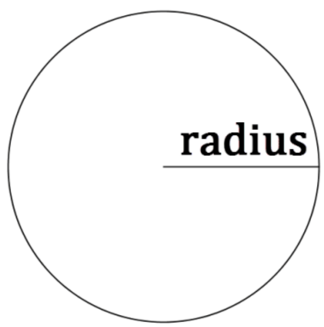
Here you will prove that all circles are similar.

Sean has two circles, one with a radius of 1 inch and another with a radius of 3 inches.

1. What is the ratio between the radii of the circles?
2. What is the scale factor between the two circles?
3. What is the ratio between the circumferences of the circles?
4. What is the ratio between the areas of the circles?
5. What do area ratios and circumference ratios have to do with scale factor?

Circles and Similarities

A **circle** is a set of points equidistant from a given point. The **radius** of a circle, r, is the distance from the center of the circle to the circle. A circle's size depends *only* on its radius.

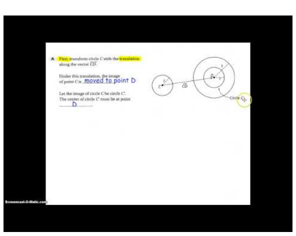


Two figures are **similar** if a similarity transformation will carry one figure to the other. A **similarity transformation** is one or more rigid transformations followed by a dilation. In the examples, you will show that a similarity transformation exists between any two circles and therefore, all circles are similar.

Recall two important formulas related to circles:

- Circumference (Perimeter) of a Circle: $C = 2\pi r$
- Area of a Circle: $A = \pi r^2$

Once you have shown that all circles are similar, you will explore how the circumferences and areas of circles are related.



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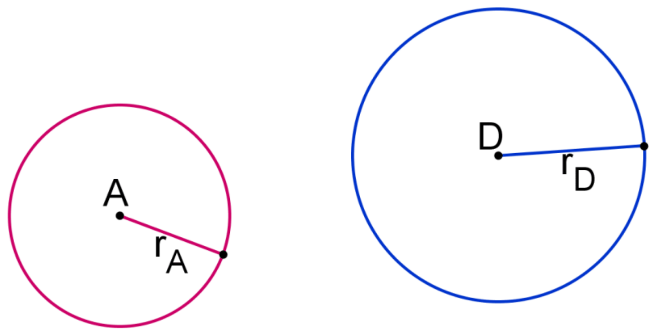
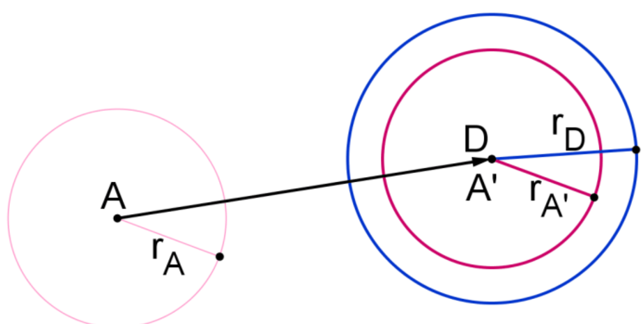
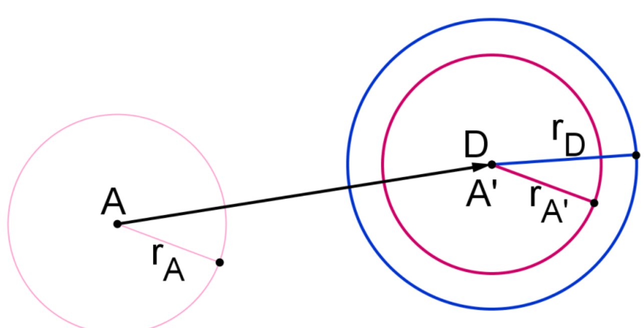
1. Consider circle A, centered at point A with radius r_A, and circle D, centered at point D with radius r_D. Perform a rigid transformation to bring point A to point D.



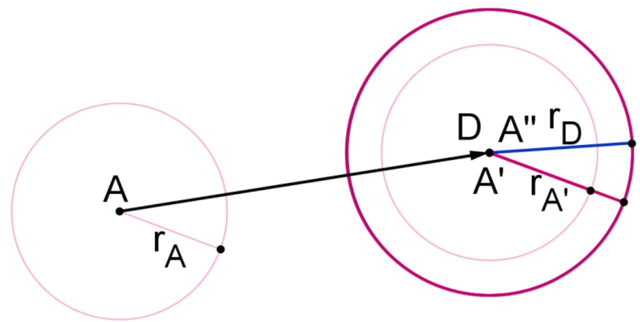
Draw a vector from point A to point D. Translate circle A along the vector to create circle A'. Note that $r_A \cong r'_A$.



2. Dilate circle A to map it to circle D. Can you be confident that the circles are similar?



Because the size of a circle is completely determined by its radius, you can use the radii to find the correct scale factor. Dilate circle A' about point A' by a scale factor of $\frac{r_D}{r_{A'}}$.



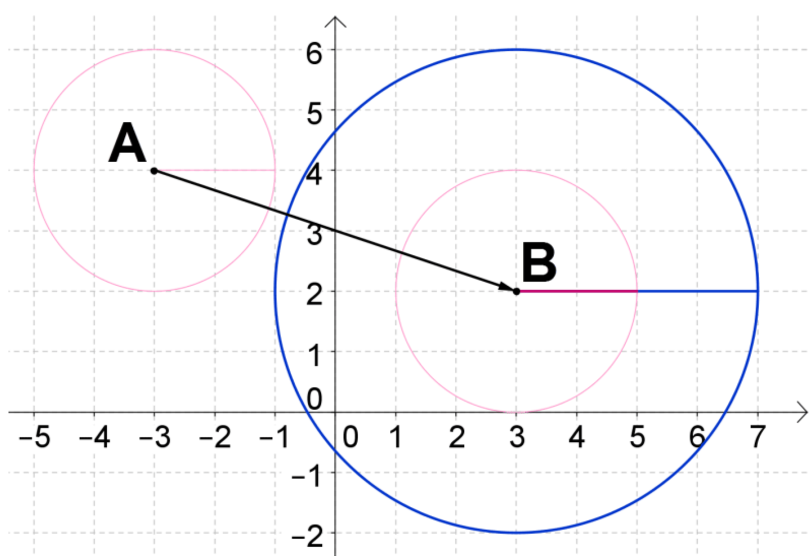
Circle A'' is the same circle as circle D. You can be confident of this because $r_{A''} = \frac{r_D}{r_{A'}} \cdot r_{A'} = r_D$ and point A'' is the same as point D. Because a circle is defined by its center and radius, if two circles have the same center and radius then they are the same circle.

This means that circle A is similar to circle D, because a similarity transformation (translation then dilation) mapped circle A to circle D.

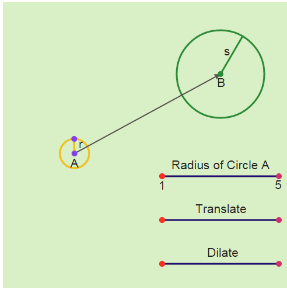
Circle A and circle D were two random circles. This proves that in general, all circles are similar.

3. Show that circle A with center $(-3,4)$ and radius 2 is similar to circle B with center $(3,2)$ and radius 4.

Translate circle A along the vector from $(-3,4)$ to $(3,2)$. Then, dilate the image about its center by a scale factor of 2. You will have mapped circle A to circle B with a similarity transformation. This means that circle A is similar to circle B.



CK-12 PLIX Interactive



PLIX

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Example 1

Earlier, you were given a problem about Sean and his two circles.

Sean has two circles, one with a radius of 1 inch and another with a radius of 3 inches.

a. What is the ratio between the radii of the circles?

The ratio between the radii is $\frac{3}{1}$.

b. What is the scale factor between the two circles?

A scale factor exists because any two circles are similar. You can use the radii to determine the scale factor. The ratio between the radii is $\frac{3}{1}$ so the scale factor is $\frac{3}{1} = 3$.

c. What is the ratio between the circumferences of the circles?

The circumference of the smaller circle is $C = 2\pi(1) = 2\pi$. The circumference of the larger circle is $C = 2\pi(3) = 6\pi$. The ratio between the circumferences is $\frac{6\pi}{2\pi} = \frac{3}{1}$.

d. What is the ratio between the area of the circles?

The area of the smaller circle is $A = \pi(1)^2 = \pi$. The area of the larger circle is $A = \pi(3)^2 = 9\pi$. The ratio between the areas is $\frac{9\pi}{\pi} = \frac{9}{1}$. Note that $\frac{9}{1} = \left(\frac{3}{1}\right)^2$.

e. What do area ratios and circumference ratios have to do with scale factor?

The area ratio is the scale factor squared, because area is a two dimensional measurement. The circumference ratio is equal to the scale factor, because circumference is a one dimensional measurement.

Example 2

Show that circle A with center $(-1,7)$ and radius 2 is similar to circle B with center $(4,6)$ and radius 3.

Translate circle A along the vector from $(-1,7)$ to $(4,6)$. Then, dilate the image about its center with a scale factor of $\frac{3}{2}$. You will have mapped circle A to circle B with a similarity transformation. This means that circle A is similar to circle B.

Example 3

The ratio of the circumference of circle D to the circumference of circle C is $\frac{4}{3}$. What is the ratio of their areas?

The ratio of the circumferences is the same as the scale factor. Therefore, the scale factor is $\frac{4}{3}$. The ratio of the areas is the scale factor squared. Therefore, the ratio of the areas is $\left(\frac{4}{3}\right)^2 = \frac{16}{9}$.

Example 4

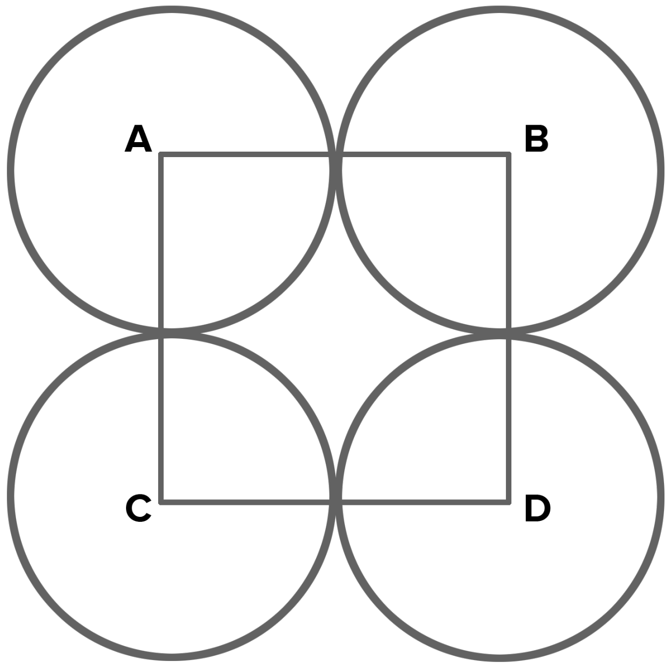
The ratio of the area of circle F to the area of circle E is $\frac{9}{4}$. What is the ratio of their radii?

The ratio of the areas is the scale factor squared. Therefore, the scale factor is $\sqrt{\frac{9}{4}} = \frac{3}{2}$. The ratio of the radii is the same as the scale factor, so the ratio of the radii is $\frac{3}{2}$.

Review

For #1-#10, show that the circles are similar by describing the similarity transformation necessary to map one circle onto the other.

1. Circle A with center $(2,7)$ and radius 4. Circle B with center $(1,-4)$ and radius 3.
2. Circle A with center $(6,4)$ and radius 3. Circle B with center $(-5,6)$ and radius 5.
3. Circle A with center $(1,4)$ and radius 2. Circle B with center $(3,-2)$ and radius 7.
4. Circle A with center $(8,1)$ and radius 6. Circle B with center $(6,-4)$ and radius 8.
5. Circle A with center $(-2,10)$ and radius 3. Circle B with center $(-1,-4)$ and radius 6.
6. Circle A with center $(-1,5)$ and radius 4. Circle B with center $(-1,5)$ and radius 5.
7. Circle A with center $(-4,-2)$ and radius 1. Circle B with center $(1,8)$ and radius 4.
8. Circle A with center $(10,3)$ and radius 5. Circle B with center $(4,2)$ and radius 8.
9. Circle A with center $(12,4)$ and radius 10. Circle B with center $(12,4)$ and radius 12.
10. Circle A with center $(-7,6)$ and radius 9. Circle B with center $(1,-4)$ and radius 9.
11. The ratio of the circumference of circle A to the circumference of circle B is $\frac{2}{3}$. What is the ratio of their radii?
12. The ratio of the area of circle A to the area of circle B is $\frac{6}{1}$. What is the ratio of their radii?
13. The ratio of the radius of circle A to the radius of circle B is $\frac{5}{9}$. What is the ratio of their areas?
14. The ratio of the area of circle A to the area of circle B is $\frac{12}{5}$. What is the ratio of their circumferences?
15. To show that any two circles are similar you need to perform a translation and/or a dilation. Why won't you ever need to use reflections or rotations?
16. In the diagram below, points A, B, C, and D are each at the center of the circle. If the circles are all congruent, what is the relationship between the area of the square ABCD and the sum of the areas of all four circles?



17. You have placed a safety light on the circumference of your bike tire. Describe the path of the light as you go down and then up a hill. Use words and a diagram.

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.1.

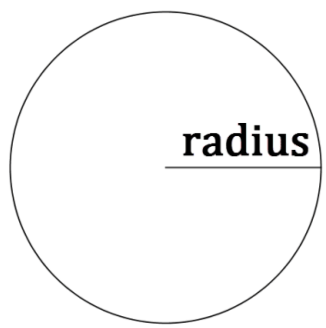
8.2 Area and Circumference of Circles

Learning Objectives

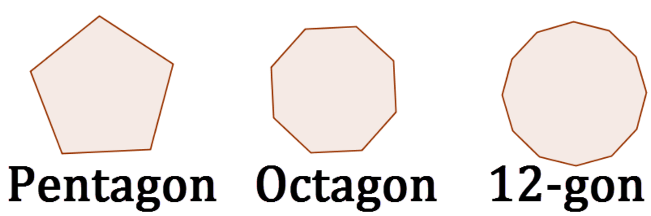
Here you will find the formulas for the area and circumference of circles by considering the area and perimeter of regular polygons. You can use regular polygons with an increasing number of sides to help explain why a circle of radius 1 unit has an area of πr^2. Where does the r^2 come from in the formula for the area of a circle?

Area and Circumference of a Circle

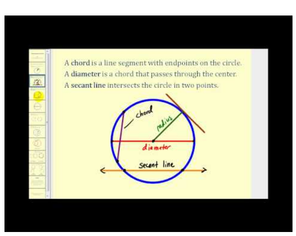
A **circle** is a set of points equidistant from a given point. The **radius** of a circle, r, is the distance from the center of the circle to the circle. All circles are geometrically **similar**.



A **regular polygon** is a closed figure that is both equilateral and equiangular. As the number of sides of a regular polygon increases, the polygon looks more and more like a circle.



Previously you have learned that the **area** of a circle with radius r is given by πr^2 and the **circumference** of a circle with radius r is given by $2\pi r$. In the examples and guided practice, you will derive these formulas by looking at the area and perimeter of regular polygons.



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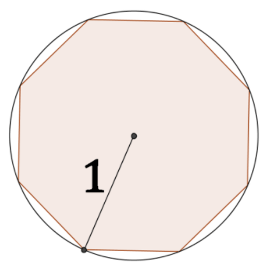
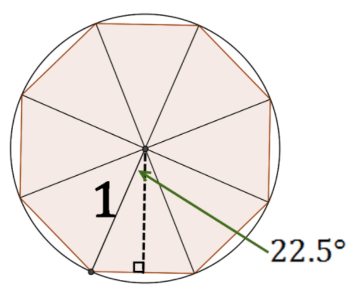
Finding the Area

1. Find the area of a regular octagon inscribed in a circle with radius 1 unit.



Break the octagon into 8 congruent triangles. You will find the area of one triangle and multiply that by 8 to find the area of the whole octagon. Draw in the height for one of the triangles. The angle formed by the height and the radius of the circle is $\frac{360^\circ}{16} = 22.5^\circ$. Remember that 360 is the number of degrees in a full circle.



Now, you can use trigonometry to find the height and base of the triangle.

- $\sin 22.5 = \frac{0.5b}{1} \rightarrow b = 2\sin 22.5 \rightarrow b \approx 0.7654$
- $\cos 22.5 = \frac{h}{1} \rightarrow h = \cos 22.5 \rightarrow h \approx 0.9239$

The area of the triangle is:

$A = \frac{bh}{2} = \frac{(0.7654)(0.9239)}{2} \approx 0.3536\ un^2$

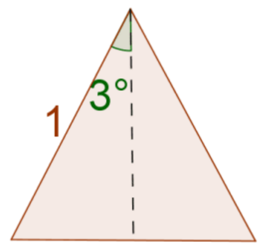
Therefore, the area of the octagon is:

$A = 8(0.3536) \approx 2.8288\ un^2$

2. Find the area of a regular 60-gon inscribed in a circle with radius 1 unit and the area of a regular 120-gon inscribed in a circle with radius 1 unit.

While you can't accurately draw a regular 60-gon or a regular 180-gon, you can use the method from #1 to find the area of each.

Regular 60-gon: Divide the polygon into 60 congruent triangles. Consider one of those triangles and draw in its height. You will focus on finding the area of this triangle. The angle formed by the height and the radius of the circle is $\frac{360^\circ}{120} = 3^\circ$.



Use trigonometry to find the height and base of the triangle.

- $\sin 3 = \frac{0.5b}{1} \rightarrow b = 2\sin 3 \rightarrow b \approx 0.1047$
- $\cos 3 = \frac{h}{1} \rightarrow h = \cos 3 \rightarrow h \approx 0.9986$

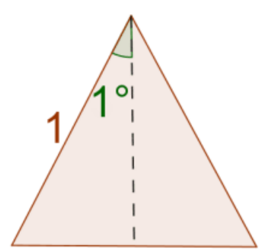
The area of the triangle is:

$A = \frac{bh}{2} = \frac{(0.1047)(0.9986)}{2} \approx 0.0523\ un^2$

Therefore, the area of the 60-gon is:

$A = 60(0.0523) \approx 3.138\ un^2$

Regular 120-gon: Divide the polygon into 180 congruent triangles. Consider one of those triangles and draw in its height. You will focus on finding the area of this triangle. The angle formed by the height and the radius of the circle is $\frac{360^\circ}{360} = 1^\circ$.



Use trigonometry to find the height and base of the triangle.

- $\sin 1 = \frac{0.5b}{1} \rightarrow b = 2\sin 1 \rightarrow b \approx 0.0349048$
- $\cos 1 = \frac{h}{1} \rightarrow h = \cos 1 \rightarrow h \approx 0.9998477$

The area of the triangle is:

$A = \frac{bh}{2} = \frac{(0.0349048)(0.9998477)}{2} \approx 0.0174482\ un^2$

Therefore, the area of the 180-gon is:

$A = 180(0.0174482) \approx 3.141\ un^2$

Now, let's explore how area can change based on the number of sides.

What happens to the area of the regular polygon as the number of sides increases? Make a conjecture about the area of a circle with radius 1 unit.

You have the following information:

TABLE 8.1:

Number of Sides	8	60	180
Area	2.8288	3.138	3.141

As the number of sides increases, the regular polygon will get closer and closer to the circle that inscribes it. Therefore, the area of the regular polygon will get closer and closer to the area of the circle. Notice that as the number of sides went from 60 to 180, the area barely changed, staying around 3.14. If you increase the number of sides to 1000, you will find that the area of the regular polygon is still approximately 3.14157. You should recognize these numbers as approximately the value of π.

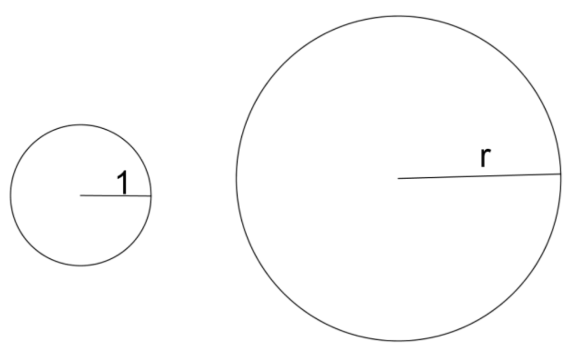
As the number of sides increases, the polygon becomes closer and closer to a circle, and the area gets closer and closer to $\pi \ un^2$. A conjecture would be that the area of a circle with radius 1 unit is $\pi \ un^2$.

Examples

Example 1

Earlier, you were asked where the r^2 came from in the formula for the area of a circle.

You can use regular polygons with an increasing number of sides to help explain why a circle of radius 1 unit has an area of $\pi \ un^2$. Remember that all circles are similar. To create another circle with radius r from a circle with radius 1 unit, apply a similarity transformation with scale factor $k = radius$.



The area of the transformed circle is k^2 times the area of the original circle. Because the area of the original circle is π and the scale factor is equal to r, the area of the transformed circle is πr^2. This is one way to explain why the formula for the area of a circle is πr^2.

Example 2

Find the perimeter of a regular octagon inscribed in a circle with radius 1 unit.

The base of one triangle was $b = 2\sin 22.5 \rightarrow b \approx 0.7654$. Therefore, the perimeter of the octagon is $P \approx 8(0.7654) = 6.1232 \ un$.

Example 3

Find the perimeter of a regular 60-gon inscribed in a circle with radius 1 unit.

The base of one triangle was $b = 2\sin 3 \rightarrow b \approx 0.1047$. Therefore, the perimeter of the 60-gon is $P \approx 60(0.1047) = 6.282$ *un*.

Example 4

Find the perimeter of a regular 180-gon inscribed in a circle with radius 1 unit.

The base of one triangle was $b = 2\sin 1 \rightarrow b \approx 0.0349048$. Therefore, the perimeter of the 180-gon is $P \approx 180(0.0349048) = 6.283$ *un*.

Example 5

Make a conjecture about the circumference of a circle with radius 1 unit.

The perimeters from #2, #3, and #4 are approaching $6.283 \approx 2\pi$. A conjecture would be that the circumference of a circle with radius 1 unit is 2π *un*.

Review

Consider a regular n-gon inscribed in a circle with radius 1 unit for questions 1-8.

1. What's the measure of the angle between the radius and the height of one triangle in terms of n?

2. What's the length of the base of one triangle in terms of sine and n?

3. What's the height of one triangle in terms of cosine and n?

4. What's the area of one triangle in terms of sine, cosine, and n?

5. What's the area of the polygon in terms of sine, cosine, and n?

6. What's the perimeter of the polygon in terms of sine, cosine, and n?

7. Let $n = 10,000$. What is the area of the polygon? What is the perimeter of the polygon? Use your calculator and your answers to #6 and #7.

8. Let $n = 1,000,000$. What is the area of the polygon? What is the perimeter of the polygon? Are you convinced that the area of a circle with radius 1 unit is π and the circumference of a circle with radius 1 unit is 2π?

9. Explain why the area of a regular polygon inscribed in a circle with radius 1 unit gets closer to π as the number of sides increases.

10. Explain why the perimeter of a regular polygon inscribed in a circle with radius 1 unit gets closer to 2π as the number of sides increases.

11. Use similarity and the fact that the circumference of a circle with radius 1 unit is 2π to explain why the formula for the circumference of a circle with radius r is $2\pi r$.

12. A circle with radius 3 units is transformed into a circle with radius 5 units. What is the ratio of their areas? What is the ratio of their circumferences?

13. The ratio of the areas of two circles is 25 : 4. The radius of the smaller circle is 2 units. What's the radius of the larger circle?

14. The ratio of the areas of two circles is 25 : 9. The radius of the larger circle is 10 units. What's the radius of the smaller circle?

15. The ratio of the areas of two circles is 5 : 4. The radius of the larger circle is 8 units. What's the circumference of the smaller circle?

16. The radius of a circular helicopter landing pad is increased by 10 feet. Determine the simplified expression for the increase in area. What is the increase in the circumference of the landing pads?

17. An octagonal hot tub seats 8 people, with each person leaning against the 4' side of the octagon. Explain how you would determine the area of the top of the octagon in order to construct a cover for the hot tub.

18. A circular plate has a diameter of 10 inches. Two cookies, each with a radius of 1.1 inches, are on the plate. Explain how to determine what percentage of the plate's area is not covered by the cookies.

Review (Answers)

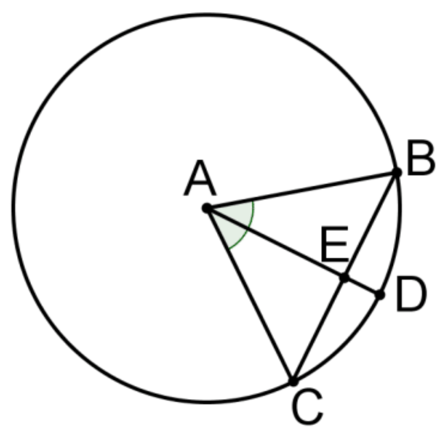
To see the Review answers, open this PDF file and look for section 8.2.

8.3 Central Angles and Chords

Learning Objectives

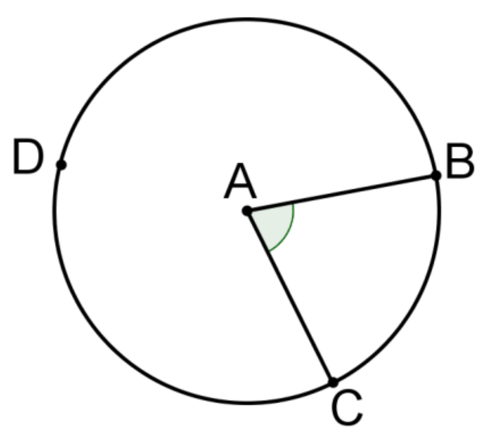
Here you will learn about central angles, arcs, and chords in circles.

Radius $\overline{AD}$ bisects $\angle BAC$ in the circle below. How does $\overline{AD}$ relate to chord $\overline{BC}$? Prove your ideas.

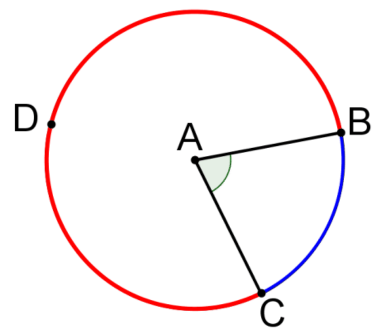


Central Angles and Chords

A **central angle** for a circle is an angle with its vertex at the center of the circle.



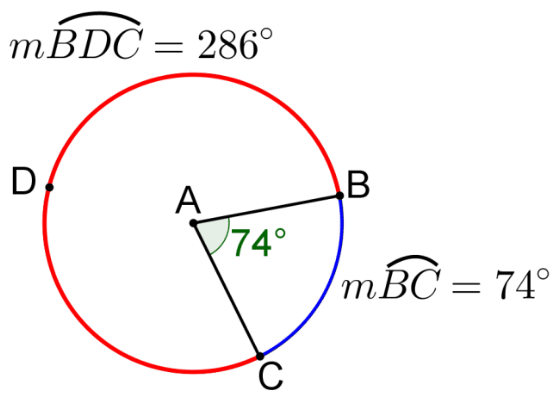
In the circle above, A is the center and $\angle BAC$ is a central angle. Notice that the central angle meets the circle at two points (B and C), dividing the circle into two sections. Each of circle portions is called an **arc**. The smaller arc (blue below) is called the **minor arc**, and is considered the arc that is **intercepted** by the central angle. The larger arc (red below) is called the **major arc**.



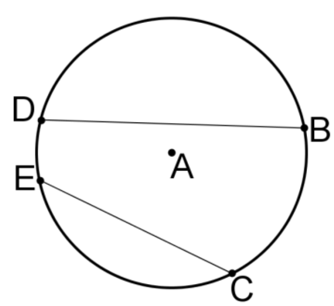
The minor arc above is named $\widehat{BC}$. Notice that B and C are the endpoints of the arc and there is an arc symbol above the letters indicating that you are referring to an arc.

The major arc above is named $\widehat{BDC}$. When naming a major arc you should use three letters so that it is clear you are referring to the larger portion of the circle.

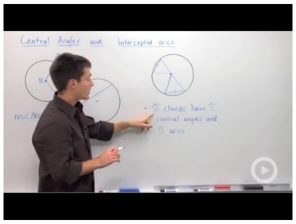
Arcs can be **measured in degrees** just like angles. In general, *the degree measure of a minor arc is equal to the measure of the central angle that intercepts it*. Because there are 360° in a circle, the sum of the measures of a minor arc and its corresponding major arc will be 360°.



A **chord** is a segment that connects two points on a circle. If a chord passes through the center of the circle then it is a **diameter**. In the circle below, both $\overline{BD}$ and $\overline{CE}$ are chords.



Notice that each chord has a corresponding arc. $\overline{CE}$ is a chord and $\widehat{CE}$ is an arc. In #2 below you will prove that two chords are congruent if and only if their corresponding arcs are congruent.



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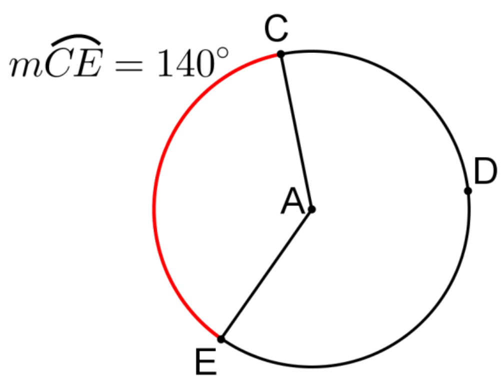
If D was not on the circle, we would not be able to tell the difference between $\widehat{BC}$ and $\widehat{BDC}$. There are 360° in a circle, where a semicircle is half of a circle, or 180°. $m\angle EFG = 180^\circ$, because it is a straight angle, so $m\widehat{EHG} = 180^\circ$ and $m\widehat{EJG} = 180^\circ$.

Let's take a look at a few problems involving central angles and chords.

1. Find $m\angle CAE$ and $m\widehat{CDE}$.



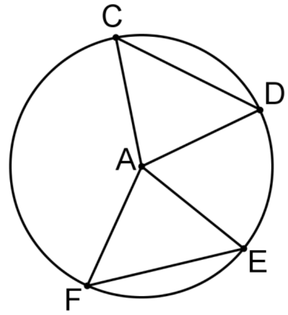
The degree measure of a minor arc is equal to the measure of the central angle that intercepts it. Therefore, $m\angle CAE = 140^\circ$. A full circle is 360°, so $m\widehat{CDE} = 360^\circ - 140^\circ = 220^\circ$.

2. Prove that two chords are congruent if and only if their corresponding arcs are congruent.

This statement has two parts that you must prove.

1. If two arcs are congruent then their corresponding chords are congruent.
2. If two chords are congruent then their corresponding arcs are congruent.

Both statements can be proved by finding congruent triangles. Consider the circle below with center A. Note that $\overline{CA}$, $\overline{DA}$, $\overline{EA}$, and $\overline{FA}$ are all radii of the circle and therefore are all congruent.



Start by proving statement #1. Assume that $\widehat{CD} \cong \widehat{FE}$. This would imply that $m\widehat{CD} = m\widehat{FE}$. Because the measure of an arc is the same as the measure of its corresponding central angle, it must be true that $m\angle CAD = m\angle FAE$ and thus $\angle CAD \cong \angle FAE$. $\overline{CA} \cong \overline{DA} \cong \overline{EA} \cong \overline{FA}$ because they are all radii of the circle. Therefore, $\Delta CAD \cong \Delta FAE$ by $SAS \cong$. This means $\overline{CD} \cong \overline{FE}$ because they are corresponding parts of congruent triangles.

Now prove the converse (statement #2). Assume that $\overline{CD} \cong \overline{FE}$. $\overline{CA} \cong \overline{DA} \cong \overline{EA} \cong \overline{FA}$ because they are all radii of the circle. Therefore, $\Delta CAD \cong \Delta FAE$ by $SSS \cong$. This means $\angle CAD \cong \angle FAE$ because they are corresponding parts of congruent triangles. This implies that the arcs intercepted by these angles are congruent and therefore $\widehat{CD} \cong \widehat{FE}$.

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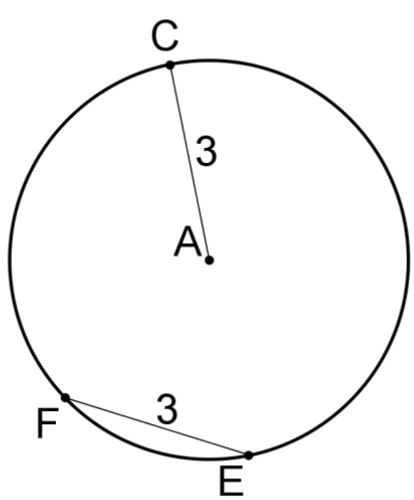
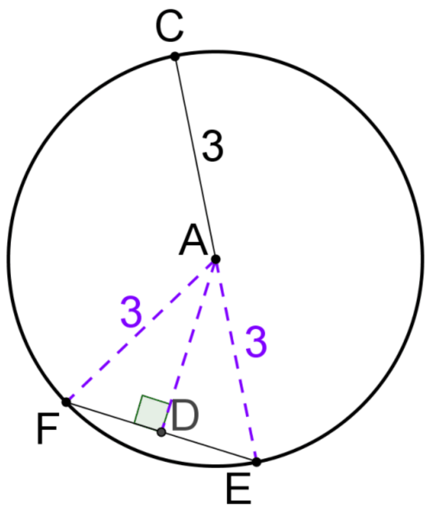
3. A is the center of the circle below. Find the shortest distance from A to $\overline{EF}$.



The shortest distance from A to $\overline{EF}$ will be the length of a segment from A to $\overline{EF}$ that is perpendicular to $\overline{EF}$. Because A is the center of the circle, $\overline{AC}$ is a radius and thus the length of any radius will be 3 units. Draw two radii from A to E and A to F. Also draw a segment from A to $\overline{EF}$ that is perpendicular to $\overline{EF}$.



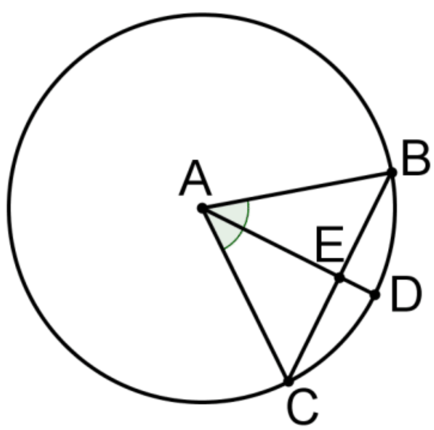
ΔAFE is an equilateral triangle because it has three sides of length 3. This means that $m\angle AFD = 60^\circ$ and ΔAFD is a 30-60-90 triangle. According to the 30-60-90 pattern, $AD = \frac{3\sqrt{3}}{2}$.

Examples

Example 1

Earlier, you were asked how $\overline{AD}$ relates to chord $\overline{BC}$.

Radius $\overline{AD}$ bisects $\angle BAC$ in the circle below. How does $\overline{AD}$ relate to chord $\overline{BC}$? Prove your ideas.



Two possible conjectures are that $\overline{AD}$ bisects $\overline{BC}$ and $\overline{AD}$ is perpendicular to $\overline{BC}$. Both of these conjectures can be proved by first proving that $\Delta ABE \cong \Delta ACE$.

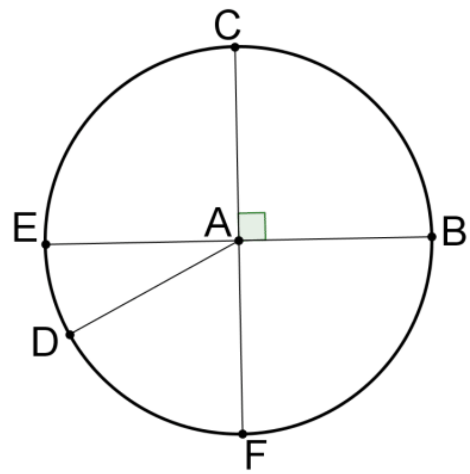
To prove that $\Delta ABE \cong \Delta ACE$, first note that $\overline{AB}$ and $\overline{AC}$ are both radii of the circle, and therefore $\overline{AB} \cong \overline{AC}$. Also note that $\overline{AE} \cong \overline{AE}$ by the reflexive property. Since it was assumed that $\overline{AD}$ bisects $\angle BAC$, $\angle BAE \cong \angle CAE$. $\Delta ABE \cong \Delta ACE$ by $SAS \cong$.

Because $\Delta ABE \cong \Delta ACE$, $\overline{BE} \cong \overline{EC}$ because they are corresponding parts. Therefore, $\overline{AD}$ bisects $\overline{BC}$. Also because $\Delta ABE \cong \Delta ACE$, $\angle CEA \cong \angle BEA$ because they are corresponding parts. $\angle CEA$ and $\angle BEA$ are supplementary because they form a line. Therefore, $m\angle CEA = m\angle BEA = 90^\circ$ and $\overline{AD}$ is perpendicular to $\overline{BC}$.

This means that if a radius bisects a central angle, then it is the perpendicular bisector of the related chord.

Example 2

In the circle below, diameters $\overline{EB}$ and $\overline{CF}$ are perpendicular and $m\angle EAD = 30^\circ$.



Find $m\widehat{BC}$.

$\overline{EB}$ and $\overline{CF}$ are perpendicular. This means that $m\angle CAB = 90^\circ$ and therefore $m\widehat{BC} = 90^\circ$

Example 3

Find $m\widehat{DF}$ in the circle under Example 2.

$m\angle EAF = 90^\circ$ because $\angle EAF$ and $\angle CAB$ are vertical angles and are therefore congruent. This means that $m\angle DAF = 60^\circ$ and therefore $m\widehat{DF} = 60^\circ$.

Example 4

Find $m\angle BAD$ in the circle under Example 2.

$m\angle BAF = 90^\circ$ because it is supplementary with $\angle BAC$. $m\angle BAD = m\angle BAF + m\angle DAF$. Therefore, $m\angle BAD = 90^\circ + 60^\circ = 150^\circ$.

CK-12 PLIX Interactive

PLIX

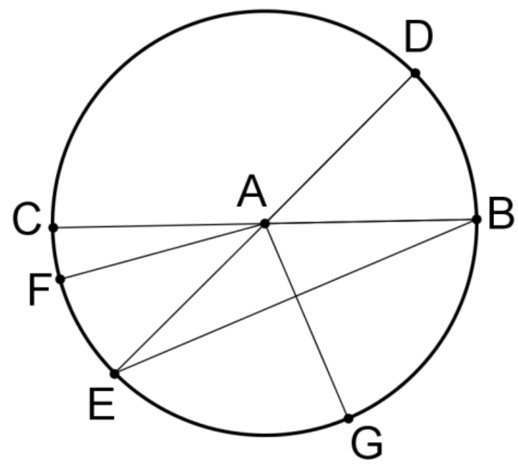
Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/chords/plix/Central-Angles-and-Arcs-5769c52d5aa4134c90f34d98

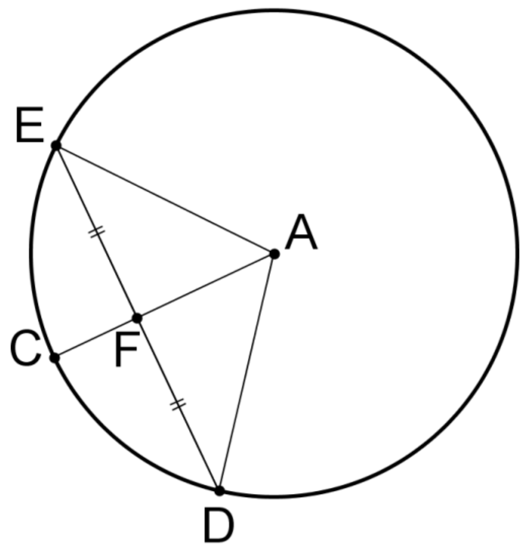
Review

1. Draw an example of a *central angle* and its *intercepted arc*.
2. What's the relationship between a central angle and its intercepted arc?
3. Draw an example of a chord.
4. A chord that passes through the center of the circle is called a _______________.

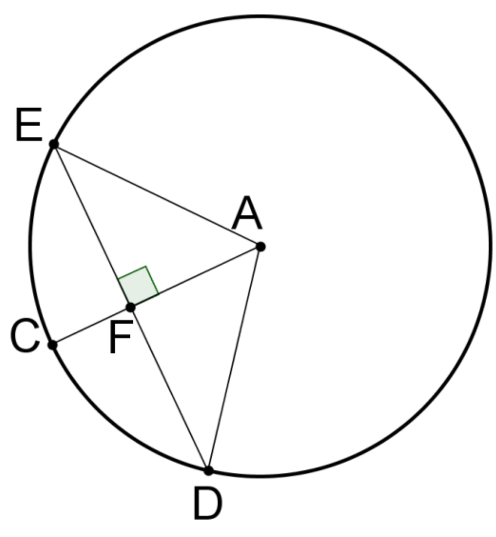
In the circle below, $\overline{CB}$ and $\overline{ED}$ are diameters, $\overline{AG}$ bisects $\angle EAB$, $m\angle DAB = 50^\circ$ and $m\angle CAF = 20^\circ$. Use this circle for #5-#9.



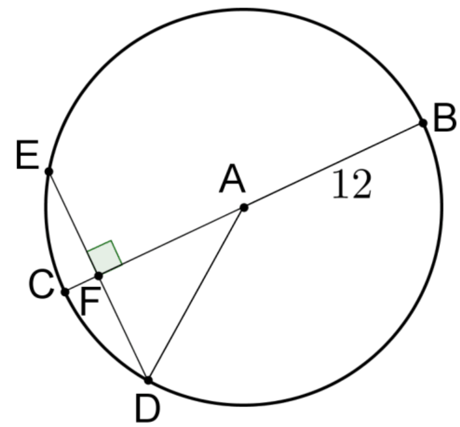
5. Find $m\widehat{FE}$.
6. Find $m\widehat{CD}$.
7. Find $m\angle EAG$.
8. Find $m\widehat{GB}$.
9. How is $\widehat{EB}$ related to $\overline{AG}$?
10. Prove that when a radius bisects a chord, it is perpendicular to the chord. *Use the picture below and prove that* $m\angle AFD = 90^\circ$.



11. Prove that when a radius is perpendicular to a chord it bisects the chord. *Use the picture below and prove that* $\overline{EF} \cong \overline{FD}$.



In the circle below with center A, $AB = 12$ and $DE = 16$.



12. Find DF.

13. Find AC.

14. Find AF.

15. Find CF.

16. Prove that no matter where a point is placed on the arc of a semicircle, the angle formed will always be a right angle.

17. Using a standard analogue clock, explain how to determine the measure of the angle formed between the two hands at three different times during the day.

18. In the diagram below, explain why $\triangle AOC$ and $\triangle BOC$ are isosceles. What can be deduced about the relationship between $\angle AOB$ and $\angle ACB$?

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.3.

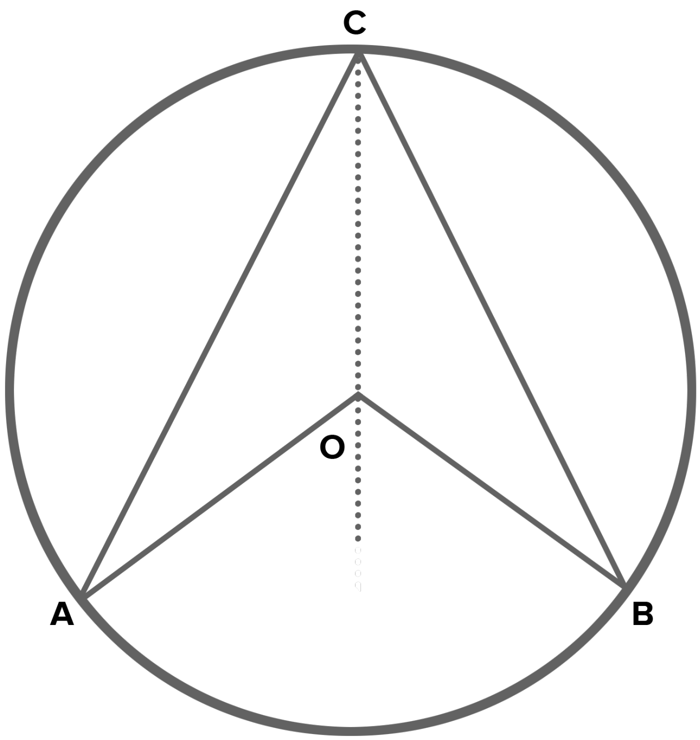


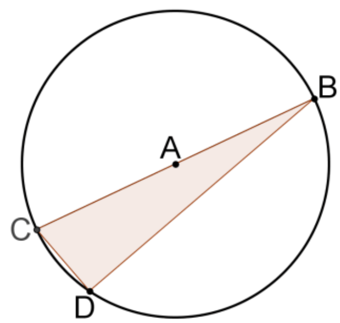
FIGURE 8.1

8.4 Inscribed Angles

Learning Objectives

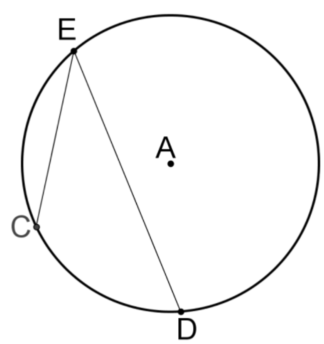
Here you will learn about inscribed angles in circles.

Point A is the center of the circle below. What can you say about ΔCBD ?



Inscribed Angles

An **inscribed angle** is an angle with its vertex on the circle. The sides of an inscribed angle will be chords of the circle. Below, $\angle CED$ is an inscribed angle.



Inscribed angles are ***inscribed*** in arcs. You can say that $\angle CED$ is ***inscribed in*** $\widehat{CED}$. You can also say that $\angle CED$ ***intercepts*** $\widehat{CD}$.

The measure of an inscribed angle is always half the measure of the arc it intercepts. You will prove and then use this theorem in the problems below.

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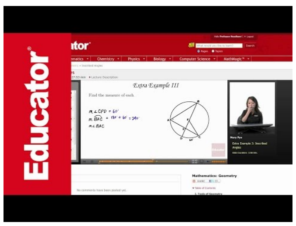
URL: http://www.ck12.org/flx/render/embeddedobject/226172



MEDIA

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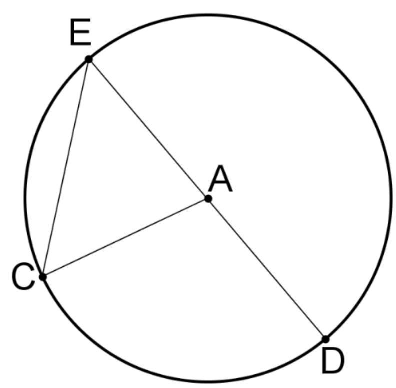
URL: http://www.ck12.org/flx/render/embeddedobject/73413

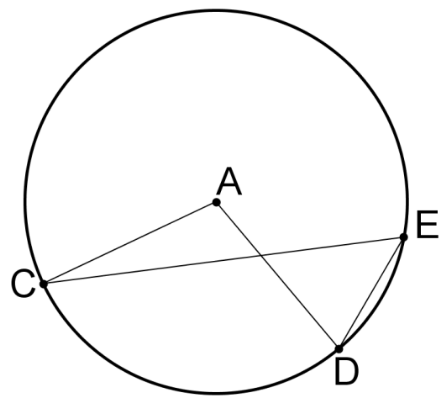
1. Consider the circle below with center at point A. Prove that $m\angle AEC = \frac{1}{2}m\angle CAD = \frac{1}{2}m\widehat{CD}$.



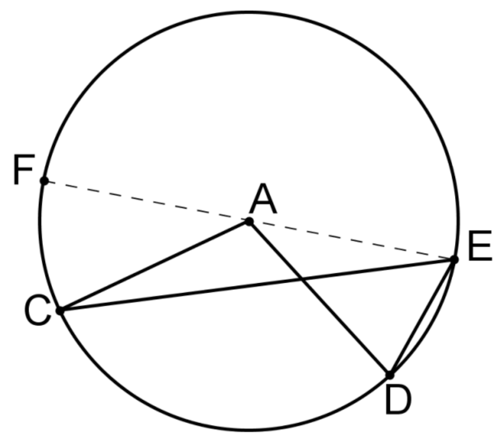
$\overline{CA}$ and $\overline{EA}$ are both radii of the circle and are therefore congruent. This means that ΔCAE is isosceles. The base angles of isosceles triangles are congruent so $m\angle ACE = m\angle AEC$. $\angle CAD$ is an exterior angle of ΔCAE, so its measure must be the sum of the measures of the remote interior angles. Therefore, $m\angle CAD = m\angle ACE + m\angle AEC$. By substitution, $m\angle CAD = m\angle AEC + m\angle AEC$. This means $m\angle CAD = 2m\angle AEC$ and $\frac{1}{2}m\angle CAD = m\angle AEC$. A central angle has the same measure as its intercepted arc, so $m\angle CAD = m\widehat{CD}$. Therefore by substitution, $\frac{1}{2}m\widehat{CD} = m\angle AEC$.

This proves that when an inscribed angle passes through the center of a circle, its measure is half the measure of the arc it intercepts.

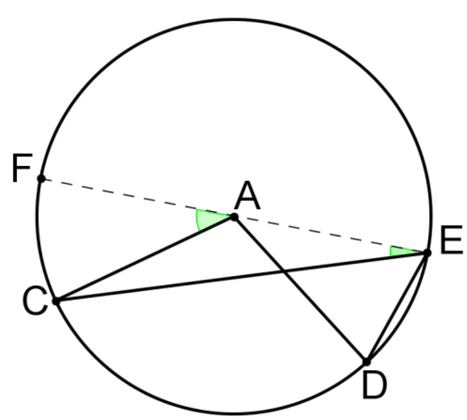
2. Use the result from the previous problem to prove that $m\angle CED = \frac{1}{2}m\widehat{CD}$.



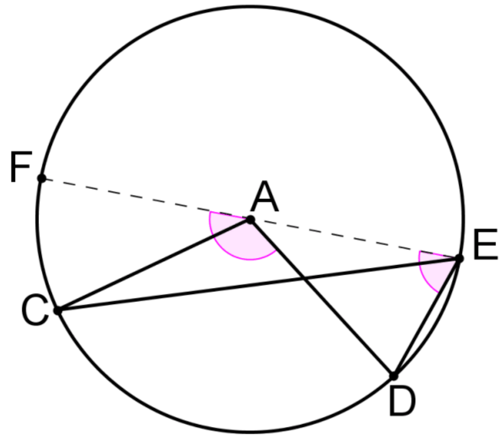
Draw a diameter through points E and A.



From #1, you know that $m\angle FEC = \frac{1}{2} m\angle FAC$.



You also know that $m\angle FED = \frac{1}{2} m\angle FAD$.



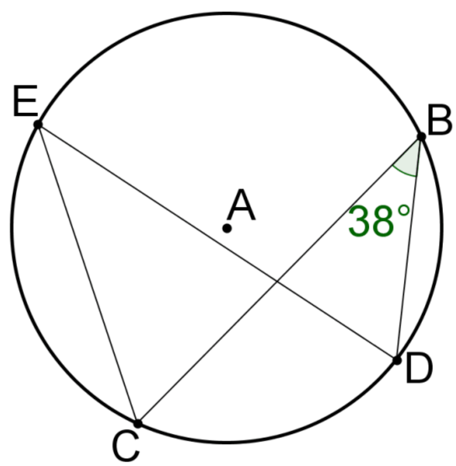
Because $m\angle FED = m\angle FEC + m\angle CED$, by substitution $\frac{1}{2}m\angle FAD = \frac{1}{2}m\angle FAC + m\angle CED$. This means $m\angle CED = \frac{1}{2}(m\angle FAD - m\angle FAC)$. $m\angle FAD - m\angle FAC = m\angle CAD$, so $m\angle CED = \frac{1}{2}m\angle CAD$.

Because $m\angle CAD = m\widehat{CD}, m\angle CED = \frac{1}{2}m\widehat{CD}$.

This proves in general that the measure of an inscribed angle is half the measure of its intercepted arc.

Now let's find the measure of an angle.

Find $m\angle CED$.



Notice that both $\angle CED$ and $\angle CBD$ intercept $\widehat{CD}$. This means that their measures are both half the measure of $\widehat{CD}$, so their measures must be equal. $m\angle CED = 38^\circ$.

MEDIA

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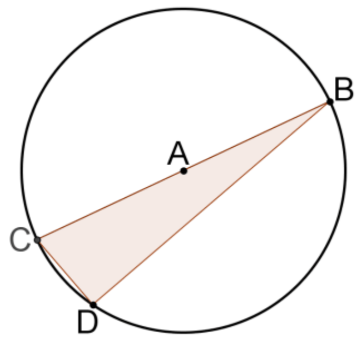
URL: http://www.ck12.org/flx/render/embeddedobject/226173

Examples

Example 1

Earlier, you were asked what can you say about ΔCBD.

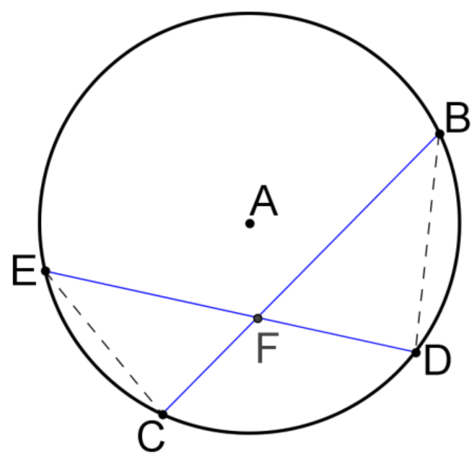
Point A is the center of the circle below. What can you say about ΔCBD ?



If point A is the center of the circle, then $\overline{CB}$ is a diameter and it divides the circle into two equal halves. This means that $m\widehat{CB} = 180^\circ$. $\angle CDB$ is an inscribed angle that intercepts $\widehat{CB}$, so its measure must be half the measure of $\widehat{CB}$. Therefore, $m\angle CDB = 90^\circ$ and ΔCBD is a right triangle.

In general, if a triangle is inscribed in a semicircle then it is a right triangle.

In #2-#3, you will use the circle below to prove that when two chords intersect inside a circle, the products of their segments are equal.



Example 2

Prove that $\Delta EFC \sim \Delta BFD$. *Hint: Look for congruent angles!*

$\angle CED \cong \angle CBD$ because both are inscribed angles that intercept the same arc $(\widehat{CD})$. $\angle EFC \cong \angle BFD$ because they are vertical angles. Therefore, $\Delta EFC \sim \Delta BFD$ by $AA \sim$.

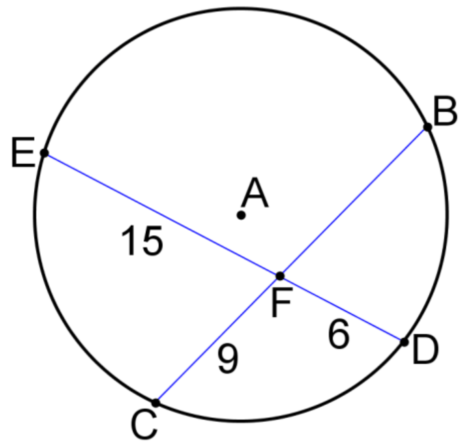
Example 3

Prove that $EF \cdot FD = BF \cdot FC$.

Because $\Delta EFC \sim \Delta BFD$, its corresponding sides are proportional. This means that $\frac{EF}{BF} = \frac{FC}{FD}$. Multiply both sides of the equation by $BF \cdot FD$ and you have $EF \cdot FD = BF \cdot FC$. **This proves that in general, when two chords intersect inside a circle, the products of their segments are equal.**

Example 4

For the circle below, find BF.



Based on the result of #2, you know that $15 \cdot 6 = BF \cdot 9$. This means that $BF = 10$.

CK-12 PLIX Interactive

PLIX

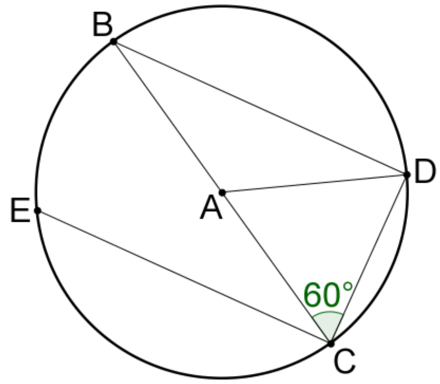
Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/segments-from-chords/plix/Segments-from-Chords-528a592e5aa4132f7ea9034c

Review

1. How are central angles and inscribed angles related?

In the picture below, $\overline{BD} \parallel \overline{EC}$. Use the picture below for #2-#6.



2. Find $m\widehat{BD}$.

3. Find $m\angle ABD$.

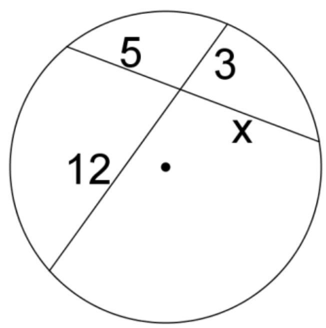
4. Find $m\widehat{EB}$.

5. Find $m\angle ECD$.

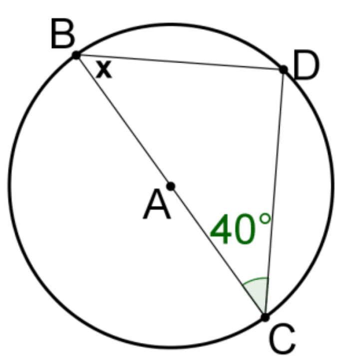
6. What type of triangle is ΔACD ?

Solve for x in each circle. If x is an angle, find the measure of the angle.

7.



8.



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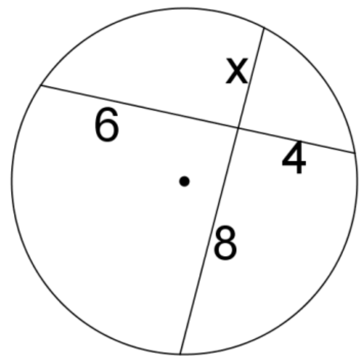
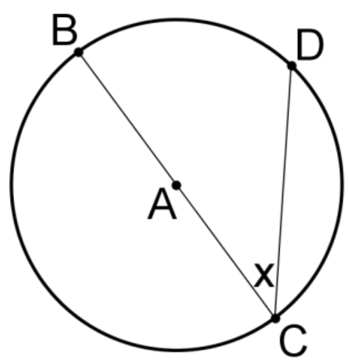


10. $m\widehat{DC} = 95^\circ$



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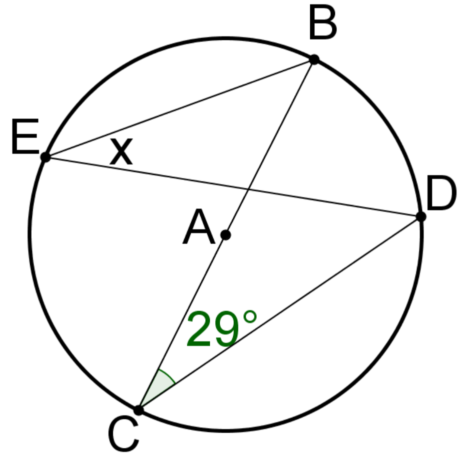

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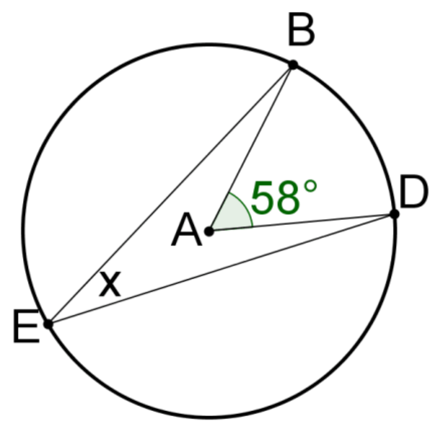

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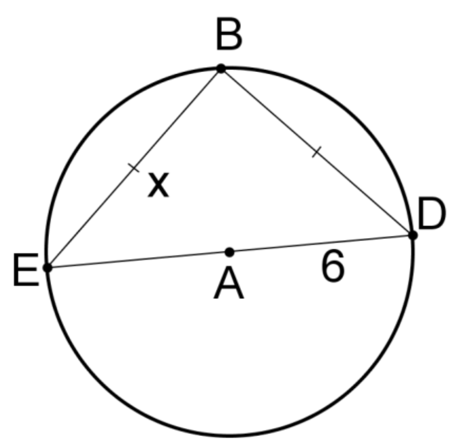

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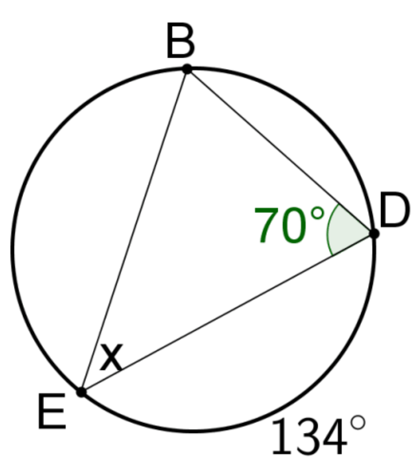
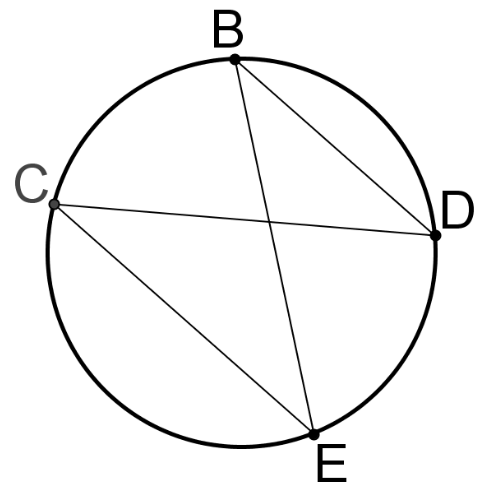
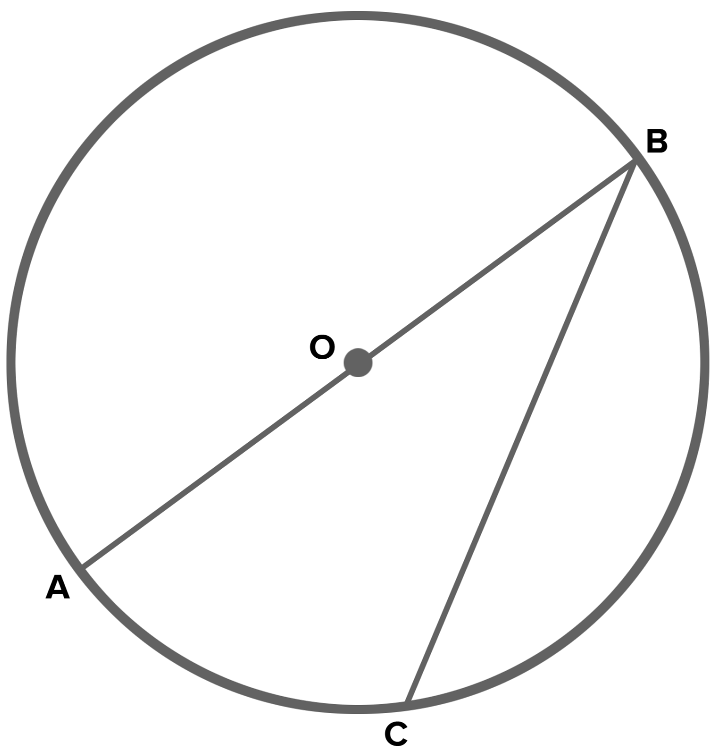


15. In the picture below, $\overline{BD} \parallel \overline{EC}$. Prove that $\widehat{BC} \cong \widehat{ED}$.

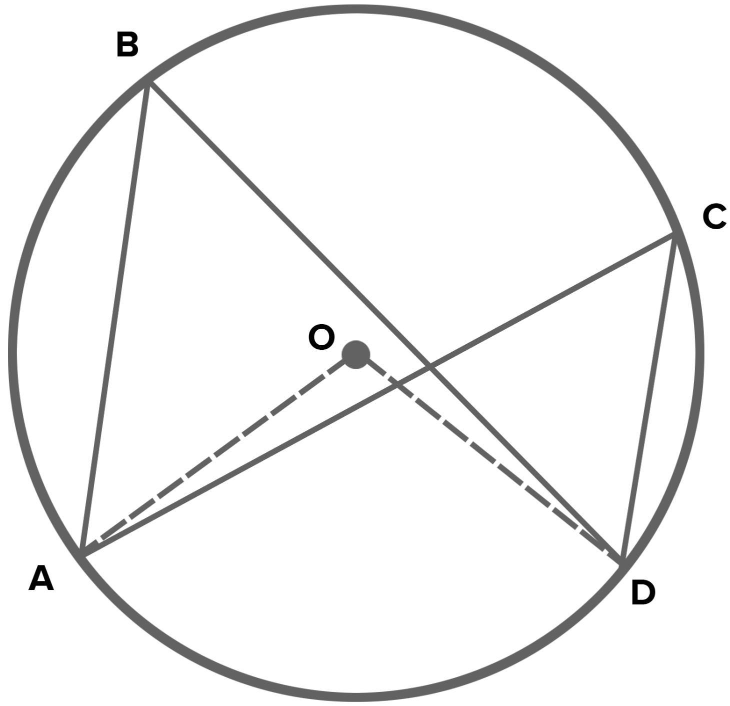


16. Using the diagram below, and knowing that the center of the circle is marked O, how could it be proved that $m\angle ABC = \frac{1}{2}mAC$?



17. Compare an inscribed angle and a central angle that intercept the same arc. How do they compare and how do they contrast?

18. Using the diagram below, explain the relationship between $\angle ABD$ and $\angle ACD$.



Review (Answers)

To see the Review answers, open this PDF file and look for section 8.4.

8.5 Inscribed and Circumscribed Circles of Triangles

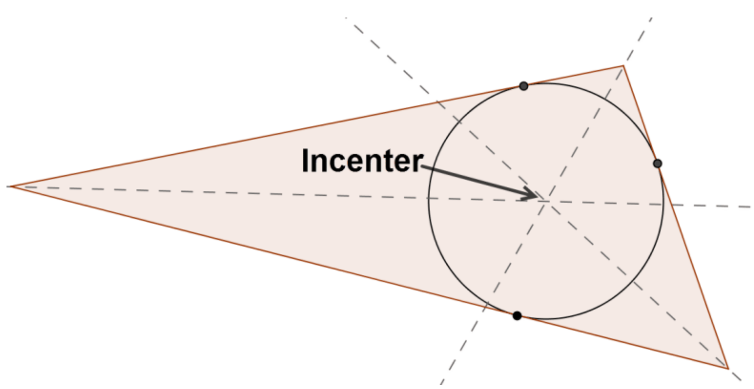
Learning Objectives

Here you will learn how to construct the inscribed and circumscribed circles of a triangle.

Given a triangle, what's the difference between the *inscribed circle* of the triangle and the *circumscribed circle* of the triangle?

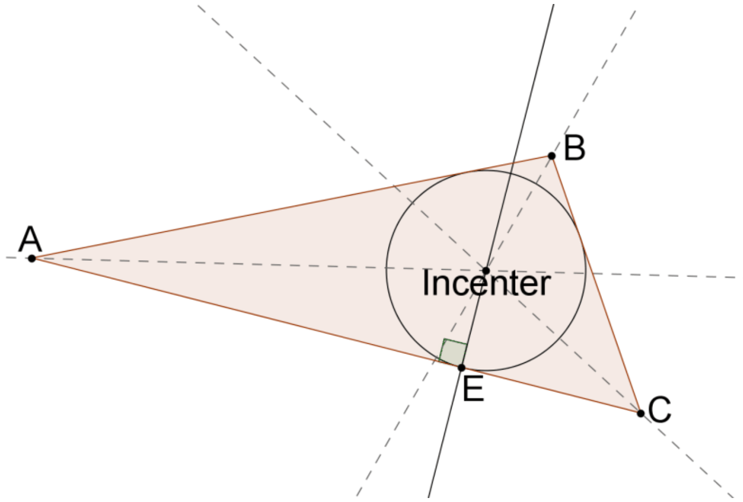
Inscribed and Circumscribed Circles of Triangles

Given a triangle, an **inscribed circle** is the largest circle contained within the triangle. The inscribed circle will touch each of the three sides of the triangle in exactly one point. The center of the circle inscribed in a triangle is the **incenter** of the triangle, the point where the angle bisectors of the triangle meet.



To construct the **inscribed circle:**

1. Construct the incenter.
2. Construct a line perpendicular to one side of the triangle that passes through the incenter. The segment connecting the incenter with the point of intersection of the triangle and the perpendicular line is the radius of the circle.
3. Construct a circle centered at the incenter with the radius found in step 2.



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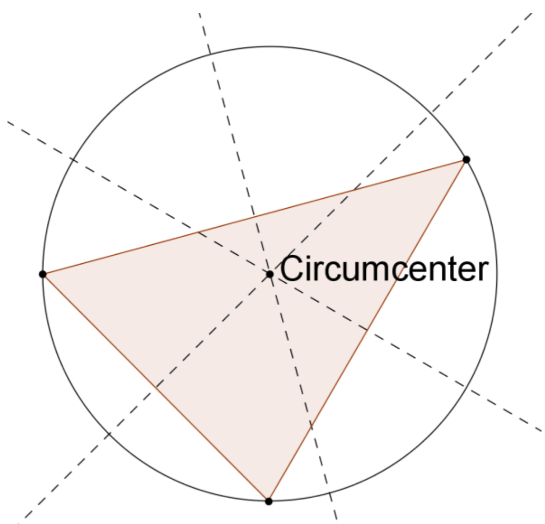
The steps for constructing the inscribed circle for a given triangle will be explored in the problems below.

Given a triangle, the **circumscribed circle** is the circle that passes through all three vertices of the triangle. The center of the circumscribed circle is the **circumcenter** of the triangle, the point where the perpendicular bisectors of the sides meet.



To construct the **circumscribed circle:**

1. Construct the circumcenter.
2. Construct a circle centered at the circumcenter that passes through one of the vertices. This same circle should pass through *all three* vertices.

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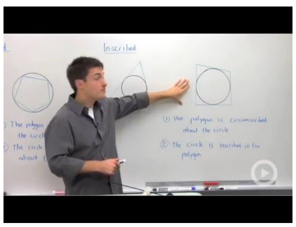
The steps for constructing the circumscribed circle for a given triangle will be explored in the Examples section.



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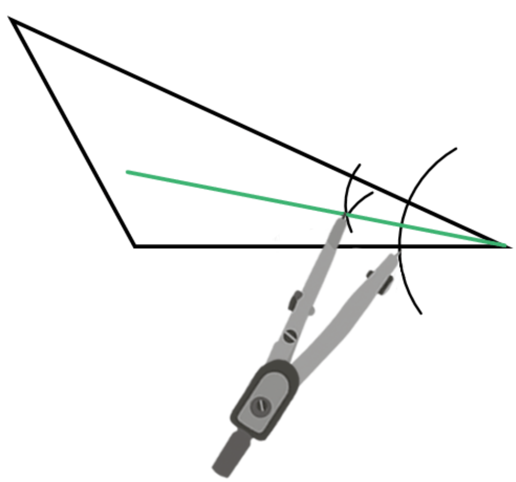
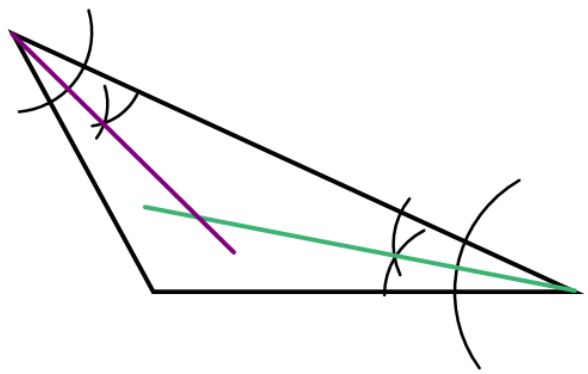
Constructing Angle Bisectors

Draw a triangle. Construct the angle bisectors of two of its angles. Why is the point of intersection of the two angle bisectors the incenter of the circle?

Use your compass and straightedge to construct the angle bisector of one of the angles.



Repeat with a second angle.

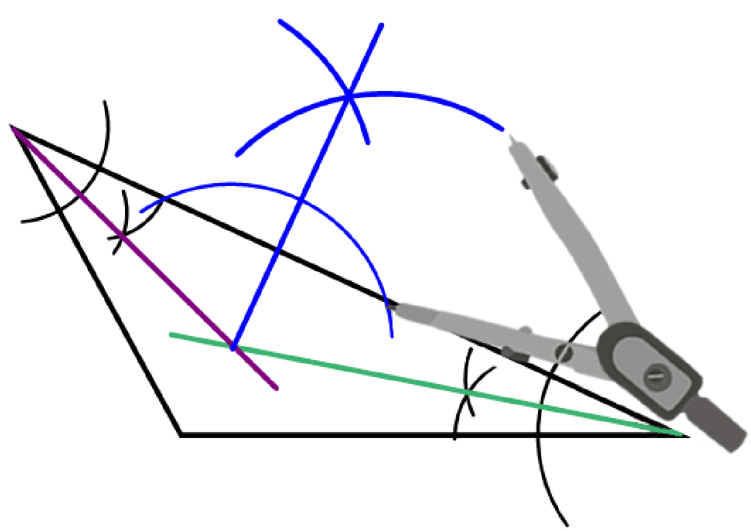


The point of intersection of the angle bisectors is the incenter. It is not necessary to construct all three angle bisectors because they all meet in the same point. The third angle bisector does not provide any new information.

Constructing Perpendicular Lines

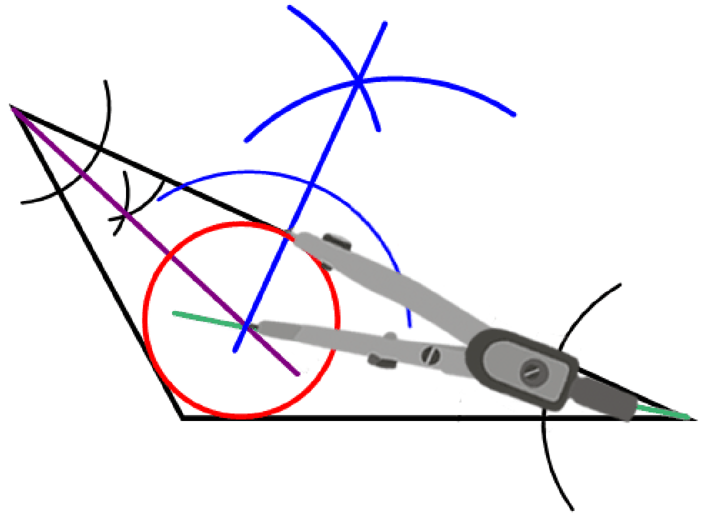
Construct a line perpendicular to one side of the triangle that passes through the incenter of the triangle.

Use your compass and straightedge to construct a line perpendicular to one side of the triangle that passes through the incenter.



Constructing Inscribed Circles

Construct a circle centered at the incenter that passes through the point of intersection of the side of the triangle and the perpendicular line from the problem above.



Note that this circle touches each side of the triangle exactly once.

Examples

Example 1

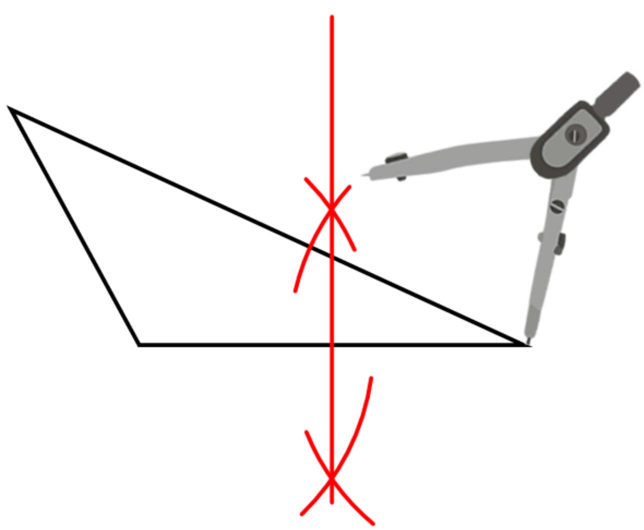
Earlier, you were asked what is the difference between the inscribed circle of the triangle and the circumscribed circle of the triangle.

The inscribed circle of a triangle is inside the triangle. The circumscribed circle of a triangle is outside the triangle.

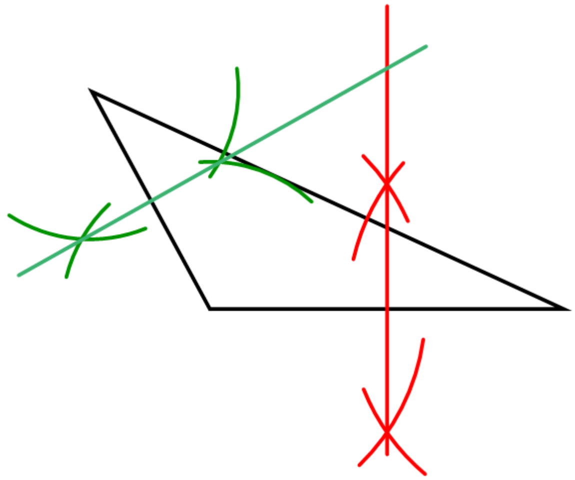
Example 2

Draw a triangle. Construct the perpendicular bisectors of two of its sides. Why is the point of intersection of the two perpendicular bisectors the circumcenter of the circle?

Use your compass and straightedge to construct the perpendicular bisector of one side.



Repeat with a second side.

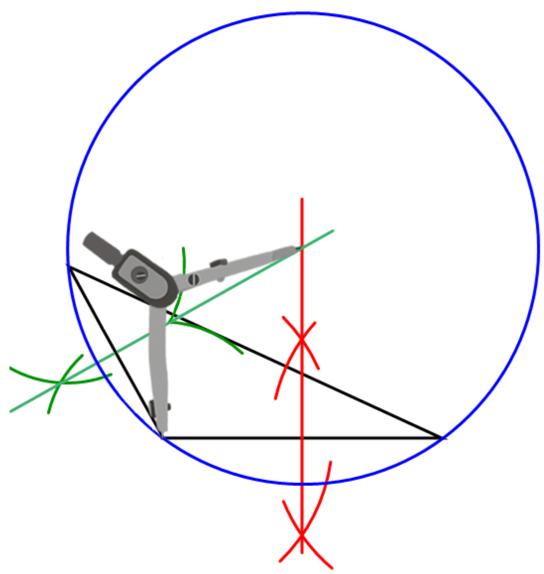


The point of intersection of the perpendicular bisectors is the circumcenter. It is not necessary to construct all three perpendicular bisectors because they all meet in the same point. The third perpendicular bisector does not provide any new information.

Example 3

Continue with your triangle from #2. Construct the circumscribed circle of the triangle.

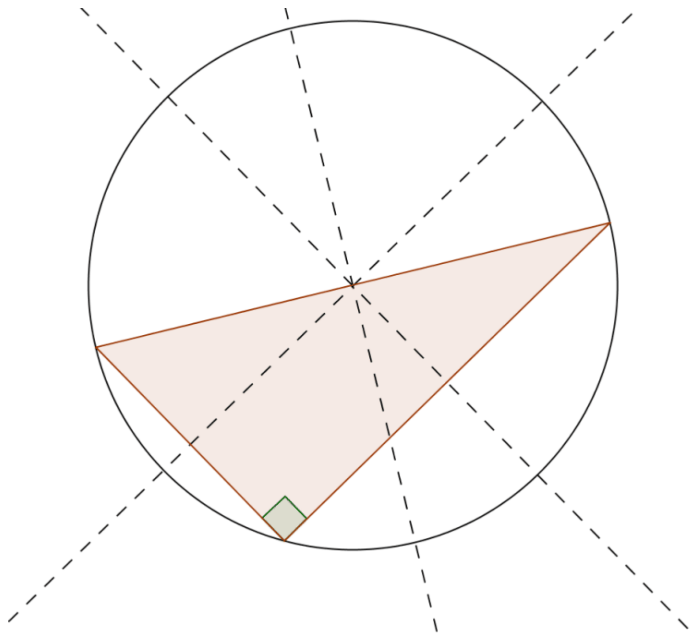
Construct a circle centered at the circumcenter that passes through one of the vertices of the triangle. This circle should pass through *all three* vertices.



Example 4

Justify the statement: *The hypotenuse of a right triangle will be a diameter of the circumscribed circle of the triangle.*

Each of the angles that make up a triangle become inscribed angles of the circumscribed circle. A 90° angle will intercept an arc of 180°, which is half a circle. Therefore, the side opposite the 90° angle of the triangle must be a diameter of the circle.



CK-12 PLIX Interactive

PLIX

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URL: http://www.ck12.org/geometry/concurrence-and-constructions/plix/Inscribed-and-Circumscribed-Circles-of-Triangles-536027f35aa4132e8a7bcae2

Review

1. Draw a triangle and construct the angle bisector of two of its angles.

2. Continue with your triangle from #1. Construct a line perpendicular to one side of the triangle that passes through the incenter of the triangle.

3. Continue with your triangle from #1 and #2. Construct the inscribed circle of the triangle.

4. Why was it not necessary to construct the angle bisector of *all three* of the angles of the triangle?

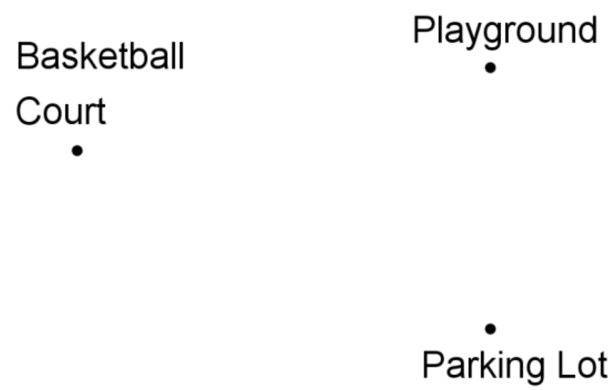
5. Explain why the incenter is equidistant from each of the sides of the triangle.

6. Draw a triangle and construct the perpendicular bisector of two of its sides.

7. Continue with your triangle from #5. Construct the circumscribed circle of the triangle.

8. Explain why the circumcenter is equidistant from each of the vertices of the triangle.

You work selling food from a food truck at a local park. You want to position your truck so that it is the same distance away from each of the three locations shown on the map below.

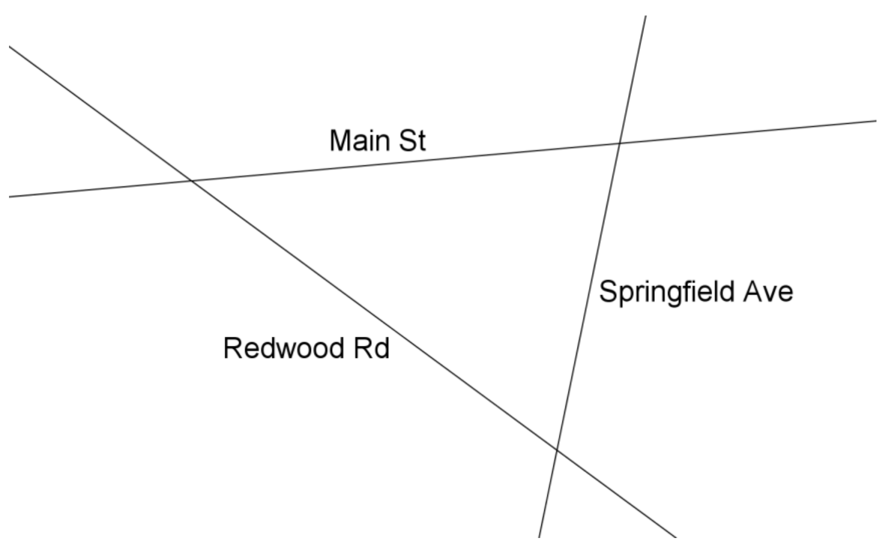


9. Is the point of interest the incenter or the circumcenter?

10. Find the point on the map that is equidistant from each of the three locations.

11. How could you fold the map in two places to find the point equidistant from each of the three locations?

A new elementary school is to be constructed in your town. The plan is to build the school so that it is the same distance away from each of the three major roads shown in the map below.



12. Is the point of interest the incenter or the circumcenter?

13. Find the point on the map that is equidistant from each of the three roads.

14. How could you fold the map in two places to find the point equidistant from each of the three roads?

15. Justify the following statement: *Given any three non-collinear points, there exists exactly one circle that passes through the points.*

16. A sprinkler is being installed in the park. The park is in the shape of a triangle with sidewalks along each side of the park. Draw a diagram to represent this problem and use your knowledge of inscribed circles to determine where the sprinkler should be placed in order to keep the ground wet and the sidewalks dry.

17. In the amusement park, lines joining the Ferris wheel, the roller coaster, and the carousal make a triangle. You want to put a refreshment stand in the park that is equidistant between these three attractions. Construct a circle that has the intersection of the perpendicular bisectors of the triangle (the circumcenter) as it center. Describe the relationship between the triangle and the circle.

18. Using the information in question 17, if the vertices of the triangle are moved to change the shape and size, how will this affect the relationship between the circle and the triangle?

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.5.

8.6 Quadrilaterals Inscribed in Circles

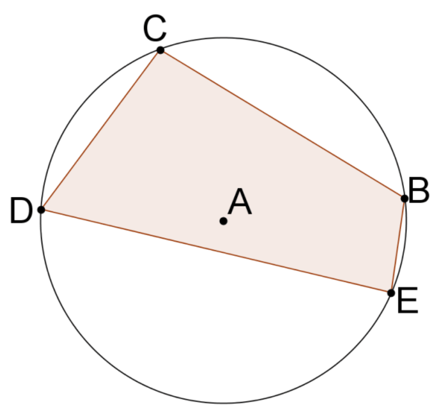
Learning Objectives

Here you will explore quadrilaterals inscribed in circles.

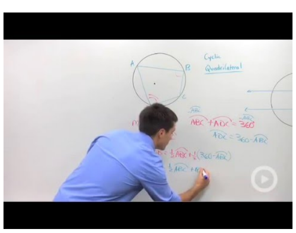
One angle of a rhombus is 30°. Can this rhombus be inscribed in a circle?

Quadrilaterals Inscribed in Circles

A quadrilateral is said to be **inscribed in a circle** if all four vertices of the quadrilateral lie on the circle. Quadrilaterals that can be inscribed in circles are known as **cyclic quadrilaterals**. The quadrilateral below is a cyclic quadrilateral.



Not all quadrilaterals can be inscribed in circles and so not all quadrilaterals are cyclic quadrilaterals. **A quadrilateral is cyclic if and only if its opposite angles are supplementary**.



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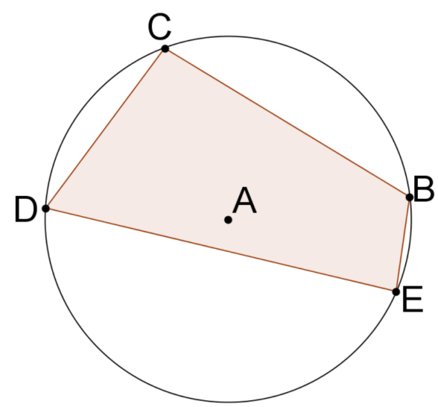
Proving Supplementary Angles

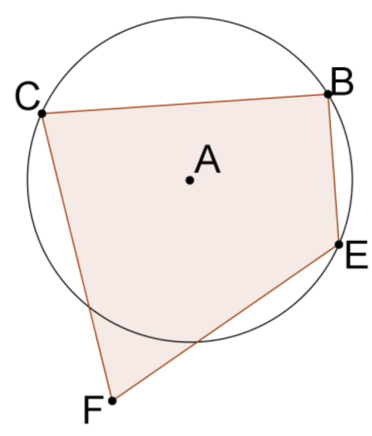
Consider the cyclic quadrilateral below. Prove that $\angle DEB$ and $\angle DCB$ are supplementary.



First note that $m\widehat{DEB} + m\widehat{DCB} = 360^\circ$ because these two arcs make a full circle. $2m\angle DEB = m\widehat{DEB}$ and $2m\angle DCB = m\widehat{DCB}$ because the measure of an inscribed angle is half the measure of its intercepted arc. By substitution, $2m\angle DEB + 2m\angle DCB = 360^\circ$. Divide by 2 and you have $m\angle DEB + m\angle DCB = 180^\circ$. Therefore, $\angle DEB$ and $\angle DCB$ are supplementary.

Finding Contradictions

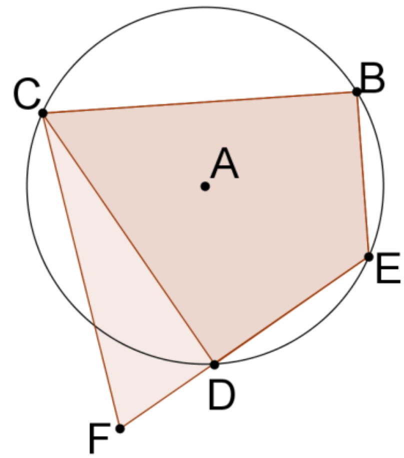
Consider the quadrilateral below. Assume that $\angle B$ and $\angle F$ are supplementary, but that F does NOT lie on the circle. Find a contradiction. What does this prove?



One method of proof is called a proof by contradiction. With a proof by contradiction you prove that something cannot **not** be true. Therefore, it must **be** true. Here, you are attempting to prove that it is ***impossible*** for a quadrilateral with opposite angles supplementary to **not** be cyclic. Therefore, such a quadrilateral must **be** cyclic.

Finding a Point of Intersection

Assume that $\angle B$ and $\angle F$ are supplementary, but F is not on the circle. Find the point of intersection of $\overline{EF}$ and the circle and call it D. Connect D with C.



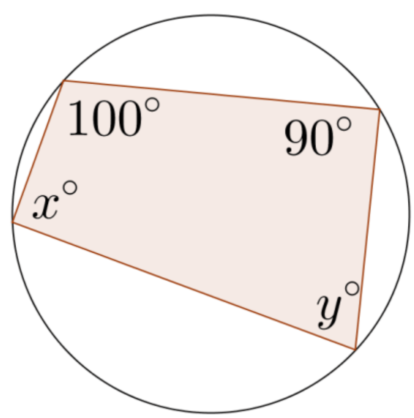
$\angle CDE$ is an exterior angle of ΔFCD, so its measure is equal to the sum of the measures of the remote interior angles of the triangle. This means that $m\angle CDE = m\angle FCD + m\angle F$. Quadrilateral $BCDE$ is cyclic, so $\angle CDE$ and $\angle B$ must be supplementary. This means that $\angle CDE$ and $\angle F$ must be congruent because they are both supplementary to the same angle.

The two highlighted statements are a contradiction – they cannot both be true. This means that your original assumption cannot exist. You cannot have a quadrilateral with opposite angles supplementary that is not cyclic. So, if opposite angles of a quadrilateral are supplementary then the quadrilateral must be cyclic.

Note: You will learn more about proof by contradiction in future courses!

Solving for Unknown Values

Solve for x and y.



Opposite angles are supplementary, so $90^\circ + x^\circ = 180^\circ$ and $100^\circ + y^\circ = 180^\circ$. This means $x = 90$ and $y = 80$.

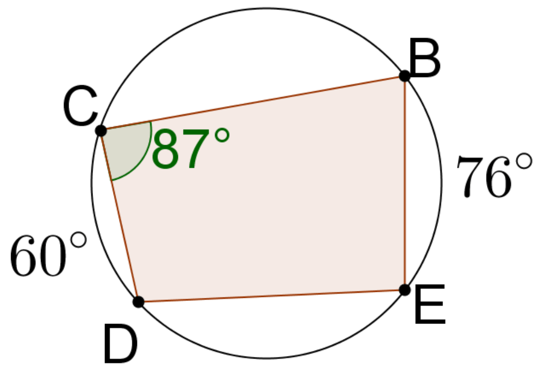
Examples

Example 1

Earlier, you were given a problem about a rhombus.

One angle of a rhombus is 30°. Can this rhombus be inscribed in a circle?

Opposite angles of a rhombus are congruent. If a rhombus has a 30° angle then it has one pair of opposite angles that are each 30° and one pair of opposite angles that are each 150°. Opposite angles are not supplementary so this rhombus cannot be inscribed in a circle.



Example 2

Find $m\widehat{DE}$.

$\angle BCD$ is the inscribed angle of $\widehat{DEB}$. This means that the measure of the arc is twice the measure of the angle. $m\widehat{DEB} = 87^\circ \cdot 2 = 174^\circ$. Since $m\widehat{BE} = 76^\circ$, $m\widehat{DE} = 174^\circ - 76^\circ = 98^\circ$.

Example 3

Find $m\angle DEB$.

$\angle BCD$ and $\angle DEB$ are opposite angles of a cyclic quadrilateral so they are supplementary. $m\angle DEB = 180^\circ - 87^\circ = 93^\circ$.

Example 4

Find $m\widehat{CB}$.

A full circle is 360°. $m\widehat{CB} = 360^\circ - 60^\circ - 98^\circ - 76^\circ = 126^\circ$.

CK-12 PLIX Interactive

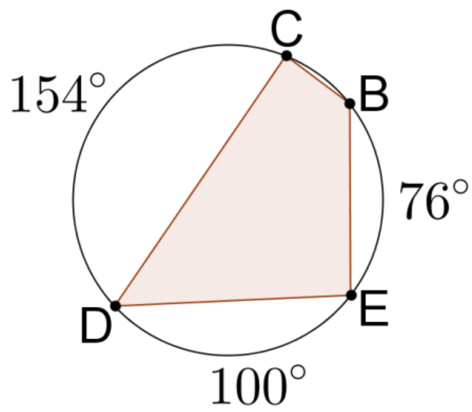
PLIX

Click image to the left or use the URL below.

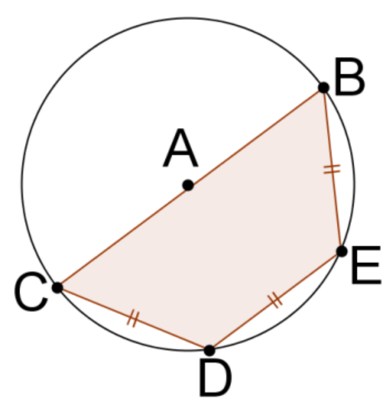
URL: http://www.ck12.org/geometry/inscribed-quadrilaterals-in-circles/plix/Quadrilateral-Inscribed-in-a-Circle-528a56fa5aa41325fb801050

Review

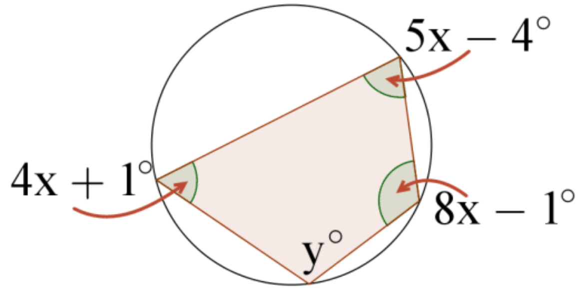
1. What is a cyclic quadrilateral?
2. A quadrilateral is cyclic if and only if its opposite angles are __________________.



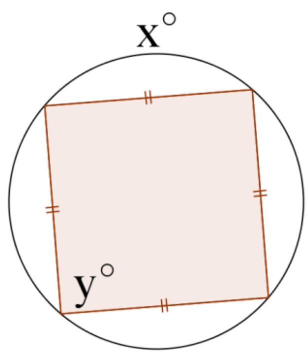
3. Find $m\angle B$.
4. Find $m\angle E$.
5. Find $m\angle D$.



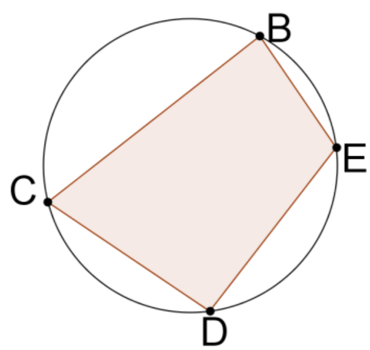
6. Find $m\widehat{CD}$.
7. Find $m\widehat{DE}$.
8. Find $m\angle CBE$.
9. Find $m\angle CEB$.



10. Solve for x.
11. Solve for y.



12. Solve for x.
13. Solve for y.
14. If a cyclic quadrilateral has a 90° angle, must it be a square? If yes, explain. If no, give a counter example.
15. Use the picture below to prove that angles B and D must be supplementary.



16. Describe a real world logo that has a quadrilateral inscribed in a circle.
17. A stained glass ornament is in the shape of a circle. The artist would like to inscribe quadrilaterals into the circle. Draw three different designs for her and describe the kinds of quadrilaterals she needs to make for each one.

Review (Answers)

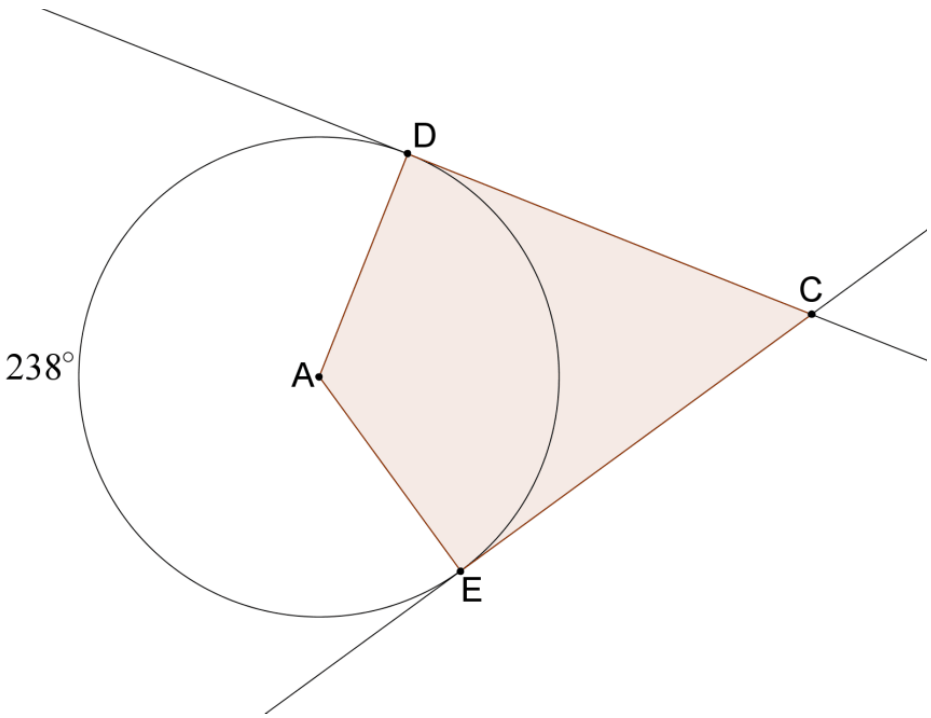
To see the Review answers, open this PDF file and look for section 8.6.

8.7 Tangent Lines to Circles

Learning Objectives

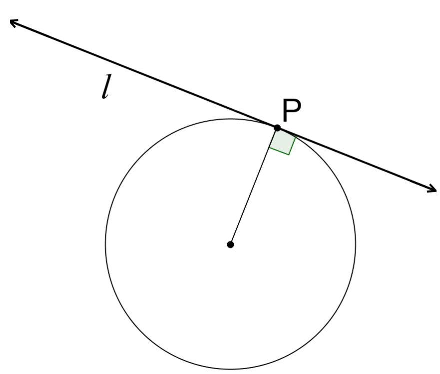
Here you will learn about lines that are tangent to circles.

$\overleftrightarrow{DC}$ and $\overleftrightarrow{CE}$ are tangent to circle A at points D and E respectively. What type of quadrilateral is $ADCE$? Can you find $m\angle DCE$?



Tangent Lines to Circles

When a line intersects a circle in exactly one point the line is said to be **tangent to the circle** or **a tangent of the circle**. Below, line l is tangent to the circle at point P.



You will prove that **if a tangent line intersects a circle at point P, then the tangent line is perpendicular to the radius drawn to point P.**

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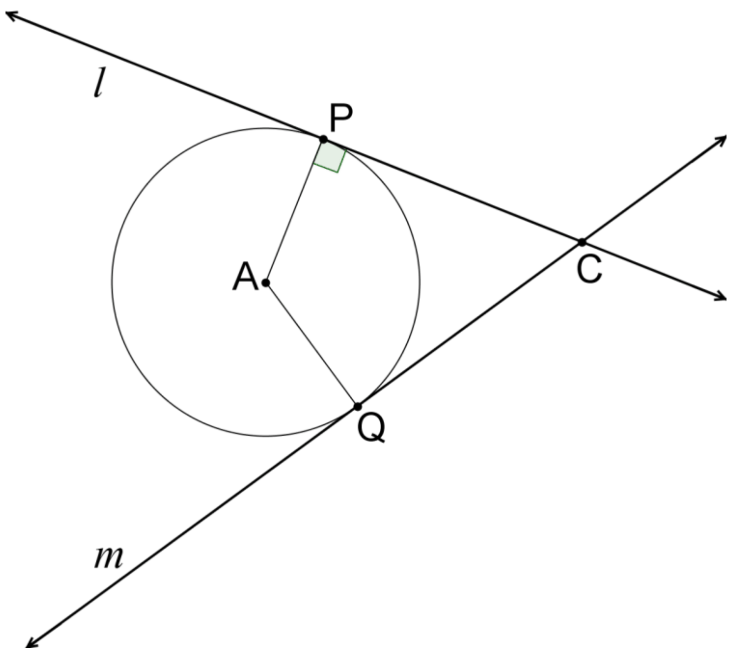
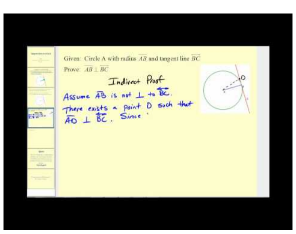
From any point outside a circle, you can drawn two lines tangent to the circle. You will learn how to construct these lines in problems later. Below, from point C both lines l and m are tangent to circle A.



In the second problem, you will show that in this situation, $\overline{PC} \cong \overline{CQ}$. In third problem, you will show that $\angle PAQ$ and $\angle PCQ$ are supplementary.



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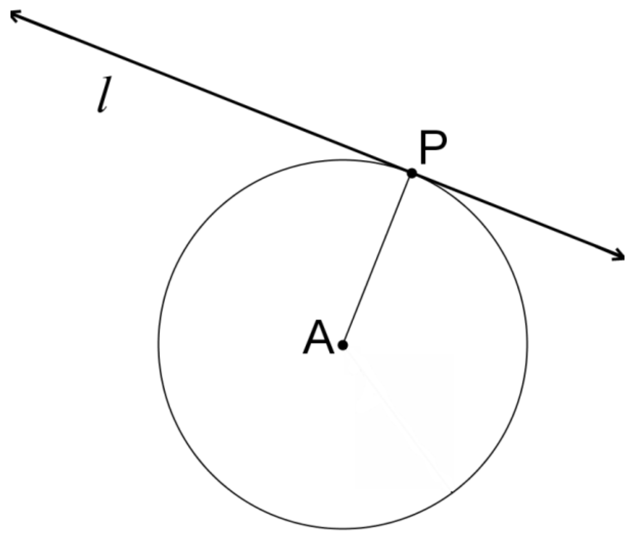
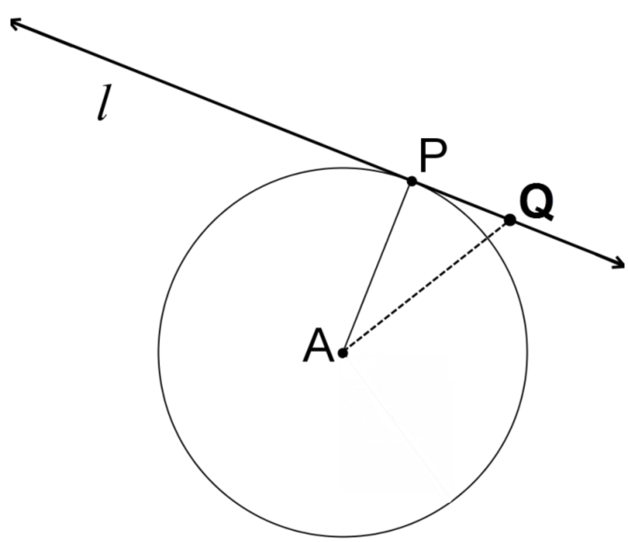
Let's look at a few example problems.

1. Line l is tangent to circle A at point P. Prove that line l is perpendicular to $\overline{AP}$.

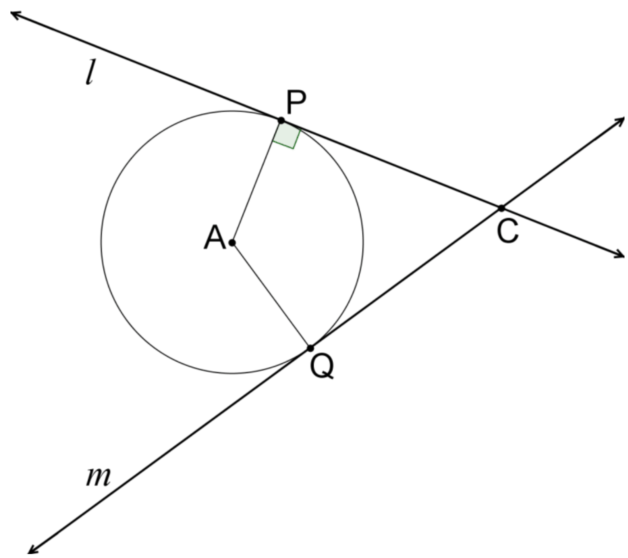


This proof relies on the fact that the shortest distance from a point to a line is along the segment perpendicular to the line.

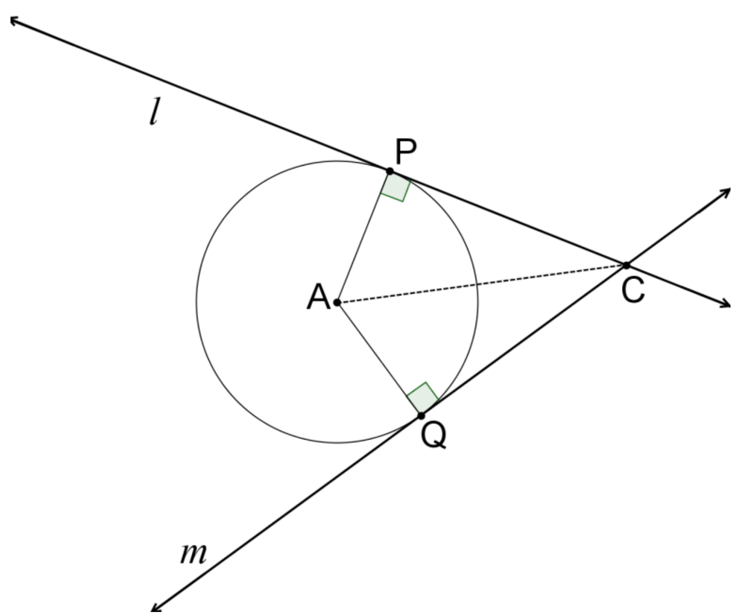
Consider a point Q on line l but not on circle A. $AQ > AP$, because Q is outside circle A. This means that the shortest distance from line l to point A is from point P to point A. Therefore, $\overline{AP}$ must be perpendicular to line l.



2. From point C, both lines l and m are tangent to circle A. Show that $\overline{PC} \cong \overline{QC}$. What does this mean in general?



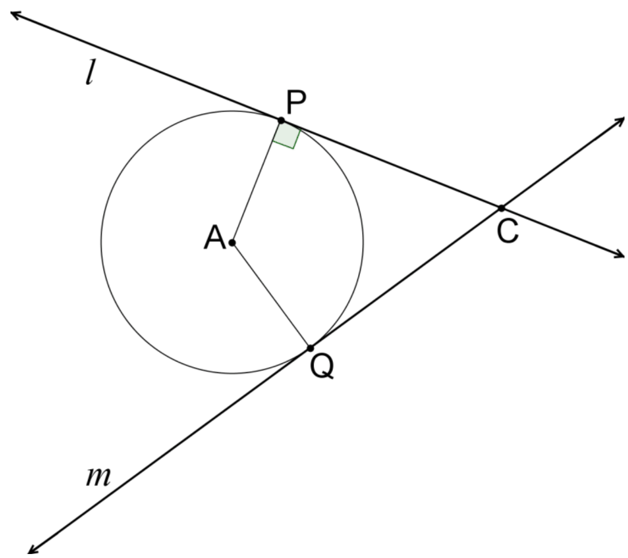
Draw a segment connecting A and C. Note that $\angle AQC$ is also a right angle.



$\overline{AC} \cong \overline{AC}$ by the reflexive property and $\overline{PA} \cong \overline{QA}$ because they are both radii of the circle. This means that $\Delta APC \cong \Delta AQC$ by $HL \cong$. $\overline{PC} \cong \overline{QC}$ because the segments are corresponding parts of congruent triangles.

$\overline{PC}$ and $\overline{QC}$ are known as **tangent segments**. **In general, two tangent segments to a circle from the same point outside the circle will always be congruent.**

3. From point C, both lines l and m are tangent to circle A. Show that $\angle PAQ$ and $\angle PCQ$ are supplementary. What does this mean in general?



$\angle ACQ$ is a right angle because line m is tangent to circle A at point Q. The sum of the measures of the interior angles of a quadrilateral is 360°. This means that $m\angle PAQ + m\angle PCQ = 360^\circ - 90^\circ - 90^\circ = 180^\circ$. Therefore, $\angle PAQ$ and $\angle PCQ$ are supplementary.

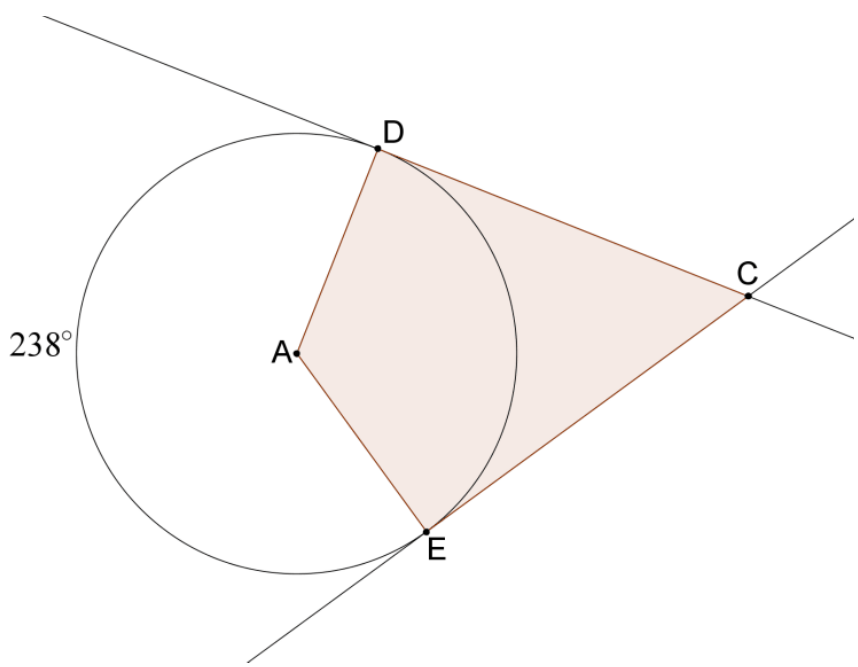
In general, the angle between two lines tangent to a circle from the same point will be supplementary to the central angle created by the two tangent lines.

Examples

Example 1

Earlier, you were given a problem about tangent lines to a circle.

$\overleftrightarrow{DC}$ and $\overleftrightarrow{CE}$ are tangent to circle A at points D and E respectively. What type of quadrilateral is $ADCE$? Can you find $m\angle DCE$?



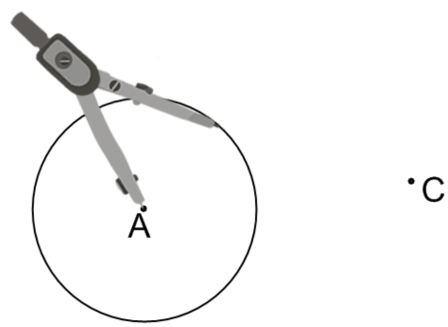
$\overline{DA}$ and $\overline{EA}$ are both radii of the circle, so they are congruent. $\overline{DC}$ and $\overline{EC}$ are both tangent segments to the circle from the same point (C), so they are congruent. The quadrilateral has two pairs of adjacent congruent segments so it is a kite.

$m\widehat{DE} = 360^\circ - 238^\circ = 122^\circ$. The means $m\angle DAE = 122^\circ$. Because $\overleftrightarrow{DC}$ and $\overleftrightarrow{CE}$ are tangent to circle A, you know that $\angle DAE$ and $\angle DCE$ are supplementary. $m\angle DCE = 180^\circ - 122^\circ = 58^\circ$.

In the following questions, you will learn how to construct lines tangent to a circle from a given point.

Example 2

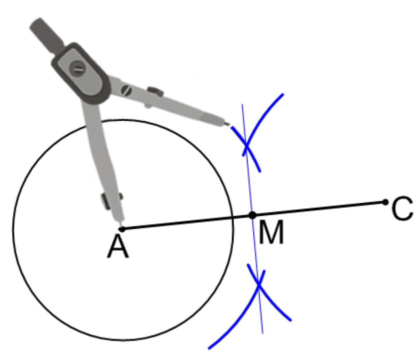
Use your compass and straightedge (or another construction device) to construct a circle and a point not on the circle. Label the center of the circle A and the point not on the circle C.



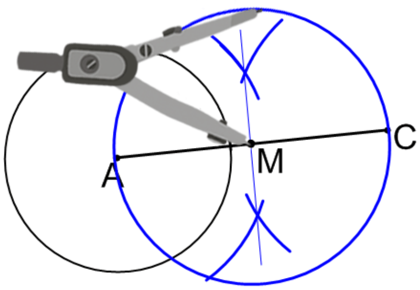
Example 3

Find the midpoint of $\overline{AC}$ and label it M. Construct a circle centered at M that passes through both A and C.

Construct the perpendicular bisector of $\overline{AC}$ in order to find its midpoint.



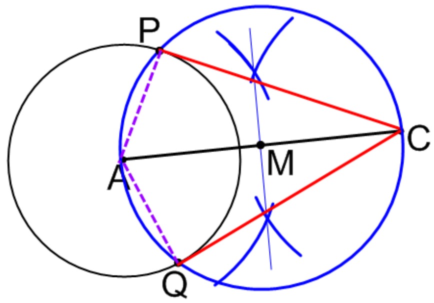
Then construct a circle centered at point M that passes through point C. The circle should also pass through point A.



Example 4

Find the points of intersection of circle M and circle A. Label the points of intersection P and Q. Connect point C with point P and point C with point Q. Why are $\overleftrightarrow{CP}$ and $\overleftrightarrow{CQ}$ tangent lines?

Find the points of intersection and connect them with point C.



Note that $\overline{AC}$ is a diameter of circle M, so it divides circle M into two semicircles. $\angle APC$ and $\angle AQC$ are inscribed angles of these semicircles, so they must be right angles. $\overline{PC}$ meets radius $\overline{AP}$ at a right angle, so $\overline{PC}$ is tangent to circle A. Similarly, $\overline{QC}$ meets radius $\overline{AQ}$ at a right angle, so $\overline{QC}$ is tangent to circle A.

CK-12 PLIX Interactive

PLIX

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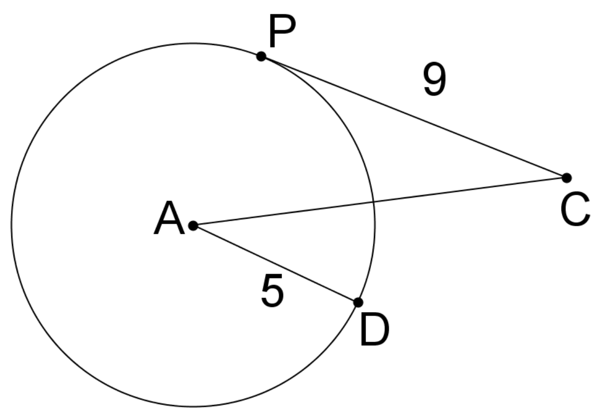
URL: http://www.ck12.org/geometry/tangent-lines/plix/Tangent-Lines-53602949da2cfe16bfc503f5

1. What is a tangent line?

For all pictures below, assume that lines that appear tangent are tangent.

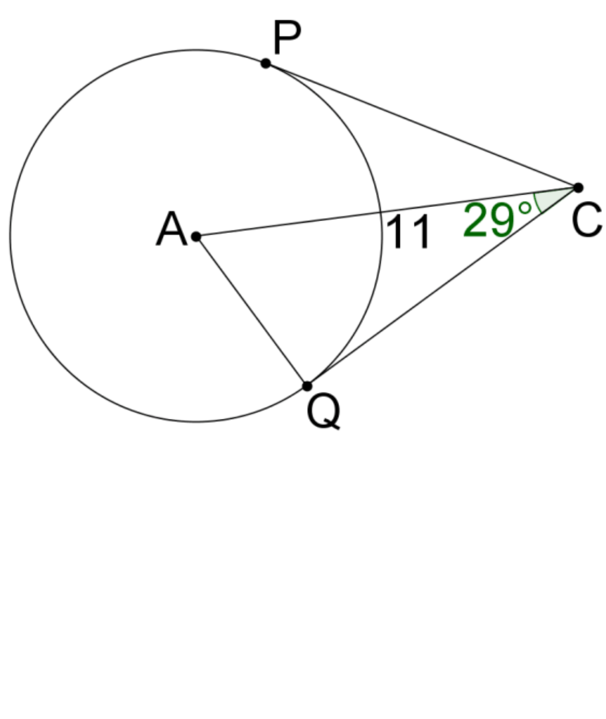
Use the image below for #2-#3.



2. Draw in $\overline{AP}$ and find its length.

3. Find AC.

Use the image below for #4-#7.



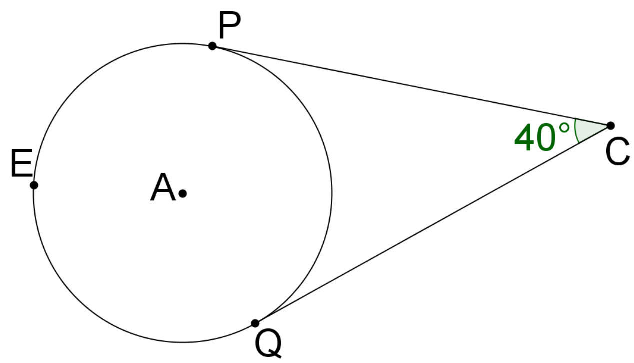
4. Find $m\angle CAQ$.

5. Find QC.

6. Find AQ.

7. Find PC.

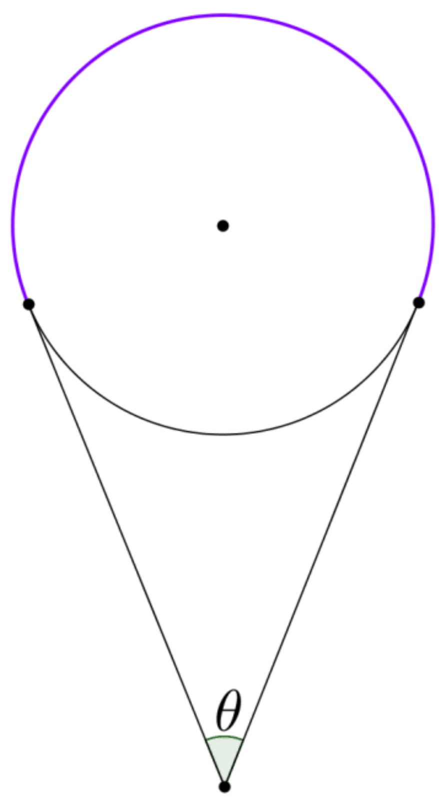
Use the image below for #8-#9.



8. Find $m\widehat{PQ}$.

9. Find $m\widehat{PEQ}$.

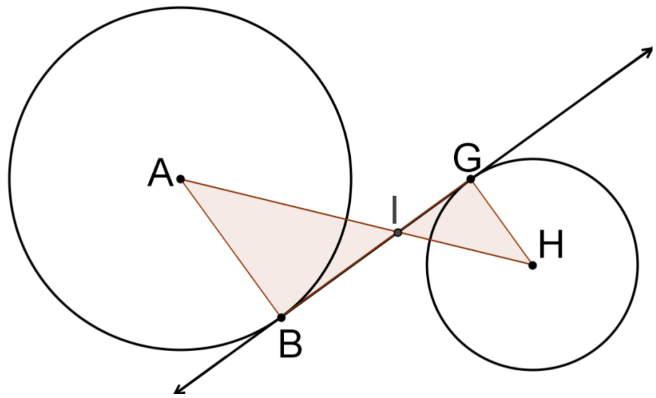
Use the image below for #10-#11. 62% of the circle is purple.



10. Find the measure of the purple arc.

11. Find the measure of angle θ.

Use the image below for #12-#13.



12. Make a conjecture about how ΔABI and ΔHGI are related.

13. Prove your conjecture from #12.

14. Use construction tools of your choice to construct a circle and a point not on the circle. Then, construct two lines tangent to the circle that pass through the point. *Hint: Look at the Guided Practice questions for the steps for this construction.*

15. Justify why your construction from #14 created tangent lines.

16. Describe, using drawings and words, how the throw of a discus by an athlete relates to tangent lines.

17. Triangle ABC is isosceles and inscribed in a circle. The tangent $[TC]$ is drawn on the circle. Prove that AN bisects $\angle TCB$.

18. How many tangents can be drawn to a circle containing a point outside the circle? Explain. What if the tangent(s) contained a point inside the circle? What if the tangent(s) contained a point on the circle?

Review (Answers)

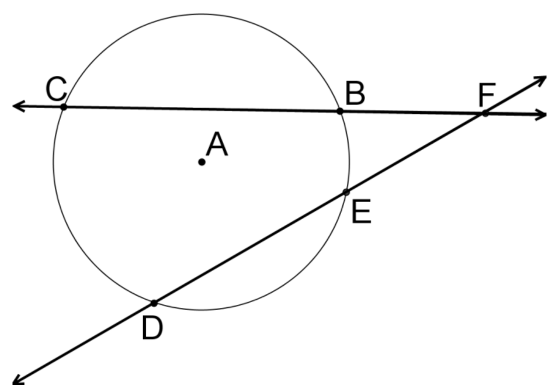
To see the Review answers, open this PDF file and look for section 8.7.

8.8 Secant Lines to Circles

Learning Objectives

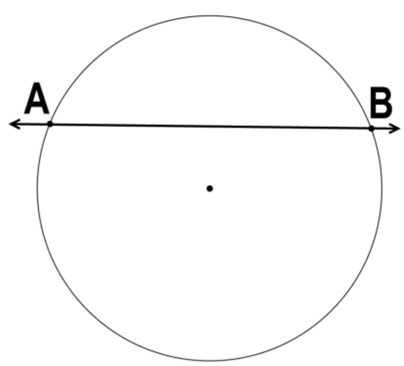
Here you will learn about secant lines to circles.

In the circle below, $m\widehat{CD} = 100^\circ$, $m\widehat{BC} = 120^\circ$, and $m\widehat{DE} = 100^\circ$. Find $m\angle BFE$.



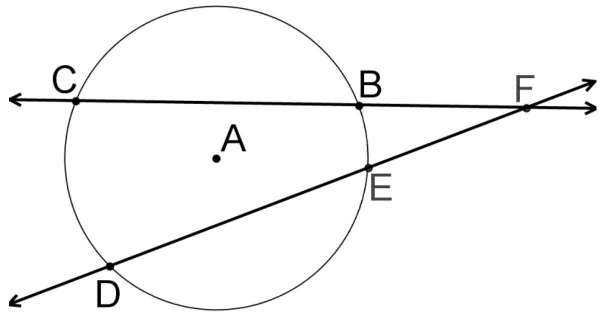
Secant Lines to Circles

Recall that a line that intersects a circle in exactly one point is called a **tangent line.** A line that intersects a circle in two points is called a **secant line.** Below, $\overleftrightarrow{AB}$ is a secant.



When two secants or a tangent and a secant are drawn, they can interact in four ways. In each case, arcs, angles and line segments have special relationships. These ideas are summarized below, and will be explored further and proved in the examples and practice.

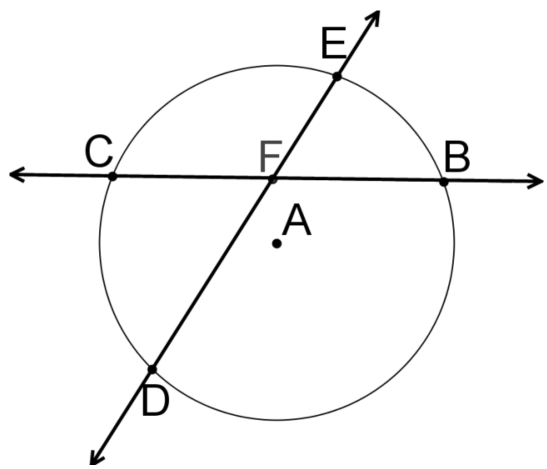
Case #1: Two secants intersect outside the circle.



Relevant Theorems:

- $BF \cdot CF = EF \cdot DF$ *(This will be explored in #1 below)*
- $\frac{m\widehat{CD} - m\widehat{BE}}{2} = m\angle BFE$ *(This will be explored in #2 below)*

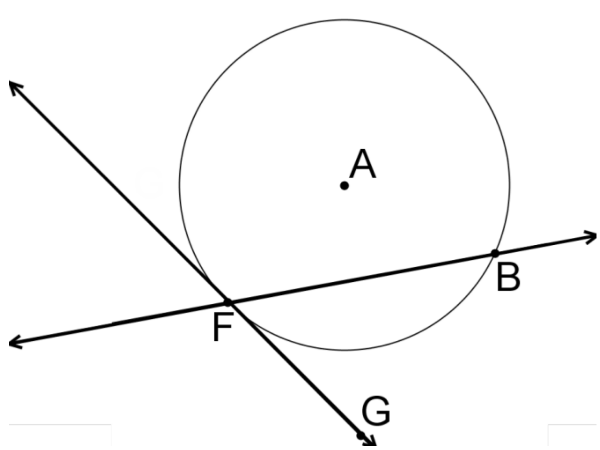
Case #2: Two secants intersect inside the circle.



Relevant Theorems:

- $CF \cdot FB = DF \cdot FE$ *(This was previously proved as a property of intersecting chords)*
- $\frac{m\widehat{CD} - m\widehat{BE}}{2} = m\angle BFE = m\angle CFD$ *(This will be explored in #3 below)*

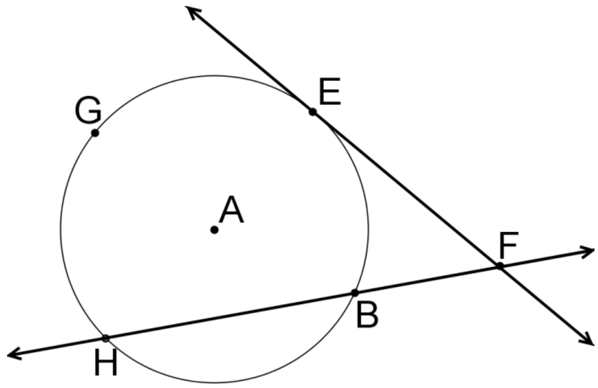
Case #3: A secant and a tangent intersect on the circle.



Relevant Theorem:

- $m\angle BFG = \frac{m\widehat{BF}}{2}$ *(This will be explored in Example 1)*

Case #4: A secant and a tangent intersect outside the circle.



Relevant Theorems:

- $FB \cdot FH = FE^2$ *(This will be explored in the Review problems)*
- $\frac{m\widehat{HGE} - m\widehat{BE}}{2} = m\angle BFE$ *(This will be explored in the Review problems)*

CK-12 PLIX Interactive

PLIX

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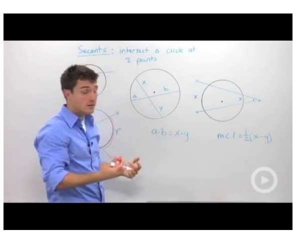

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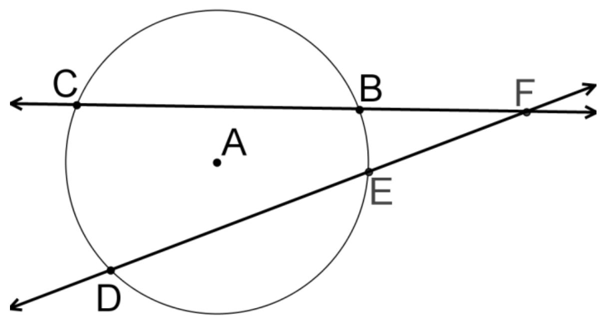
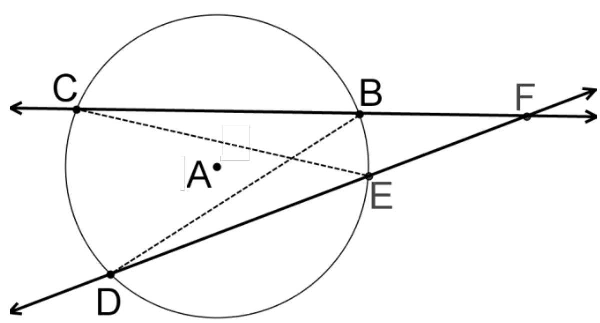
Let's take a look at some problems involving secant lines.

1. Prove that $BF \cdot CF = EF \cdot DF$.

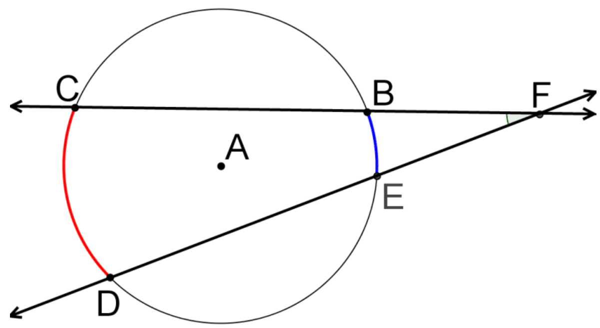


Draw chords $\overline{CE}$ and $\overline{DB}$.

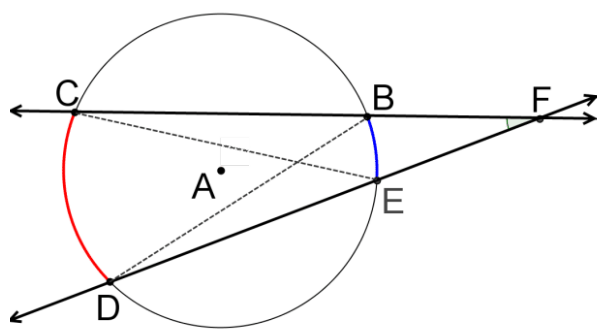


Two triangles are created, ΔCEF and ΔDBF. Note that both triangles share $\angle F$. Also note that both $\angle BCE$ and $\angle EDB$ are inscribed angles of $\widehat{BE}$. Therefore, $\angle BCE \cong \angle EDB$. Because ΔCEF and ΔDBF have two pairs of congruent angles, they are similar by $AA \sim$. This means that corresponding sides of the triangles are proportional. In particular, $\frac{BF}{EF} = \frac{DF}{CF}$. This means that $BF \cdot CF = EF \cdot DF$.

2. Prove that $\frac{m\widehat{CD} - m\widehat{BE}}{2} = m\angle BFE$.



You are trying to prove that the measure of the angle is equal to half the difference between the measures of the red arc and the blue arc. As in #1, draw chords $\overline{CE}$ and $\overline{DB}$.



Consider how the angles are arcs are related.

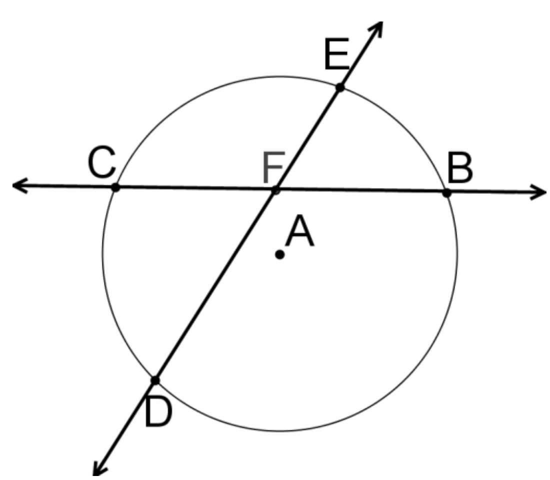
- $m\angle CED = \frac{m\widehat{CD}}{2}$ (inscribed angle)
- $m\angle ECF = \frac{m\widehat{BE}}{2}$ (inscribed angle)
- $m\angle CED = m\angle ECF + m\angle BFE$ (exterior angle equals the sum of the remote interior angles)

Make two substitutions and you have:

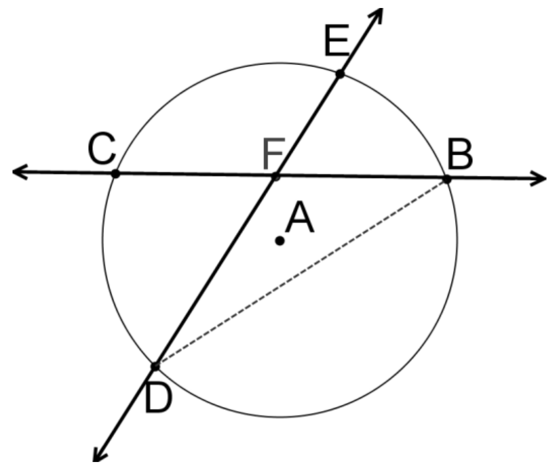
$\frac{m\widehat{CD}}{2} = \frac{m\widehat{BE}}{2} + m\angle BFE$

Therefore, $m\angle BFE = \frac{m\widehat{CD}}{2} - \frac{m\widehat{BE}}{2} = \frac{m\widehat{CD} - m\widehat{BE}}{2}$

3. Prove that $\frac{m\widehat{CD} + m\widehat{BE}}{2} = m\angle BFE = m\angle CFD$.



This logic of this proof is similar to the logic used in #2. Start by drawing chord $\overline{DB}$.



Consider how the angles and arcs are related.

- $m\angle CBD = \frac{m\widehat{CD}}{2}$ (inscribed angle)
- $m\angle BDE = \frac{m\widehat{BE}}{2}$ (inscribed angle)
- $m\angle CFD = m\angle CBD + m\angle\angle BDE$ (exterior angle equals the sum of the remote interior angles)

Make two substitutions and you have:

$m\angle CFD = \frac{m\widehat{CD}}{2} + \frac{m\widehat{BE}}{2}$

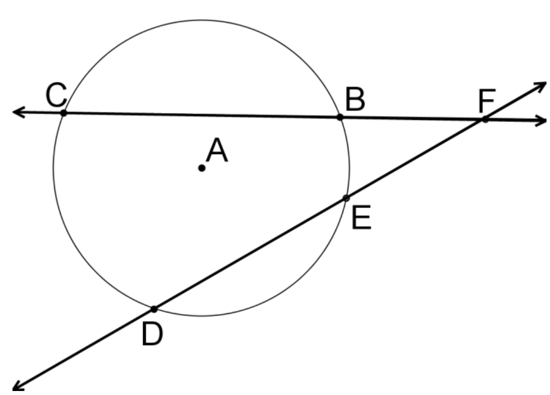
Therefore, $\frac{m\widehat{CD}+m\widehat{BE}}{2} = m\angle CFD$. Because $\angle CFD$ and $\angle BFE$ are vertical angles, they are congruent and have equal measures. This means $\frac{m\widehat{CD}+m\widehat{BE}}{2} = m\angle BFE = m\angle CFD$.

Examples

Example 1

Earlier, you were given a problem about a secant line to a circle.

In the circle below, $m\widehat{CD} = 100^\circ$, $m\widehat{BC} = 120^\circ$, and $m\widehat{DE} = 100^\circ$. Find $m\angle BFE$.



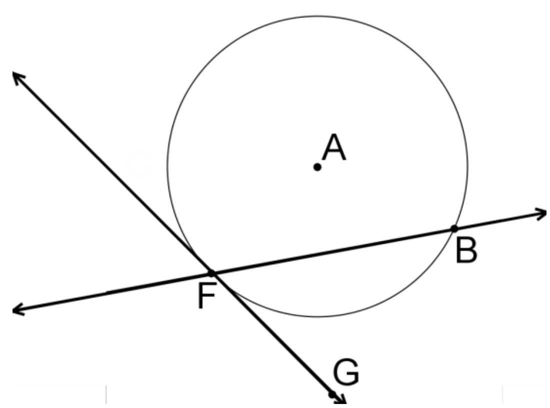
This is an example of two secants intersecting outside the circle. The intersection angle of the two secants is equal to half the difference between their intercepted arcs. In other words, $m\angle BFE = \frac{m\widehat{CD}-m\widehat{BE}}{2}$. You are given $m\widehat{CD} = 100^\circ$, but you don't know $m\widehat{BE}$. Use the fact that a full circle is 360° to find $m\widehat{BE}$.

$$m\widehat{BE} = 360^\circ - m\widehat{CD} - m\widehat{BC} - m\widehat{DE}$$
$$m\widehat{BE} = 360^\circ - 100^\circ - 120^\circ - 100^\circ$$
$$m\widehat{BE} = 40^\circ$$

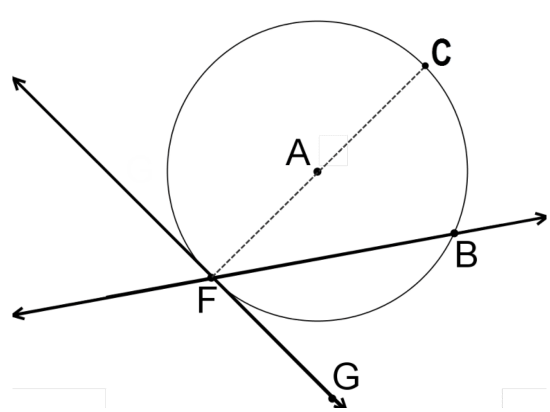
Now, solve for the measure of $\angle BFE$.

Example 2

$\overleftrightarrow{FG}$ is tangent to circle A at point F. Prove that $m\angle BFG = \frac{m\widehat{BF}}{2}$.



Draw a diameter through points F and A. This segment will be perpendicular to $\overleftrightarrow{FG}$.



First note that $m\widehat{CB} + m\widehat{BF} = 180^\circ$ because the two arcs make a semicircle. This means that $\frac{m\widehat{CB}}{2} + \frac{m\widehat{BF}}{2} = 90^\circ$ and thus $\frac{m\widehat{BF}}{2} = 90^\circ - \frac{m\widehat{CB}}{2}$.

Now consider other angle and arc relationships:

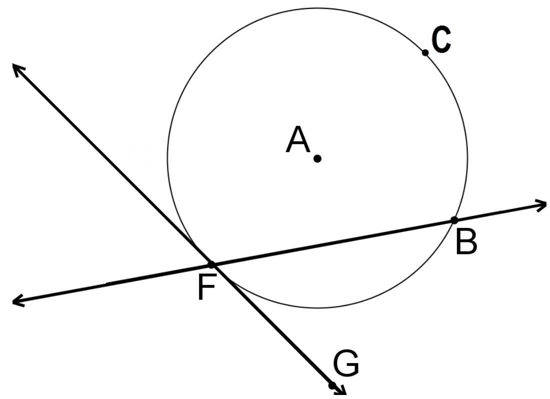
- $m\angle CFB = \frac{m\widehat{CB}}{2}$ (inscribed angle)
- $m\angle CFB + m\angle BFG = 90^\circ$ (two angles make a right angle)

By substitution, $\frac{m\widehat{CB}}{2} + m\angle BFG = 90^\circ$. Therefore, $m\angle BFG = 90^\circ - \frac{m\widehat{CB}}{2}$.

Consider the two highlighted statements. Both $\frac{m\widehat{BF}}{2}$ and $m\angle BFG$ are equal to $90^\circ - \frac{m\widehat{CB}}{2}$. Therefore, $m\angle BFG = \frac{m\widehat{BF}}{2}$.

Example 3

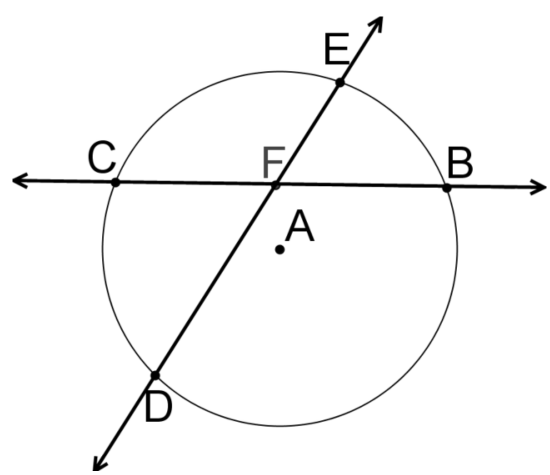
$m\widehat{FCB} = 280^\circ$. Find $m\angle BFG$.



If $m\widehat{FCB} = 280^\circ$, then $m\widehat{FB} = 360^\circ - 280^\circ = 80^\circ$. Therefore, $m\angle BFG = \frac{80^\circ}{2} = 40^\circ$.

Example 4

$m\widehat{CD} = 70^\circ$ and $m\widehat{BE} = 40^\circ$. Find $m\angle CFE$.



$m\widehat{CD} = 70^\circ$ and $m\widehat{BE} = 40^\circ$. $m\angle CFD$ is the average of the measure of the intercepted arcs.

$$m\angle CFD = \frac{70^\circ + 40^\circ}{2} = 55^\circ$$

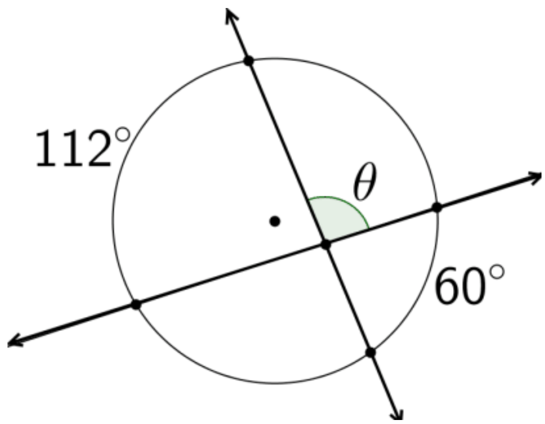
Therefore, $m\angle CFE = 180^\circ - 55^\circ = 125^\circ$.

Review

1. What's the difference between a secant and a tangent?

Use the relationships explored in this concept to solve for x or θ in each circle.

2.



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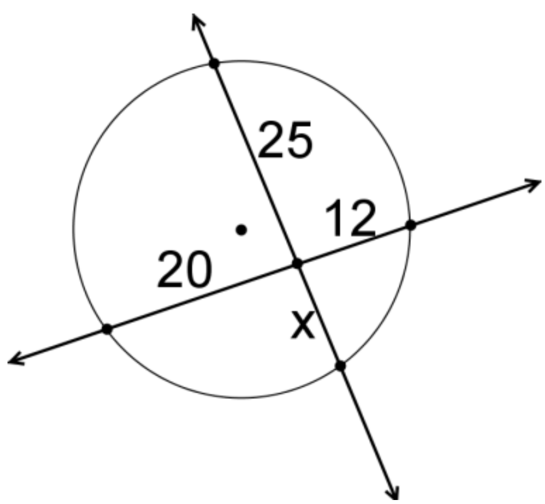

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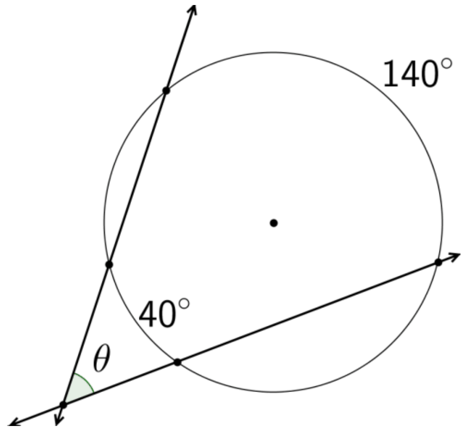

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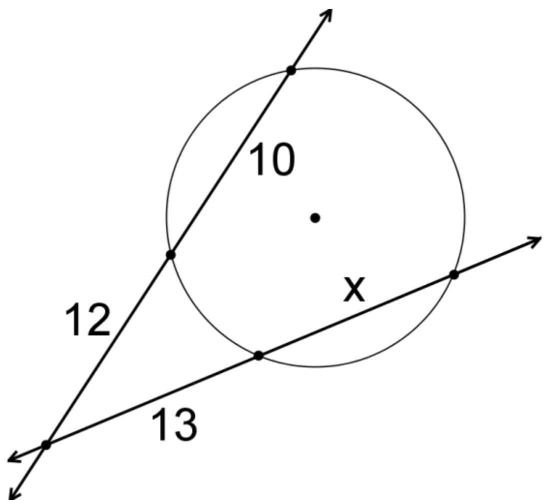

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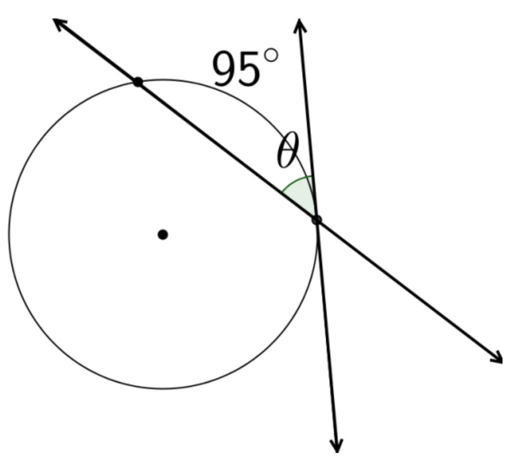

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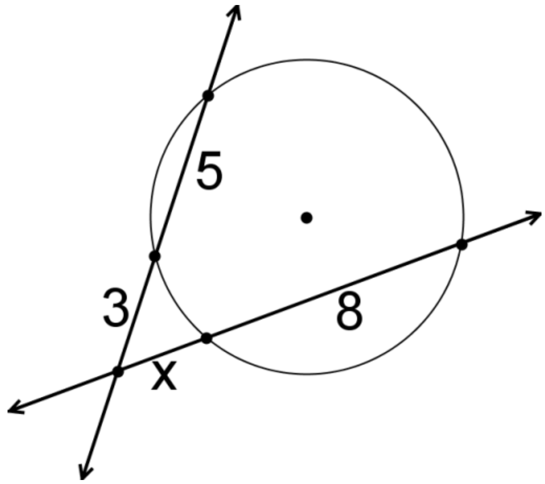
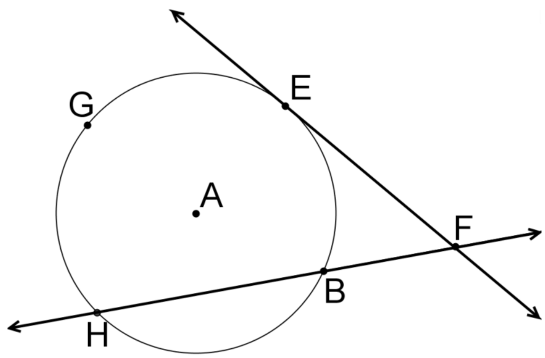


In #8-#12 you will explore *Case #4: A secant and a tangent intersect outside the circle.*



8. Draw chord $\overline{BE}$. Explain why $\angle FEB \cong \angle EHB$.

9. Prove that $\Delta EHF \sim \Delta BEF$.

10. Prove that $FB \cdot FH = FE^2$.

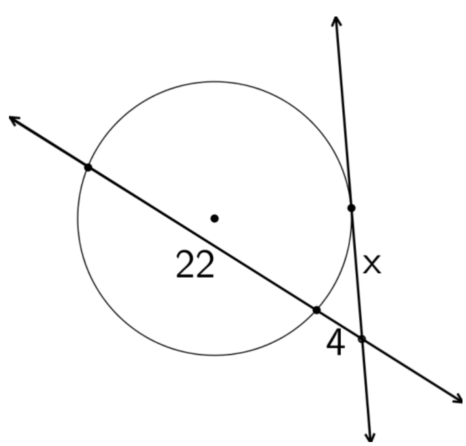
11. Prove that $\frac{m\widehat{HGE}}{2} = \frac{m\widehat{BE}}{2} + m\angle BFE$ (Use Example B to help).

12. Prove that $\frac{m\widehat{HGE} - m\widehat{BE}}{2} = m\angle BFE$.

13. How is the theorem proved in #11-#12 related to the theorem proved in Examples B?

Solve for x or θ in each circle.

14.



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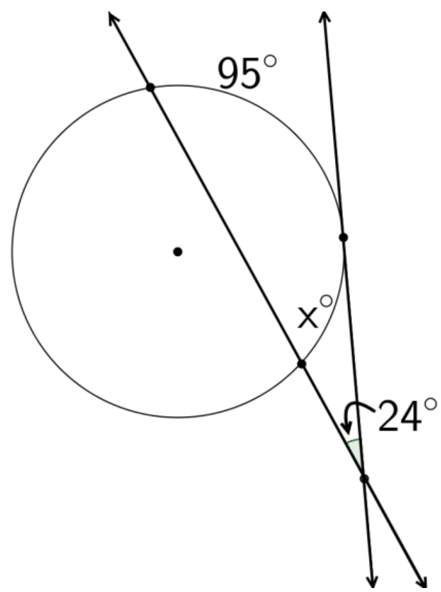


16. Rainbows are created when droplets of water bend (or refract) the sunlight as it passes through them. The different angles of refraction give different wavelengths of light to make the rainbow. Look at the figure below:

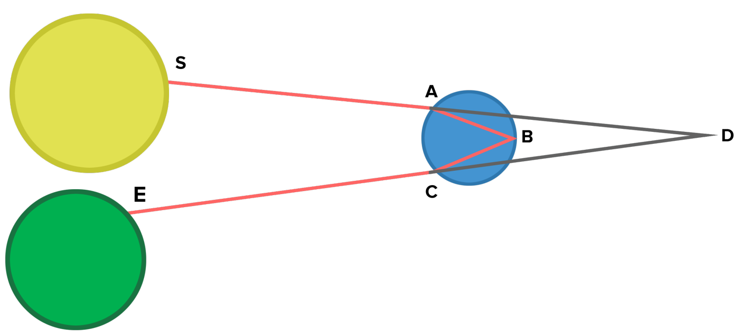


FIGURE 8.2

Light from the sun leaves point S and travels through the air and reaches a drop of water at point A, where it bends. The light goes to the back of the drop of water (point B) and it is reflected to then move out of the drop of water at C. From here the light heads to earth (point E). In the diagram point D shows how much this ray of light has deviated from its original path. Explain how you could calculate the angle at point D.

17. In the famous movies attributed to Star Trek, there is a specific logo considered to be the Starfleet Insignia. Research this logo and use your knowledge of circles and secants to describe it in words. Can you find another logo that uses the properties of secants?

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.8.

8.9 Arc Length

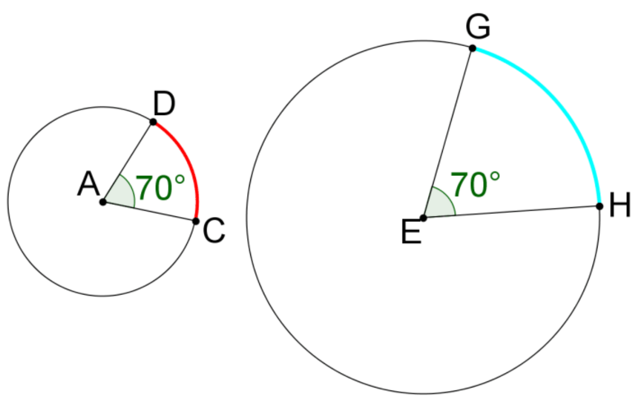
Learning Objectives

Here you will learn how to find the length of an arc and how to measure angles in radians.

A radian is another unit of measurement for angles just like degrees. Radians are especially convenient for measuring angles that have to do with circles. There are 360° in a circle and 90° in a right angle. How many radians are in a circle and how many radians make up a right angle?

Arc Length

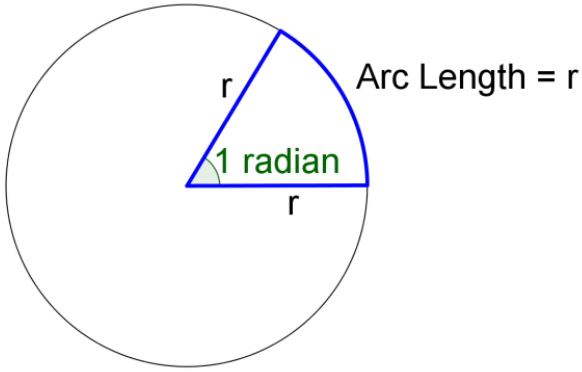
Recall that a portion of a circle is called an **arc**. One way to measure an arc is with degrees. The measure of an arc is equal to the measure of its corresponding central angle. Below, $m\widehat{DC} = 70^\circ$ and $m\widehat{GH} = 70^\circ$.



When you measure an arc in degrees, it tells you the **relative size** of the arc compared to the whole circle. It does not tell you anything about the **absolute size** of the arc or the circle it came from. Both arcs above have the same measure, but $\widehat{GH}$ is physically *longer*, due to circle E being bigger.

This leads to another way of describing the size of an arc. **Arc length** measures the distance (in units such as inches or centimeters) along a circle between the endpoints that define the arc. Above, $\widehat{GH}$ has a greater arc length than $\widehat{DC}$. Because the radius of a circle is what determines the circle's size, **the length of an arc intercepted by a given angle will be directly proportional to the radius of the circle.** *You will derive this fact in Examples A and B.*

Arc length suggests a new way of measuring angles in circles. Previously, you have measured all angles in degrees, but you can also measure angles in radians. **1 radian is the angle that creates an arc that has a length equal to the radius**. Below, the arc has a length equal to the radius. The angle that is created is 1 radian.



You can use the formula for the circumference of a circle to show that there are 2π radians in a circle (You will justify this in the Examples). This means that:

$$2\pi \ radians = 360^\circ$$

Therefore, you have the following conversions:

$$1 \ radian = \left(\frac{180}{\pi}\right) degrees \ (\approx 57.3^\circ)$$
$$1 \ degree = \left(\frac{\pi}{180}\right) radians \ (\approx 0.17 \ radians)$$

Note that if a given angle has no units, it is assumed to be in radians.

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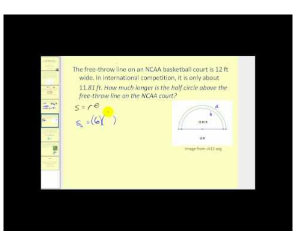

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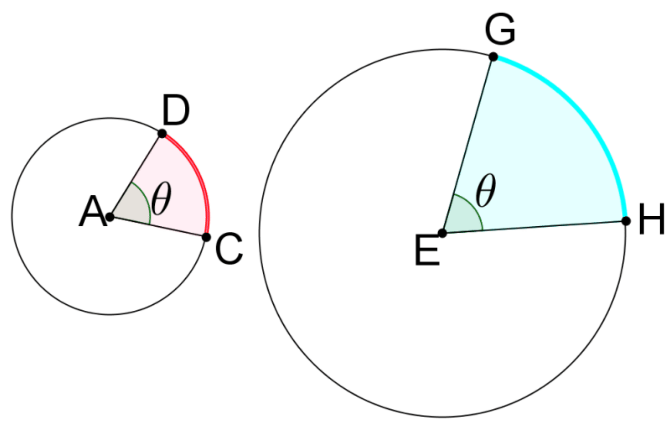
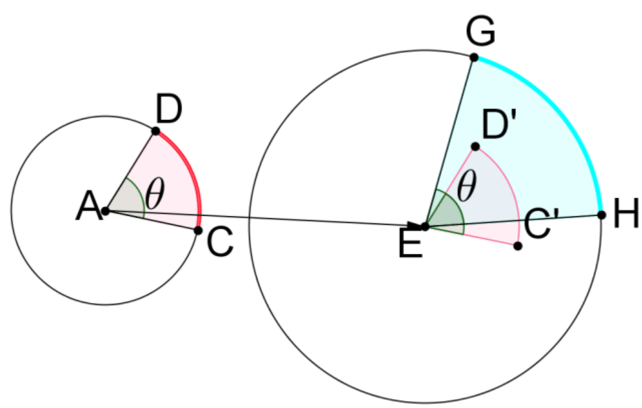
A **sector** is a portion of a filled circle bounded by two radii and an arc (a “wedge” of a circle).

Let's do a problem involving sector similarity.

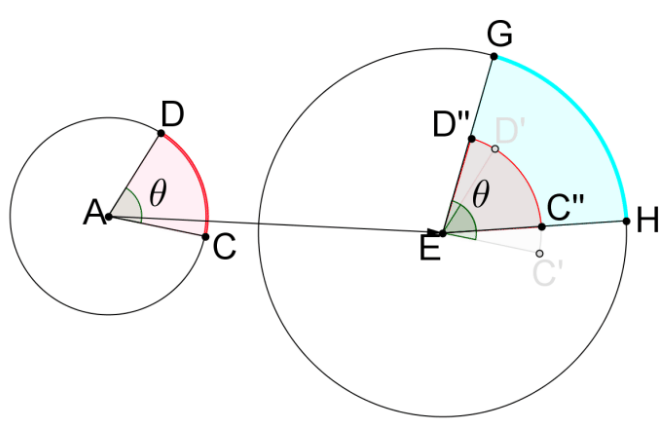
Show that two sectors with the same central angle are similar by using transformations to explain why the two shaded sectors below are similar.



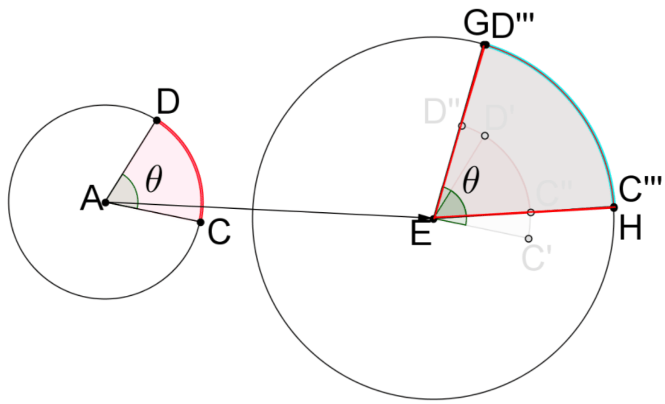
Two shapes are similar if a similarity transformation exists between them. Draw a vector from point A to point E. Translate the red sector along the vector.



Rotate the image so that $\overline{EC'}$ lies on $\overline{EH}$. Because angles are preserved with translations and rotations, the image of $\overline{ED'}$ will lie on $\overline{EG}$.



Dilate the image about point E by a factor of $\frac{EH}{AC}$.

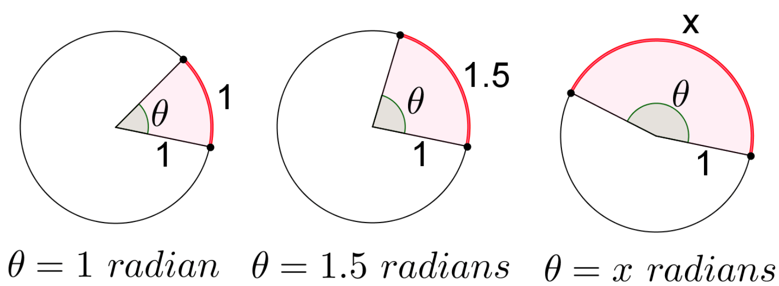


The two sectors are similar because a sequence of rigid transformations followed by a dilation carried one sector to the other.

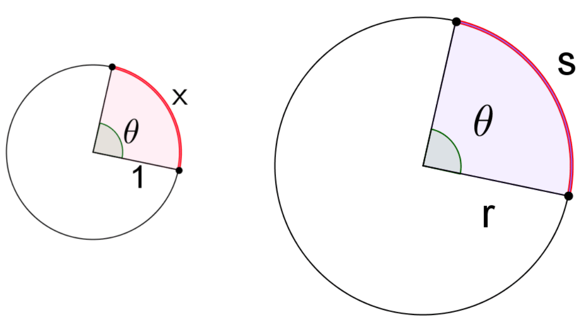
Next, let's look at a problem involving arc length.

Explain why the length of arc s is equal to θr, where θ is the central angle in radians and r is the radius of the circle.

In a circle of radius 1, the measure of a central angle in radians will be equal to the length of the intercepted arc. This is because the number of radians equals the number of radii that make up the arc.



If a sector has radius r, it is similar to a sector of radius 1 with the same central angle (as shown in the previous problem about sectors).



Because the sectors are similar, corresponding lengths are proportional:

$$\frac{r}{1} = \frac{s}{x}$$
$$s = xr$$

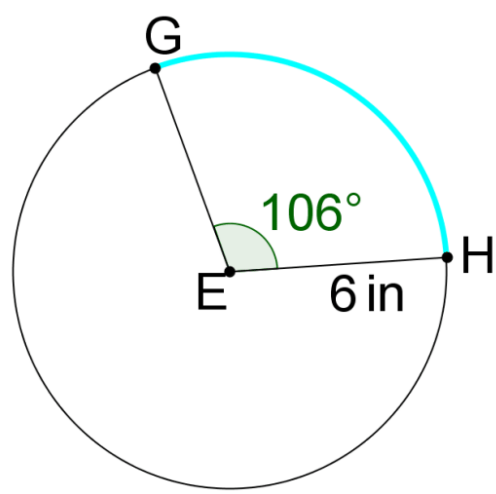
x must be equal to the measure of θ in radians. Therefore:

$$s = \theta r$$

Why does this make sense? Remember that if θ is in radians, then θ is equal to the number of radii that fit around the arc. The number of radii that fit around the arc multiplied by the length of the radius will equal the length of the arc.

Finally, lets look at a problem where we measure arc length

Find $m\widehat{GH}$ and the length of $\widehat{GH}$.



$m\widehat{GH} = 106°$. To find the length of the arc, multiply the radius (6 in) by the measure of the central angle in radians.

Remember that 1 $degree = \left(\frac{\pi}{180}\right)$ $radians$. This means that $106° = 106 \cdot \left(\frac{\pi}{180}\right) \approx 1.85$ $radians$.

Now you can find the length of the arc:

Don't forget that your angle must be in radians in order to use the formula $s = \theta r$!

Examples

Example 1

Earlier, you were asked about radians in cir

There are 360° in a circle and 90° in a right angle. How many radians are in a circle and how many radians make up a right angle?

There are 2π radians in a circle. A right angle is 90° and 90° is $\frac{1}{4}$ of a circle, so there are $\frac{2\pi}{4} = \frac{\pi}{2}$ radians in a right angle.

Example 2

Explain why a circle has 2π *radians*.

The circumference of a circle with radius 1 is $2\pi(1) = 2\pi$. Therefore, 2π radii fit around a circle with radius 1. All circles are similar, so 2π radii must fit around all circles. 1 radian is the angle that creates an arc that has a length equal to the radius, so 2π radians is the angle that creates an arc with a length equal to 2π radii. Therefore, a circle is 2π radians because 2π radii fit around any circle.

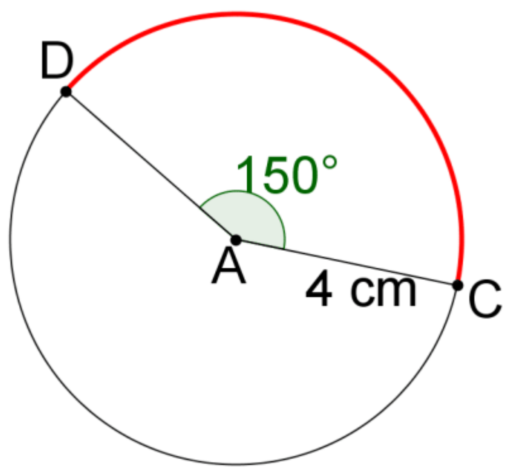
Example 3

How many radians are there in 150°?

Remember that $1\ degree = \left(\frac{\pi}{180}\right)\ radians$. This means that $150^\circ = 150 \cdot \left(\frac{\pi}{180}\right) \approx 2.62\ radians$.

Example 4

Find the length of $\widehat{CD}$.



$s = \theta r$ and $\theta = 150^\circ = 2.62\ radians$. The length of $\widehat{CD}$ is $s = (2.62)(4) \approx 10.48\ cm$.

CK-12 PLIX Interactive

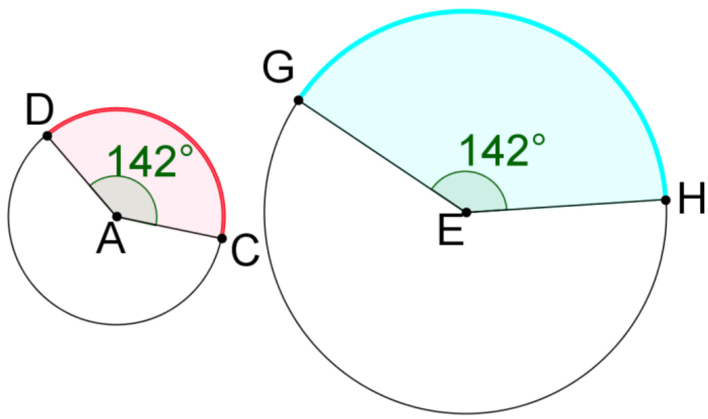
PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/trigonometry/length-of-an-arc/plix/Pi-Hour-57223bd2da2cfe5bd8087af3

Review

1. What's the difference between finding the **measure** of an arc and the **length** of an arc?
2. What is the relationship between a radian and a radius?
3. How are radians and degrees related?
4. Explain why the two shaded sectors below are similar.



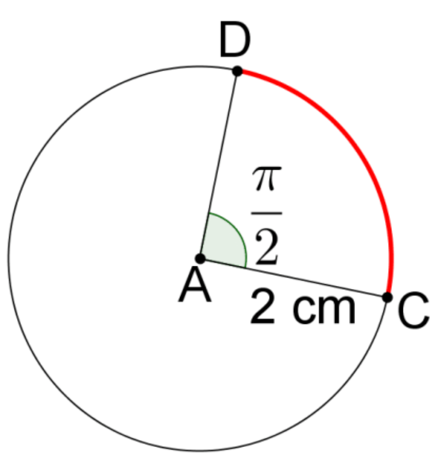
5. Justify why the length of arc s is equal to θr, where θ is the central angle in radians and r is the radius of the circle.

Convert each angle measured in degrees to an angle measured in radians. Leave answers in terms of π.

6. 180°
7. 360°
8. 90°
9. 60°
10. 30°

Find the **measure in degrees** and **length in centimeters** of $\widehat{CD}$ in each circle.

11.



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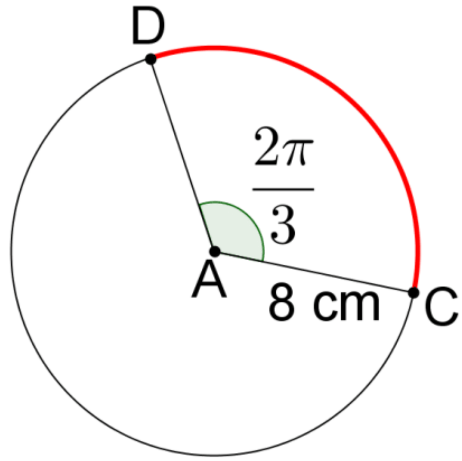

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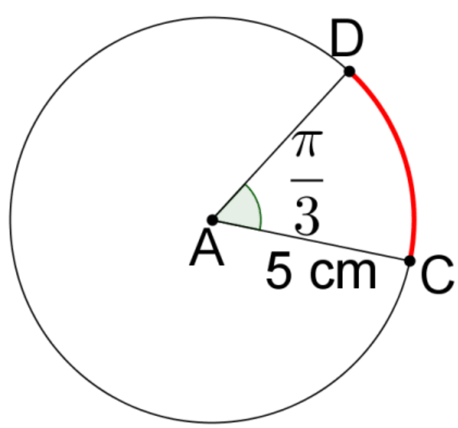

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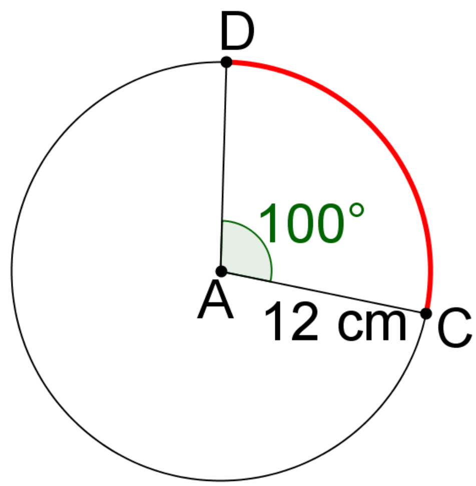


15. Explain how to find the length of an arc when given the central angle in radians. How does this compare to the process of finding the length of an arc when given the central angle in degrees?

16. Sheldon's grandmother made him and six friends a strawberry-rhubarb pie. He cut a piece that had a crust length of 3 inches. How big of a pie did his grandmother make if each friend had same size piece of pie? Explain your answer.

17. A stop sign is in the shape of a regular octagon. If it was inscribed in a circle, how could you determine the measure for each arc? Explain your reasoning. How would your determination change if you had a school zone sign which is in the shape of a regular pentagon?

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.9.

8.10 Sector Area

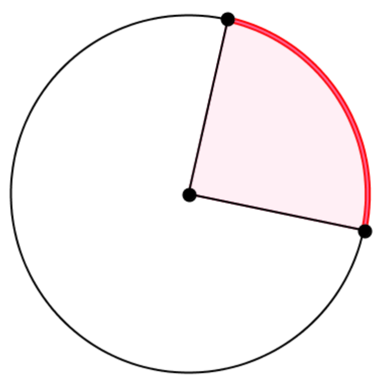
Learning Objectives

Here you will learn how to find the area of a sector of a circle.

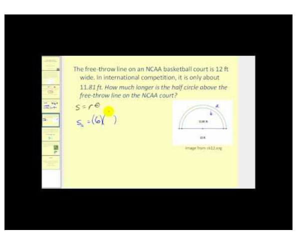
You make a $10''$ diameter cheesecake for a party and coat the top in a layer of chocolate. You divide the cheesecake up into equal slices so that the central angle of each slice is 15°. How many square inches of chocolate are on the top of each slice?

Sector Area

A **sector** is a portion of a filled circle bounded by two radii and an arc. A sector is like a "wedge" of a circle. Below, the portion of the circle shaded red is a sector.



Because a sector is two dimensional, you can calculate its area. The area of a whole circle with radius r is πr^2. The area of a sector represents a fraction of this whole circle area. The measure of the central angle helps to tell you what fraction of the circle the sector is. In the problems below, you will show that $\textit{Sector Area} = \frac{r^2\theta}{2}$, where r is the radius of the circle and θ is the measure of the central angle in radians.



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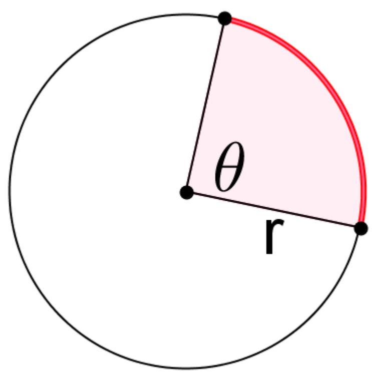
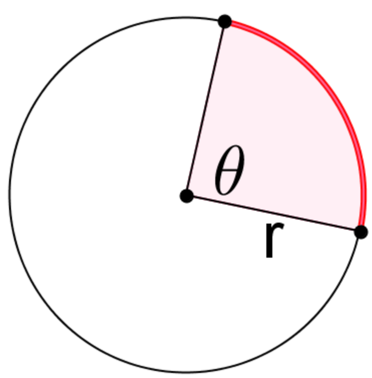
Let's take a look at some problems about sector area.

1. In terms of θ, what fraction of the circle is the red sector? Assume θ is in radians.



Thinking in radians, a full circle is 2π radians. The sector is therefore $\frac{\theta}{2\pi}$ of the whole circle.

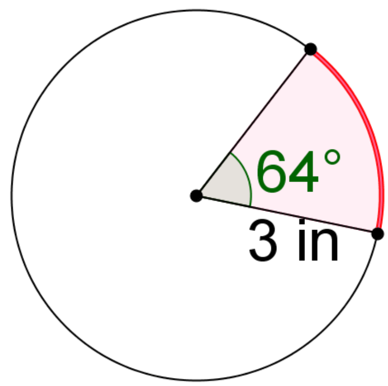
2. Explain why the area of the sector below is $\frac{r^2\theta}{2}$ where θ is the measure of the central angle in radians.



The area of the whole circle is πr^2 and this sector represents $\frac{\theta}{2\pi}$ of the whole circle. Therefore, the area of the sector is:

$$\pi r^2 \cdot \frac{\theta}{2\pi} = \frac{r^2\theta}{2}$$

3. Find the area of the red sector below.



The angle is given in degrees, so first convert it to radians. Remember that $1\ degree = \left(\frac{\pi}{180}\right) radians$. This means that $64^\circ = 64 \cdot \left(\frac{\pi}{180}\right) \approx 1.12\ radians$

$$Sector\ Area = \frac{r^2\theta}{2} \approx \frac{(3^2)(1.12)}{2} \approx 5.03\ in^2$$

Examples

Example 1

Earlier, you were given a problem about dividing a cheesecake.

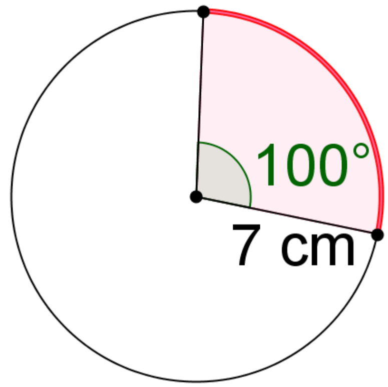
You make a $10''$ diameter cheesecake for a party and coat the top in a layer of chocolate. You divide the cheesecake up into equal slices so that the central angle of each slice is 15°. How many square inches of chocolate are on each slice?

The top of each slice covered in chocolate is a sector with radius 5 inches and a central angle of 15°. To find the area of the sector, first convert 15° to radians. Remember that $1\ degree = \left(\frac{\pi}{180}\right) radians$. This means that $15^\circ = 15 \cdot \left(\frac{\pi}{180}\right) \approx 0.26\ radians$.

$$Chocolate\ Area = \frac{r^2\theta}{2} \approx \frac{(5^2)(0.26)}{2} \approx 3.27\ in^2$$

Example 2

Find the area of the sector below:

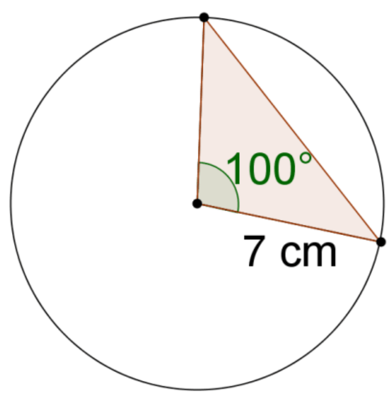


To find the area of the sector, first convert 100° to radians. Remember that 1 $degree = \left(\frac{\pi}{180}\right)$ $radians$. This means that $100^\circ = 100 \cdot \left(\frac{\pi}{180}\right) \approx 1.75$ $radians$.

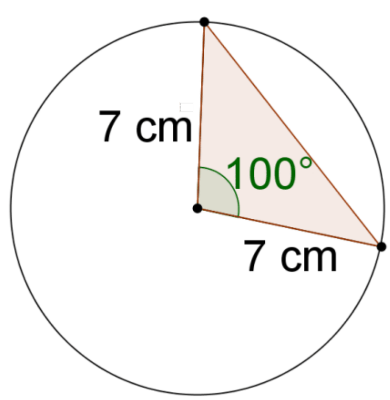
$$Sector\ Area = \frac{r^2\theta}{2} \approx \frac{(7)^2(1.75)}{2} \approx 42.76\ cm^2$$

Example 3

Find the area of the triangle below:



Because the radius of the circle is 7 cm, two sides of the triangle are length 7 cm.

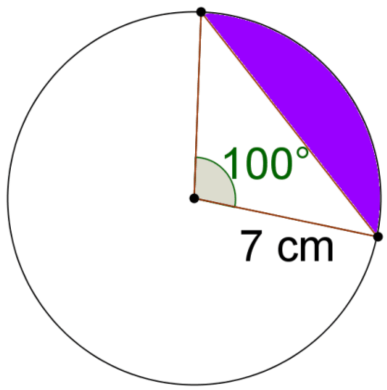


You know two sides and an included angle so you can find the area using the sine area formula.

$$\begin{aligned}
Triangle\ Area &= \frac{1}{2}(a)(b)(\sin C) \\
Triangle\ Area &= \frac{1}{2}(7)(7)\sin 100^\circ \\
Triangle\ Area &\approx 24.12\ cm^2
\end{aligned}$$

Example 4

Use your answers to #2 and #3 to find the area of the circular segment below (shaded in purple):



The circular segment is the portion of the sector not included in the triangle. To find its area, subtract the area of the triangle from the area of the sector.

$$Segment\ Area = 42.76\ cm^2 - 24.12\ cm^2 = 18.64\ cm^2$$

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/area-of-sectors-and-segments/plix/Circular-Motion-and-Dimensional-Analysis-Center-Pivot-Irrigation-55d267ceda2cfe560cb41c82

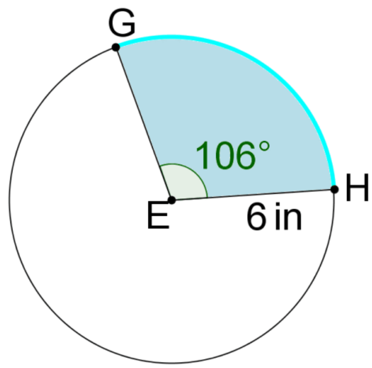
Review

1. Explain why the area of a sector is $\frac{r^2\theta}{2}$ where θ is the measure of the central angle in radians.

2. How do you find the area of a sector if the central angle is given in degrees?

Find the area of each region shaded in blue.

3.



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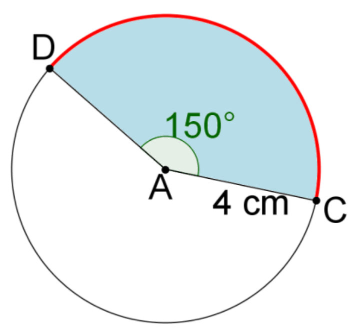

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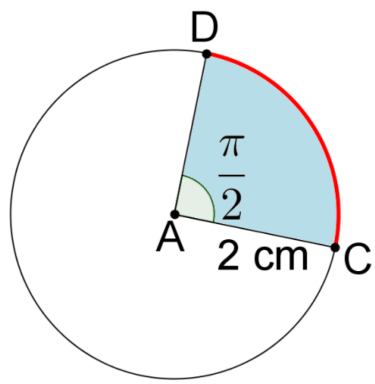

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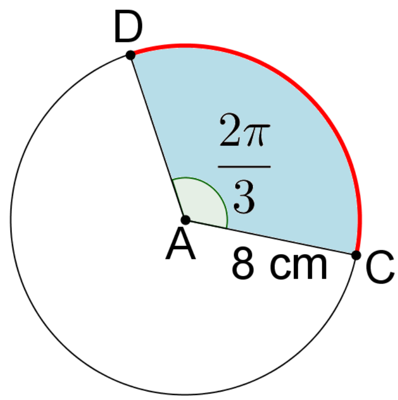

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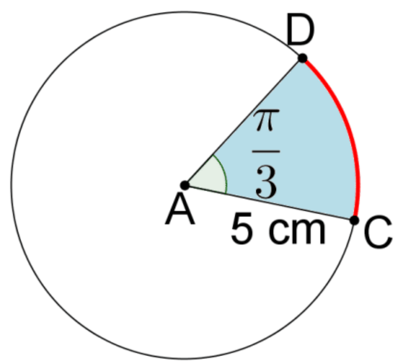

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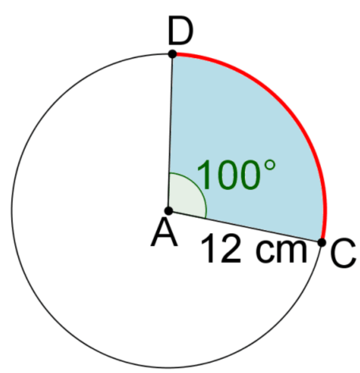

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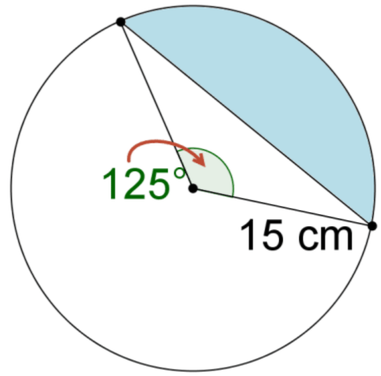

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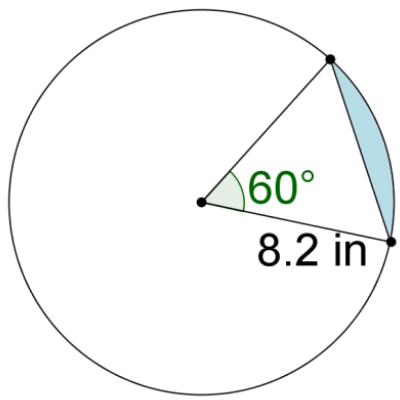

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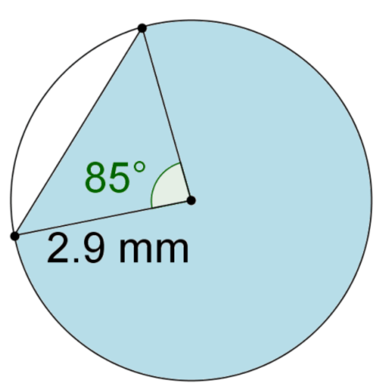

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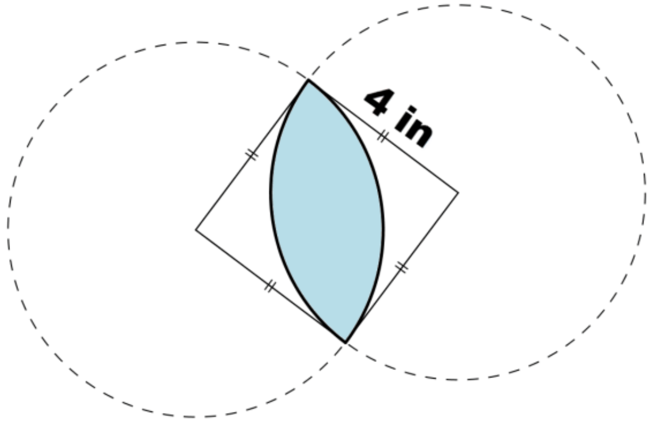

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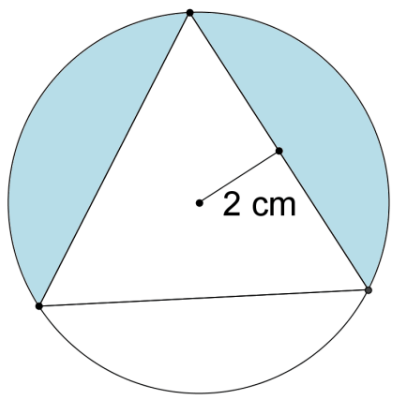

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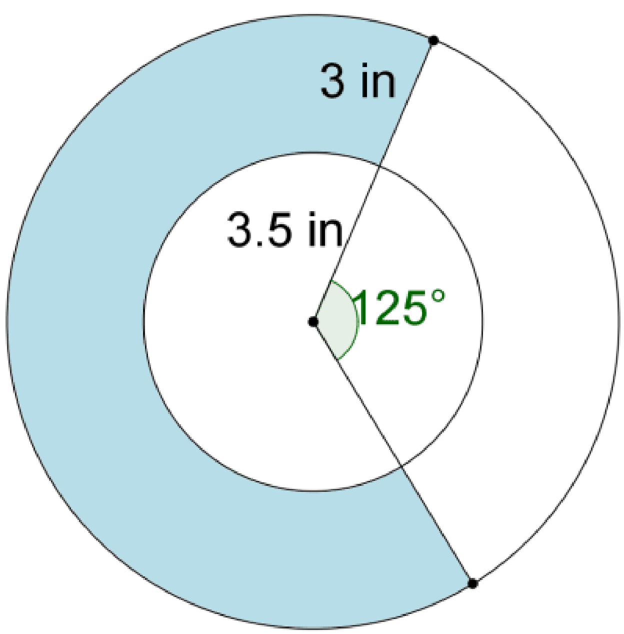

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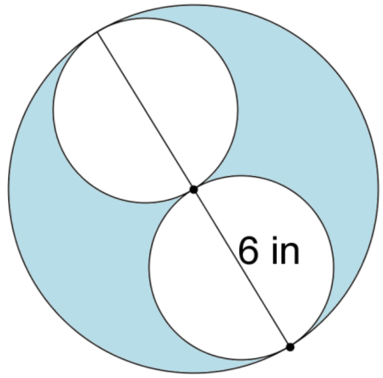

16. A lawn sprinkler is set in the corner of the backyard. Thomas has the sprinkler set to rotate at 90 degrees. He knows that the sprinkler will spray 30 feet. If his backyard is 850 square feet, does Thomas have the sprinkler set to reach most of his yard? Explain your reasoning and any suggestions you might have for Thomas if you find he does not have the sprinkler set properly.

17. A windshield wiper usually rotates between 80 and 95 degrees if your car uses only one wiper. You are deciding on switching to a more sporty 7 inch wiper from your current 10 inch wiper. Is this a good idea? Explain your reasoning.

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.10.

8.11 References

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CHAPTER 9 Three Dimensions

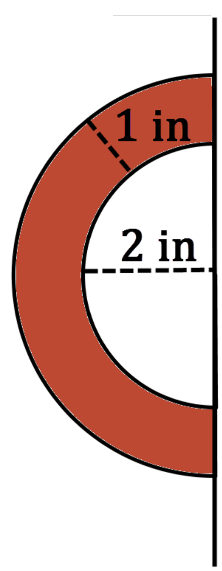
Chapter Outline

9.1 Connections Between Two and Three Dimensions

Learning Objectives

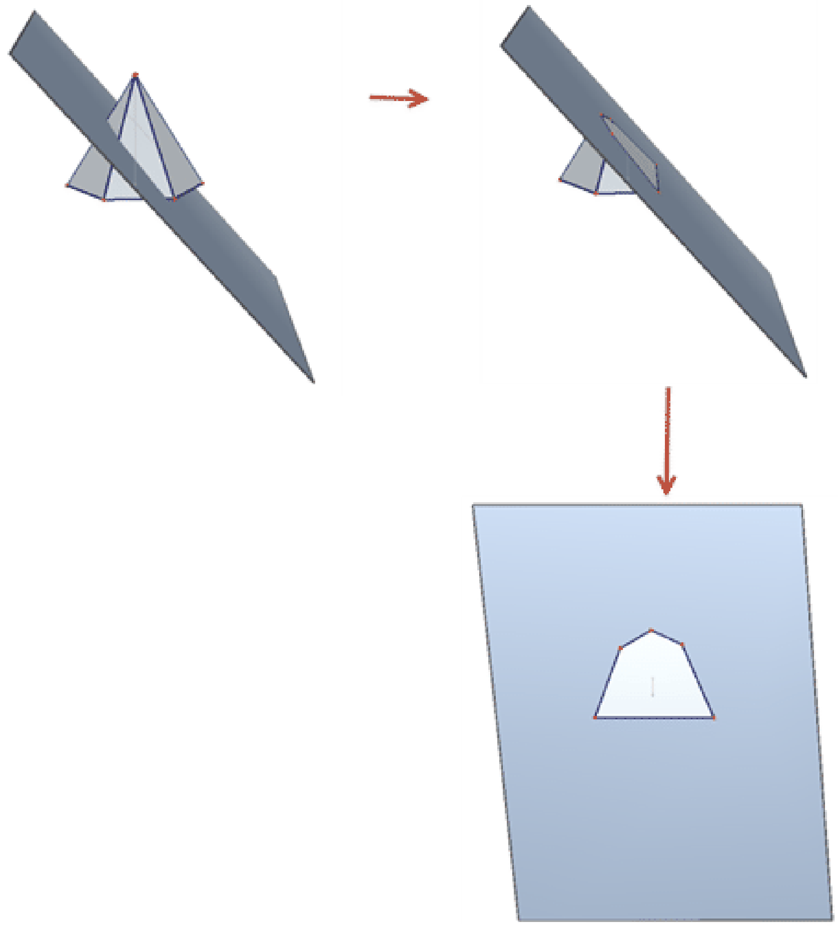
Here you will review cross sections of three dimensional objects. You will also practice identifying three dimensional objects generated by rotations of two dimensional shapes.

The shaded figure below is rotated around the line. What is the volume of the solid that is created?



Cross Sections

Recall that a **cross section** is the shape you see when you make one slice through a solid. A solid can have many different cross sections depending on where you make the slice. Consider a hexagonal pyramid. Cross sections *perpendicular* to the base will be triangles. Cross sections *parallel* to the base will be hexagons. It is also possible to take cross sections using planes that are neither parallel nor perpendicular to the base. Below, a hexagonal pyramid has been sliced at a slant. The cross section is a pentagon.



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Cross sections are one way that two dimensional objects are connected to three dimensional objects.

A second connection between two and three dimensions comes from the fact that three dimensional solids can be created by rotating two dimensional objects around a line.

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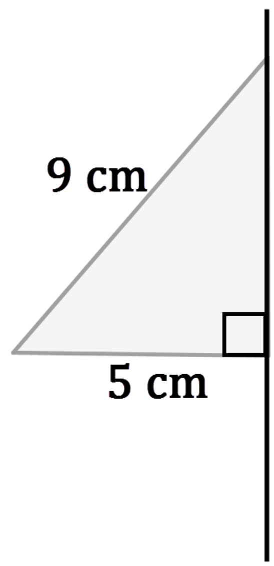
Let's look at some problems about cross sections.

1. Identify the solid that is created when the following shape is rotated around the line.



The solid is a cone with radius 5 cm.

2. Find the volume of the solid from #1.

The volume of a cone is $V = \frac{\pi r^2 h}{3}$. The radius is 5 cm. To find the height of the cone, use the Pythagorean Theorem:

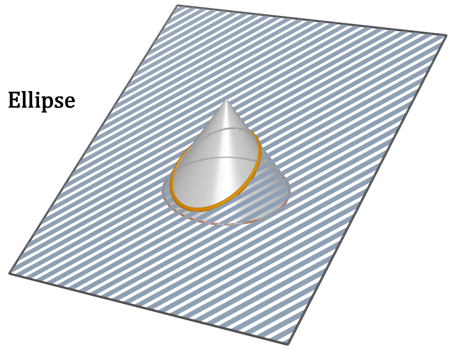
$$5^2 + h^2 = 9^2$$
$$h = \sqrt{56} \approx 7.48\ cm$$

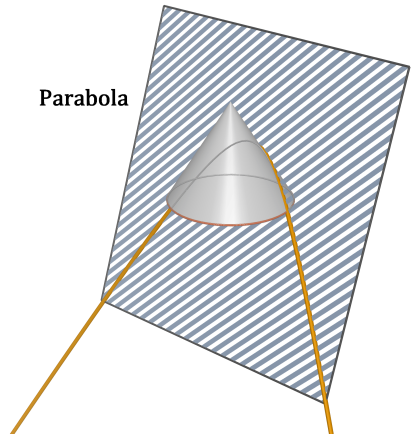
The volume of the cone is:

$$V = \frac{\pi(5^2)(7.48)}{3} \approx 195.83\ cm^3$$

3. Identify at least three two dimensional shapes created by cross sections of the solid from first problem.

Cross sections taken parallel to the base will be circles. Cross sections taken perpendicular to the base will be triangles. When the cross section is taken at a slant, there are many other possibilities. Two additional cross sections are an ellipse or a filled in parabola.



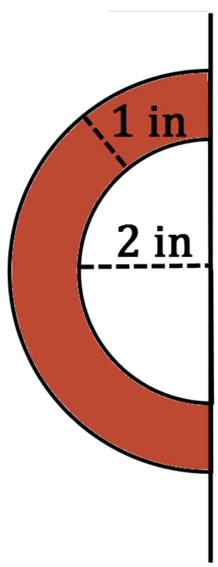


Examples

Example 1

Earlier, you were asked what is the volume of the solid that was created.

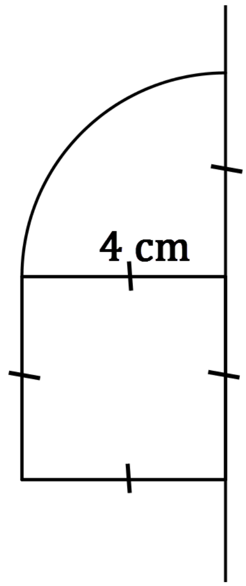
The shaded figure below is rotated around the line.



The resulting solid is a sphere with a sphere removed from the center. The volume of the large sphere is $\frac{4\pi(3^3)}{3} = 36\pi\ in^3$. The volume of the small sphere is $\frac{4\pi(2^3)}{3} = \frac{32\pi}{3}\ in^3$. The volume of the resulting solid is:

$$V = 36\pi - \frac{32\pi}{3} = \frac{76\pi}{3}\ in^3$$

Use the picture below for #2-#4.



Example 2

Identify the solid that is created when the figure above is rotated around the line.

The solid is a cylinder with a hemisphere on top. The radius of each is 4 cm.

Example 3

Find the volume of the solid from #2.

The volume of the cylinder is $\pi(4^2)(4) = 64\pi \ cm^3$. The volume of the hemisphere is $\frac{4\pi(4^3)}{6} = \frac{128\pi}{3} \ cm^3$. The volume of the whole solid is $\frac{320\pi}{3} \ cm^3$.

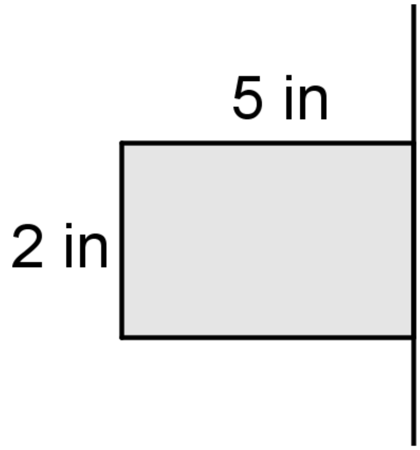
Example 4

Identify at least three two dimensional shapes created by cross sections of the solid from #2.

Possible answers: Cross sections taken parallel to the base will be circles. Cross sections taken perpendicular to the base will be rectangles with half circles on top. Some cross sections taken at a slant will be ellipses.

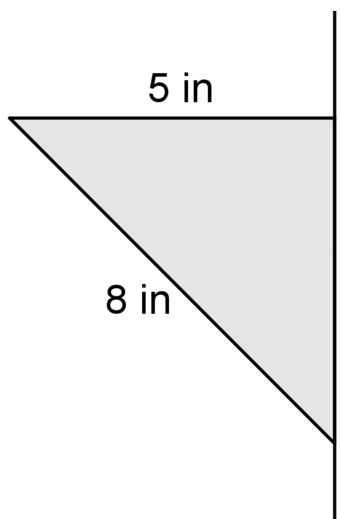
Review

Use the picture below for #1-#3.



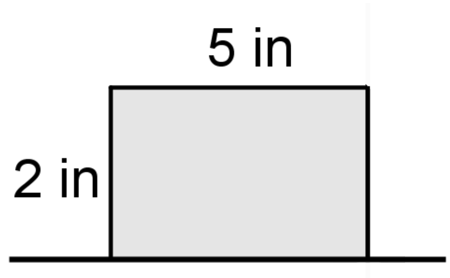
1. Describe the solid that is created when the figure above is rotated around the line.
2. Find the volume of the solid.
3. Identify at least 3 two dimensional shapes created by cross sections of the solid.

Use the picture below for #4-#6.



4. Describe the solid that is created when the figure above is rotated around the line.
5. Find the volume of the solid.
6. Identify at least 3 two dimensional shapes created by cross sections of the solid.

Use the picture below for #7-#9.

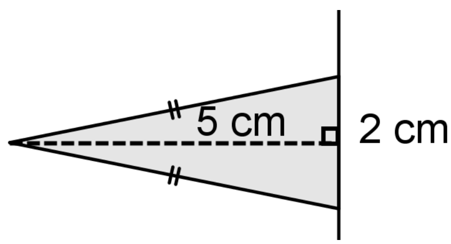


7. Describe the solid that is created when the figure above is rotated around the line.

8. Find the volume of the solid.

9. Identify at least 2 two dimensional shapes created by cross sections of the solid.

Use the picture below for #10-#12.

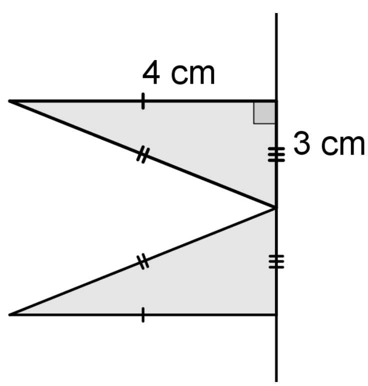


10. Describe the solid that is created when the figure above is rotated around the line.

11. Find the volume of the solid.

12. Identify at least 2 two dimensional shapes created by cross sections of the solid.

Use the picture below for #13-#15.



13. Describe the solid that is created when the figure above is rotated around the line.

14. Find the volume of the solid.

15. Identify at least 2 two dimensional shapes created by cross sections of the solid.

16. Below are cross-sections of various solids. Find the area of each cross-section. (On the isometric grid paper shown, a side of an equilateral triangle represents 1 unit of length in any direction.)

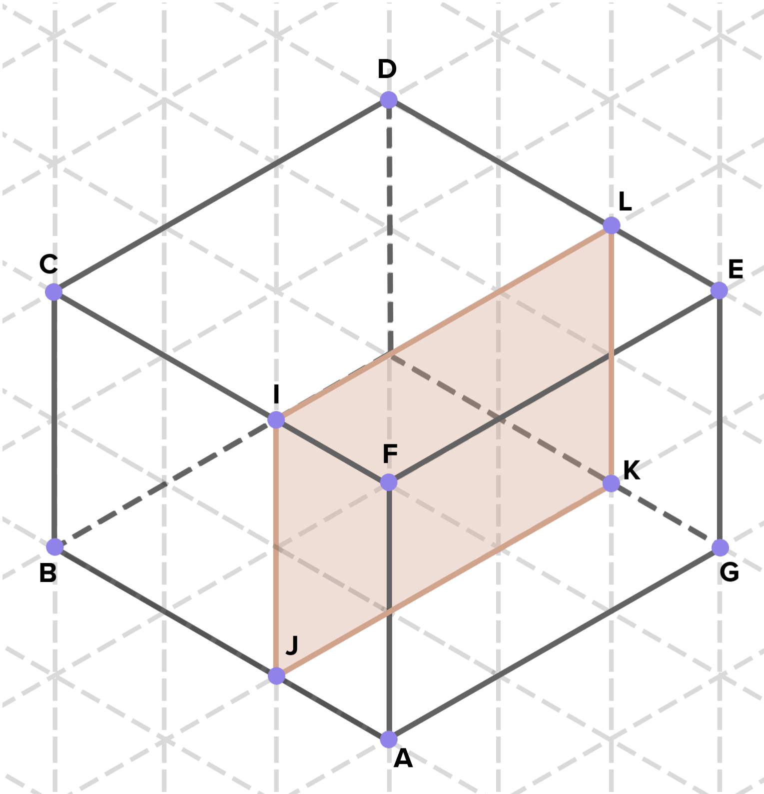


FIGURE 9.1

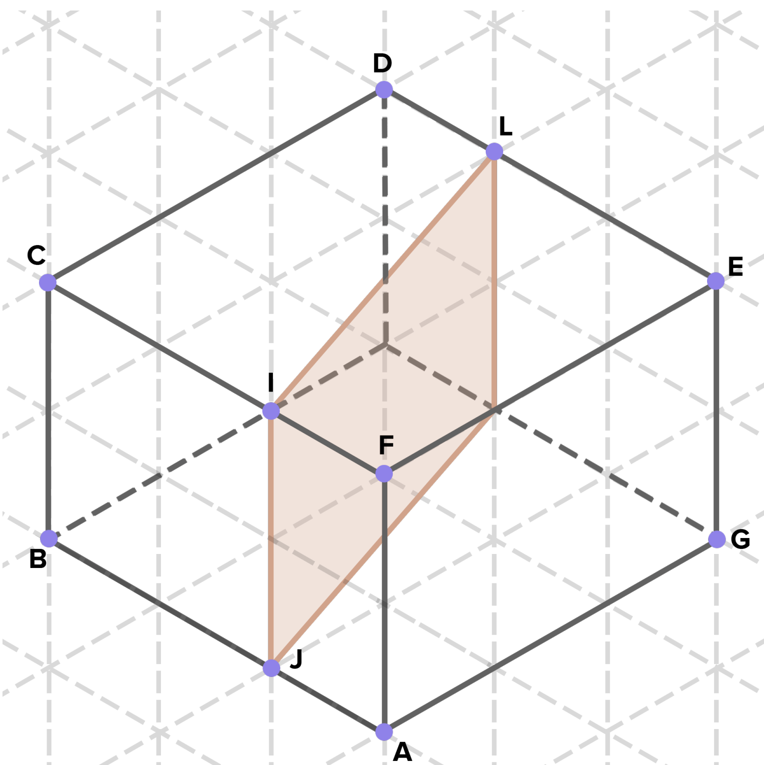


FIGURE 9.2

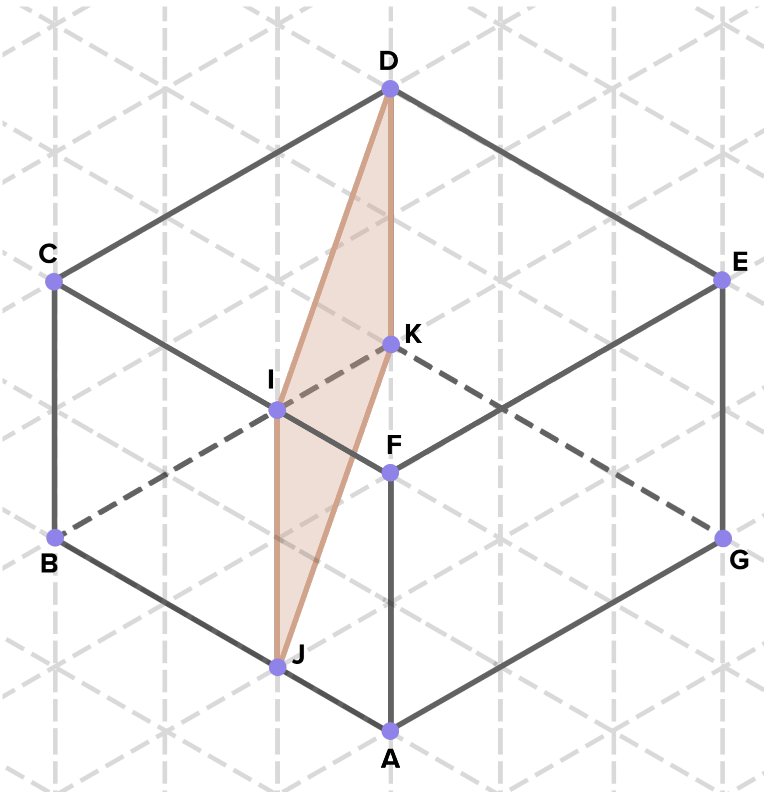


FIGURE 9.3

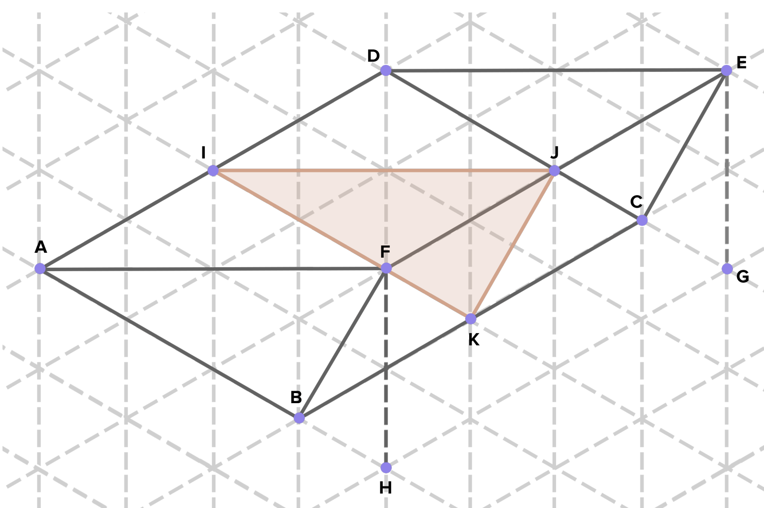


FIGURE 9.4

Review (Answers)

To see the Review answers, open this PDF file and look for section 9.4.

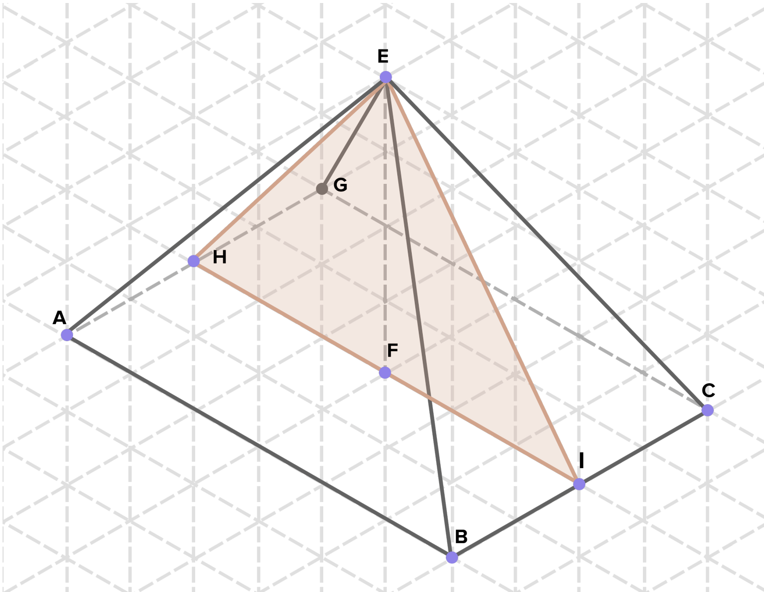


FIGURE 9.5

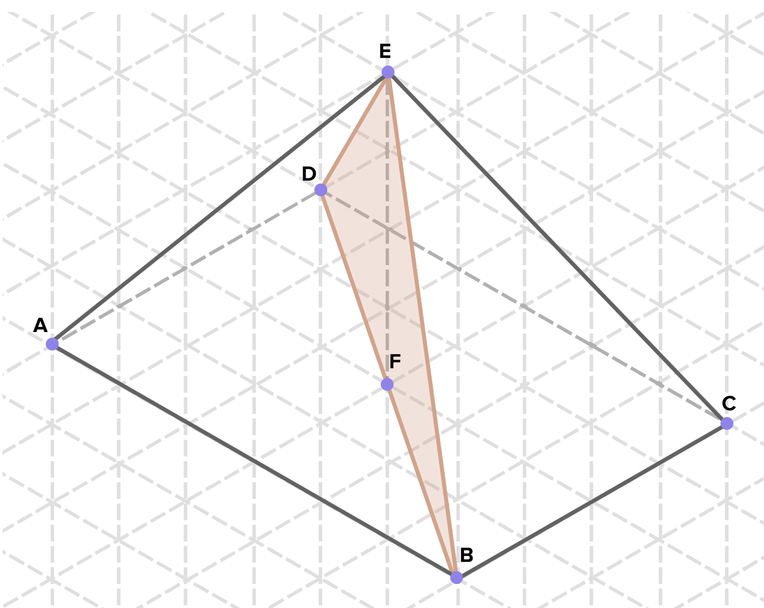


FIGURE 9.6

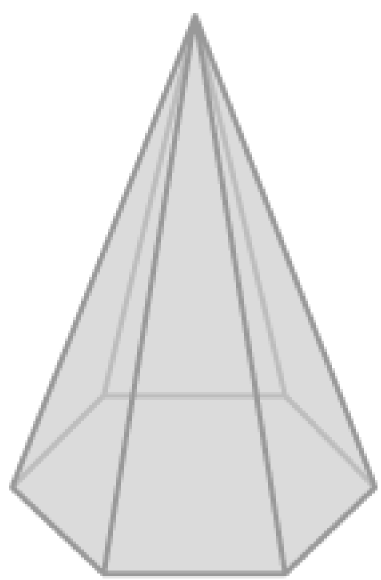
9.2 Cross Sections of Solids

Learning Objectives

Here you will review cross sections of solids.

Cross Sections

A cross section is the intersection of a figure in three-dimensional space with a plane. It is the face you obtain by making a "slice" through a solid object. A cross section is two-dimensional. The figure (face) obtained from a cross section depends upon the orientation (angle) of the plane doing the cutting. Consider the following hexagonal pyramid.



Cross sections perpendicular to the base and through the vertex will be triangles. Below, you can see a plane cutting through the pyramid, part of the pyramid removed, and the cross section.

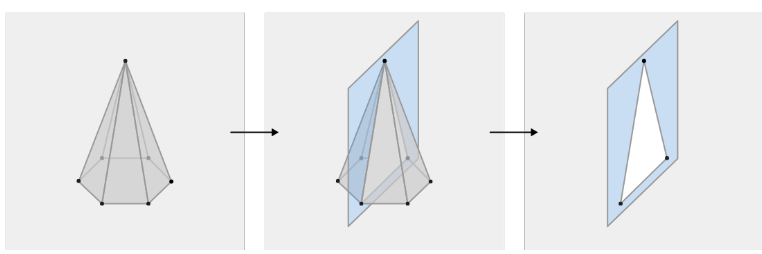


FIGURE 9.7

You could also take a slice parallel to the base. Cross sections parallel to the base will be hexagons.

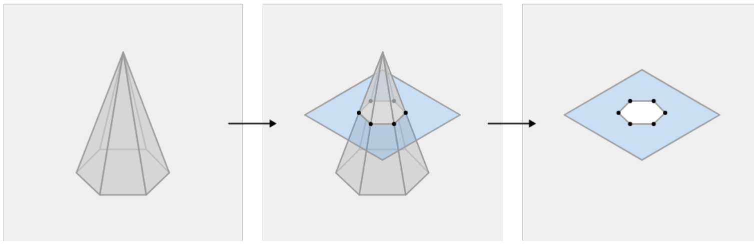


FIGURE 9.8

It is also possible to take cross sections using planes that are neither parallel nor perpendicular to the base. These can be much more difficult to visualize. Physical models or dynamic geometry software are extremely helpful. For example, the same hexagonal pyramid has been sliced at a slant below. The cross section is a hexagon.

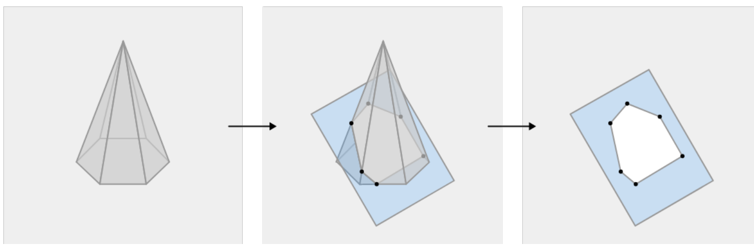


FIGURE 9.9

Cross sections are one way of representing three-dimensional objects in two dimensions.

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Cross Sections of Solids

Explore the different cross sections of the solids given below.

Solid: _____ Cross Section: _____

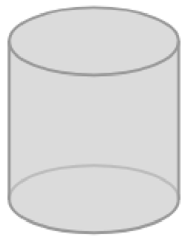


FIGURE 9.10

Constructing Cross Sections

Using planes parallel to the base, what cross sections can you construct from a rectangular prism?

Considering only planes that are parallel to the base, all cross sections are rectangles. This is shown below.

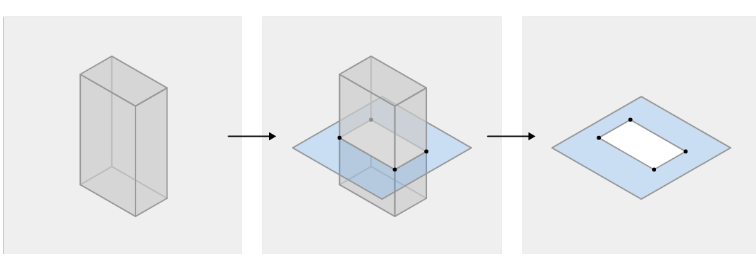


FIGURE 9.11

Using planes perpendicular to the base, what cross sections can you construct from a rectangular prism?

Considering only planes that are perpendicular to the base, all cross sections are rectangles. This is shown below.

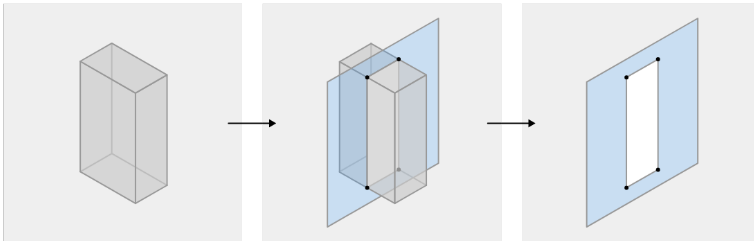


FIGURE 9.12

Describing Cross Sections

1. A rectangular pyramid is sliced by a plane parallel to its base. Describe the cross section. How does moving the plane so that it is closer or further from the base change the cross section?

The cross section will always be a rectangle. The closer the plane is to the base, the bigger the rectangle.

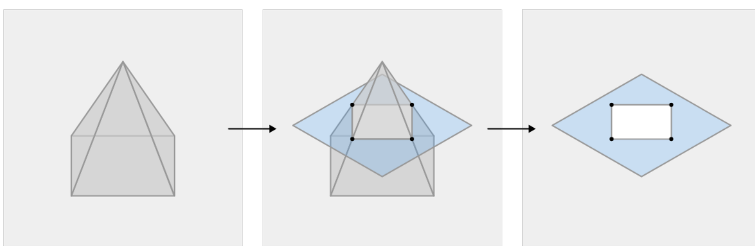


FIGURE 9.13

2. Describe how to create a triangular cross section of a rectangular pyramid.

Slice the pyramid with a plane that is perpendicular to the base of the pyramid.

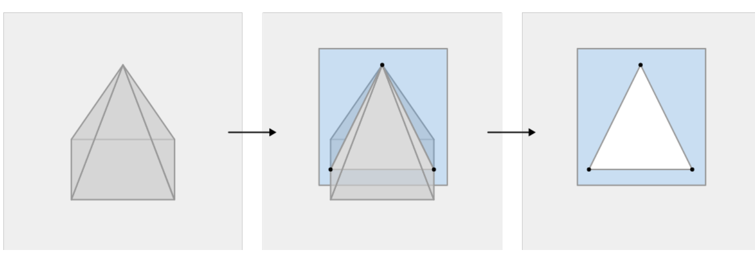


FIGURE 9.14

Examples

Example 1

A pentagonal prism is sliced with a plane parallel to the base. The area of the cross section is 15 square inches. If the height of the prism is 20 inches, what is the volume of the prism?

A pentagonal prism is sliced with a plane parallel to the base. This means that the cross section is a pentagon that is congruent to the base. The area of the cross section is 15 square inches. This means that the area of the base is 15 square inches. Since the height of the prism is 20 inches, the volume of the prism is:

$$V = A_{\text{base}} \cdot h$$
$$V = (15)(20)$$
$$V = 300 \text{ in}^3$$

Example 2

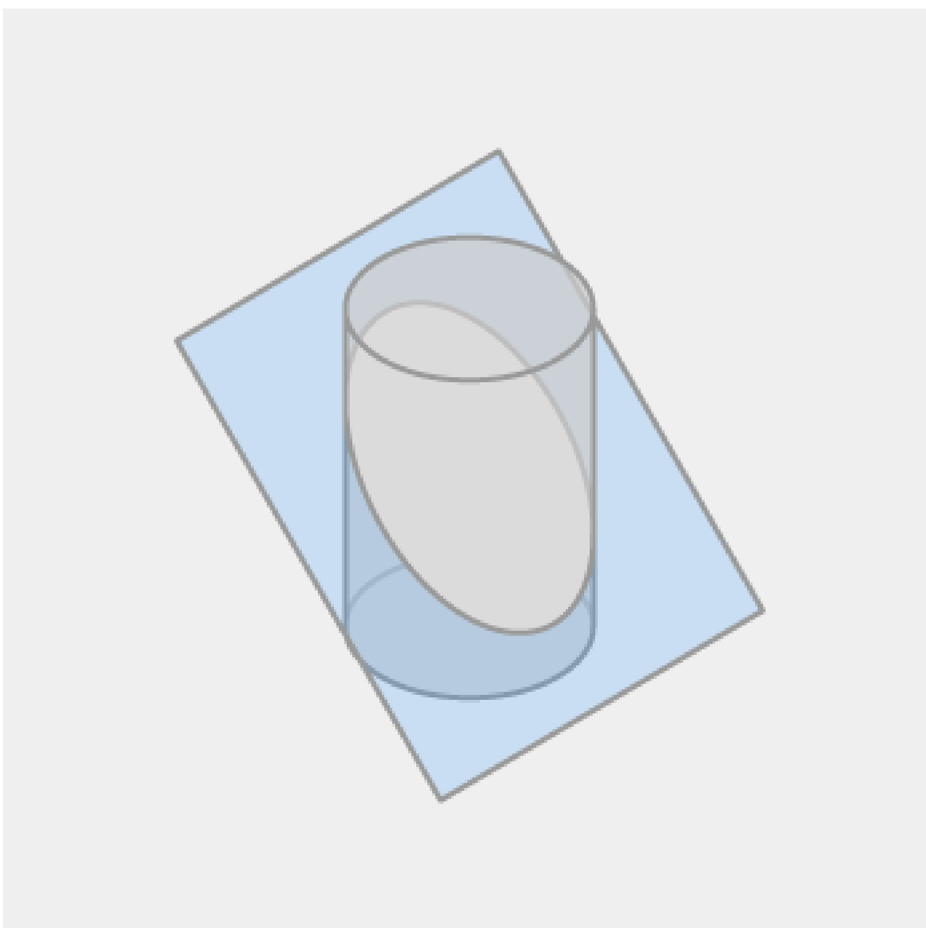
Describe the cross sections of a cylinder that are perpendicular or parallel to the base of the cylinder.

Rectangles and circles, respectively.

Example 3

Could the cross section of a cylinder be an ellipse (oval)?

Yes. If the slice is slanted as shown below:



Example 4

A cylinder is sliced parallel to its base. The area of the cross section is 36π in^2. What is the radius of the cylinder?

The cross section of the cylinder is a circle.

The area of cross section is:

$$
\begin{aligned}
A &= \pi r^2 \\
36\pi &= \pi r^2 \\
36 &= r^2 \\
6 &= r
\end{aligned}
$$

Therefore, the radius of the circle = 6 inches.

Review

1. Describe the cross sections of a sphere.
2. Describe the cross sections of a pentagonal prism that are parallel or perpendicular to the base.
3. Describe the cross sections of a pentagonal pyramid that are parallel or perpendicular to the base.
4. Could the cross section of a cube be a triangle? Explain.
5. A cross section of a pyramid is taken parallel to its base. What are the connections between the cross section and the base?
6. A cross section of a pyramid is taken perpendicular to its base. What shape is the cross section? Does the shape of the base matter?
7. For a prism, information about what type of cross section can help you to determine the volume?
8. A cylinder with radius 4 inches is sliced parallel to its base. What is the area of the cross section?
9. The volume of a cylinder is $360\pi\ in^2$. The cylinder is sliced parallel to its base. If the height of the cylinder is 10 inches, what is the area of the cross section?

Use the solid below for 10-12.

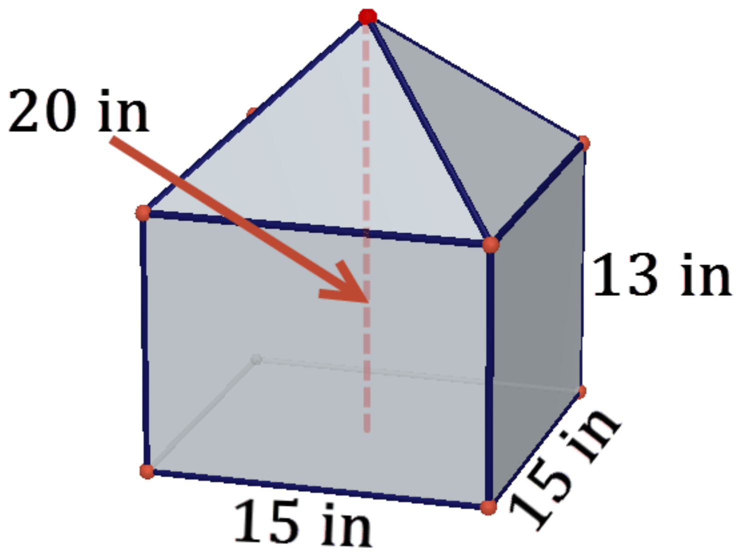


FIGURE 9.15

10. A plane slices the solid parallel to its base. Describe the cross section. How does the cross section change as the plane moves further from the base?

11. A plane slices the solid perpendicular to its base through the center of the base. Describe the cross section.

12. Find the area of the cross section from #11.

Use the solid below for 13-15.

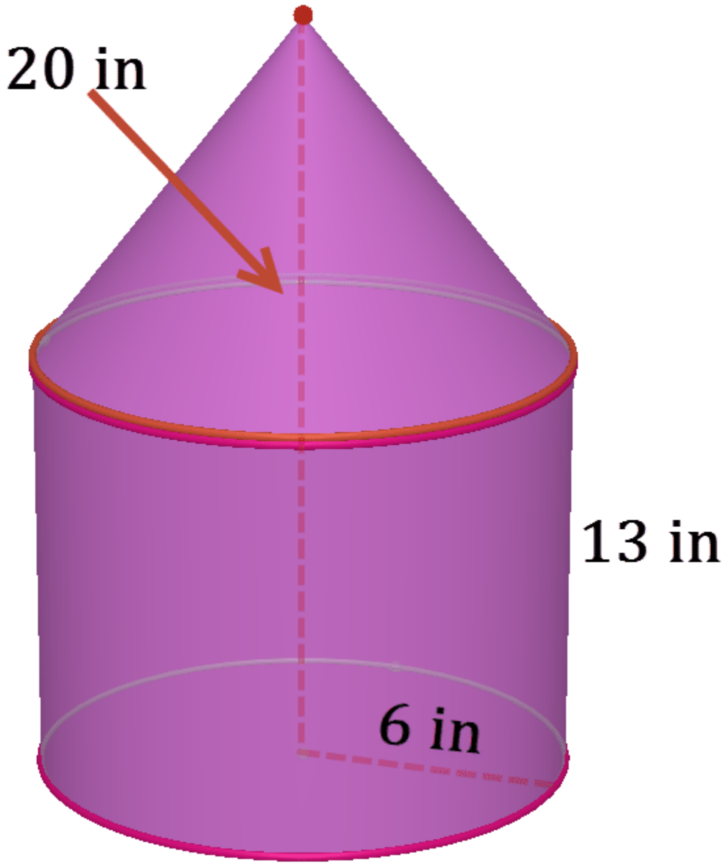


FIGURE 9.16

13. A plane slices the solid parallel to its base. Describe the cross section. How does the cross section change as the plane moves further from the base?

14. A plane slices the solid perpendicular to its base through the center of the base. Describe the cross section.

15. Find the area of the cross section from #14.

16. Find the area of the cross-section below. (The isometric grid paper is composed of equilateral triangles whose sides are of unit length 1) How does the area of this cross section compare with the parallel faces? Are there other cross-sections with areas the same as the parallel faces? Describe them and find their area.

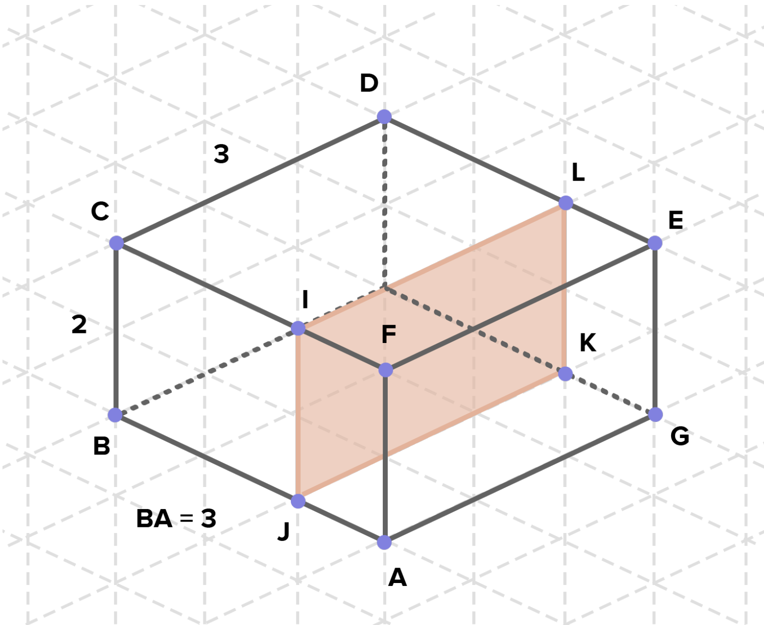


FIGURE 9.17

17. Use isometric grid paper, interactive geometry software, or a free-hand sketch to investigate the following questions, and support your answers with drawings and explanations. Given a rectangular prism:

a. How many cross sections are parallel to bases?
b. How many cross sections intersect two complete edges of the prism?
c. How many cross sections intersect one complete edge of the prism?
d. Is it possible to create a cross-section that intersects only one vertex of the prism?
e. Is it possible to create a cross-section that intersects no vertices of the prism?
f. Is it possible to create a cross-section that intersects two vertices but no complete edges?
g. Is it possible to create a cross-section that passes through only three faces? Four faces? Five faces?
h. What is the maximum number of faces a cross-section can pass through?
i. What is the relationship between the number of faces a cross-section passes through, and the number of sides of the resulting polygon?

18. Find the area of the cross-section below.

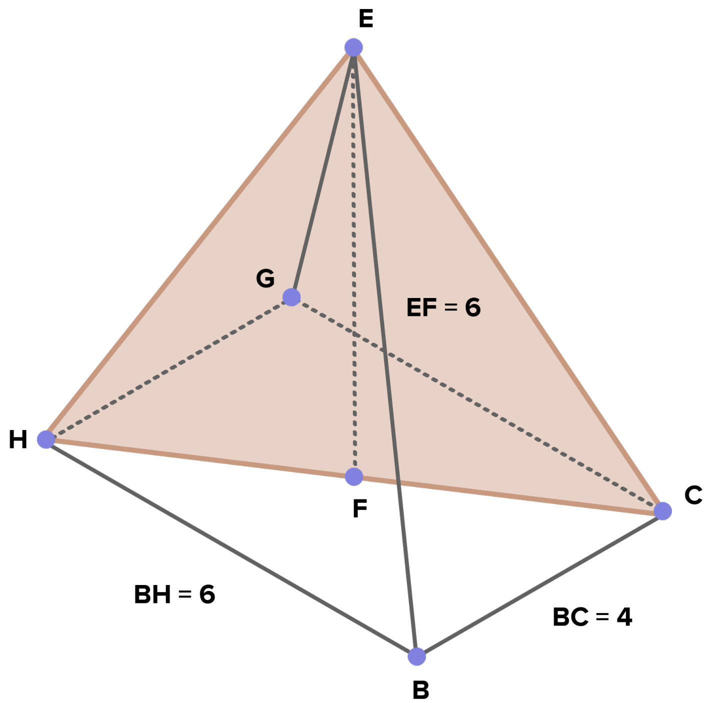


FIGURE 9.18

19. Sketch a cone and answer the following questions. Explain your answers.

1. Sketch the shape of a cross-section that is not parallel to the base and only intersects the lateral face of the cone.
2. What is the shape of a cross-section that is parallel to the base?
3. What is the shape of a cross-section that is perpendicular to the base and intersecting the apex of the cone?
4. Sketch and describe the shape of a cross section that is perpendicular to the base but doesn't intersect the apex.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.12.

9.3 Surface Area and Nets

Learning Objectives

Here you will review surface area and nets of prisms, pyramids, and composite solids. In geometry, a net is a 2-dimensional shape that can be folded to form a 3-dimensional shape or a solid. Or a net is a drawing made when the surface of a 3-dimensional figure is laid out flat showing each face and edge of the figure in 2-dimension.

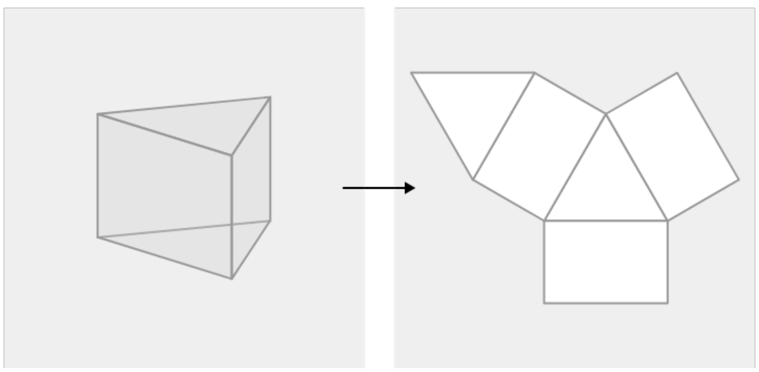


FIGURE 9.19

Here are some steps to determine whether a net forms a solid:

1. Make sure that the solid and the net have the same number of faces and that the shapes of the faces of the solid match the shapes of the corresponding faces in the net.

2. Visualize how the net is to be folded to form the solid and make sure that all the sides fit together properly.

Nets are helpful when we need to find the surface area of the solids. In the image above, you can see a triangular prism when unfolded consists of two triangles and three rectangles. The triangles are the bases of the prism and the rectangles are the lateral faces.

Surface Area of a Solid

The surface area of a 3-dimensional object is the measure of the total area of all its faces. This means that one way to find the surface area of a solid is to find the area of its net.

Surface area of rectangular prism $= 2ab + 2ac + 2bc = 2(ab + ac + bc)$

The surface area of a rectangular prism is the area of the six rectangles that cover it. But we don't have to figure out all six because we know that the top and bottom are the same, the front and back are the same, and the left and right sides are the same.

Nets of Solids

Explore the nets and see how the 2d shape transforms into a 3d shape.

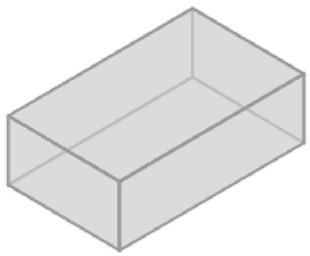
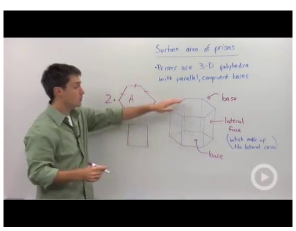


FIGURE 9.20

Note that there could be a couple different interpretations of any net. Most prisms have multiple nets.

Prisms



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Find the approximate surface area of the triangular prism below. The base is an equilateral triangle.

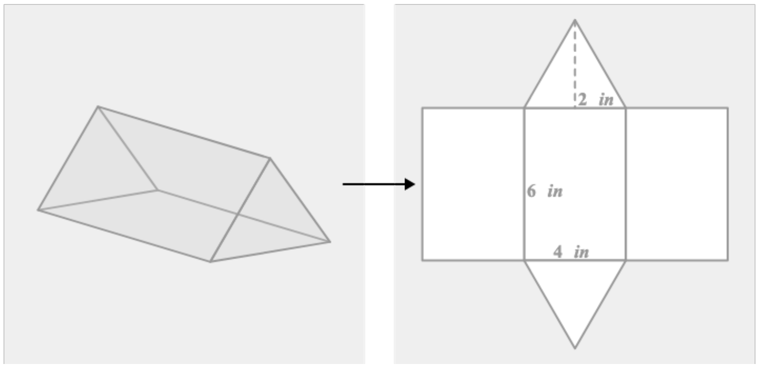


FIGURE 9.21

The net is made of three congruent rectangles and two congruent equilateral triangles.

The surface area is the sum of the areas of the five shapes. To find the area of the triangle, you need to know the height of the triangle. Using the Pythagorean Theorem, you can determine that the height is approximately 3.46 inches.

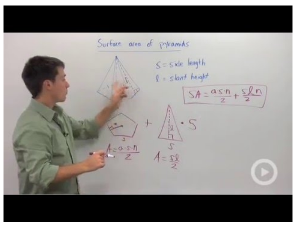
$H^2 = P^2 + B^2$
$4^2 = P^2 + 2^2$
$P^2 = 16 - 4 = 12$
$P = 3.46$ inches

$$
\begin{aligned}
Area_{triangle} &= \frac{1}{2}\, bh \\
Area_{triangle} &= \frac{1}{\cancel{2}}\left(\overset{2}{\cancel{4}}\right)(3.46) \\
Area_{triangle} &= 6.93\ in^2 \\
Area_{rectangle} &= bh \\
Area_{rectangle} &= (4)\ (6) \\
Area_{rectangle} &= 24\ in^2
\end{aligned}
$$

$$
\begin{aligned}
&Total\ \ surface\ \ area \\
&= 2\, Area_{triangle} + 3\, Area_{rectangle} \\
&= 2\ (6.93) + 3(24) \\
&= 13.86 + 72 \\
&= 85.86\ in^2
\end{aligned}
$$

FIGURE 9.22

Pyramids



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URL: http://www.ck12.org/flx/render/embeddedobject/222876

Draw the net of a square pyramid.

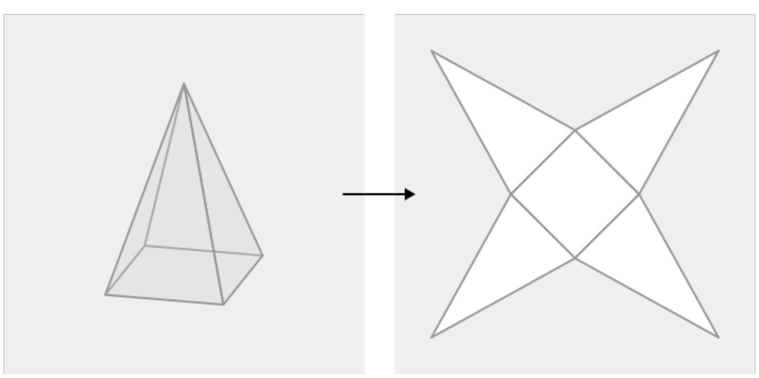

FIGURE 9.23

A square pyramid has a height of 20 inches. Each side of the square base is 12 inches. What is the surface area of the pyramid?

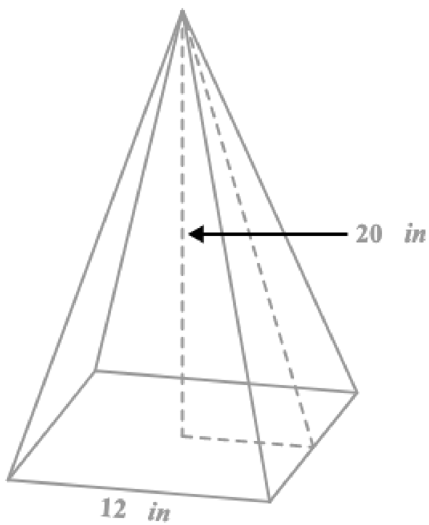

FIGURE 9.24

In order to find the surface area, you will need to determine the area of the triangle faces. In order to find the area of each triangle face, you will need the base and height of the triangle. In order to do this, imagine a right triangle standing upright in the pyramid.

The height of this triangle is 20 inches and the base of the triangle is 6 inches. It's hypotenuse, which is the height of the triangle face, can be determined with the Pythagorean Theorem.

Now you can find the area of each of the five shapes that make up the net in order to find the surface area.

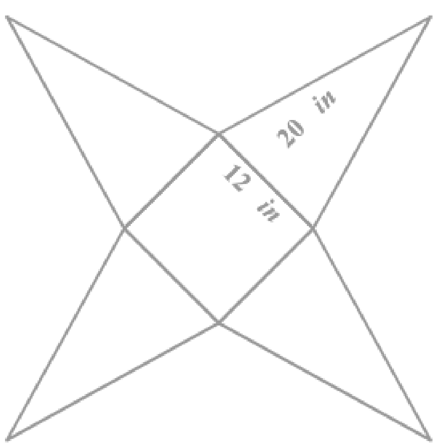

FIGURE 9.25

$H^2 = P^2 + B^2$
$H^2 = 20^2 + 6^2$
$H^2 = 400 + 36$
$H = \sqrt{436} = 20.88$ inches

$$
\begin{aligned}
Area_{triangle} &= \frac{1}{2}bh \\
Area_{triangle} &= \frac{1}{2}(12)(20.88) \\
Area_{triangle} &= 125.28\ in^2
\end{aligned}
$$

$$
\begin{aligned}
Area_{square} &= sides^2 \\
Area_{square} &= (12)^2 \\
Area_{square} &= 144\ in^2
\end{aligned}
$$

$$
\begin{aligned}
&Total\ surface\ area \\
&= 4\,Area_{triangle} + Area_{square} \\
&= 4\,(125.28) + (144) \\
&= 645.12\ in^2
\end{aligned}
$$

FIGURE 9.26

Identify the solid whose net is given below, based on the base shape.

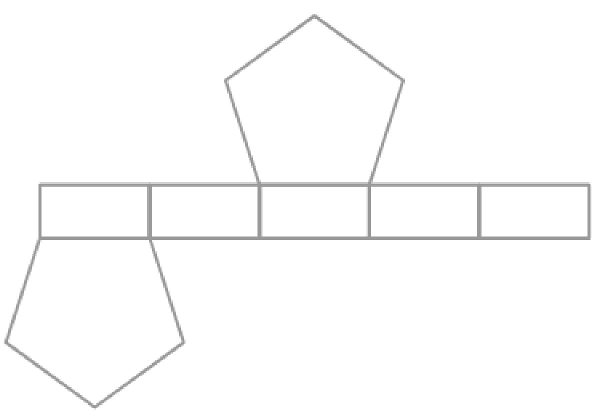

FIGURE 9.27

The solid is _____.

CK-12 PLIX: Cross-Sections and Nets

PLIX
Click image to the left or use the URL below.
URL: http://www.ck12.org/geometry/cross-sections-and-nets/plix/Cross-Sections-and-Nets-Cereal-Box-Blueprint-536a66985aa4132ffa250114

Examples

Example 1

A cup of paint covers about 22 square feet. You need to paint all faces of a cube to use as a prop in a play. Each edge of the cube is 2.5 feet long. How much paint will you need to buy?

In order to figure out how much paint you will need, you should find the surface area of the cube.

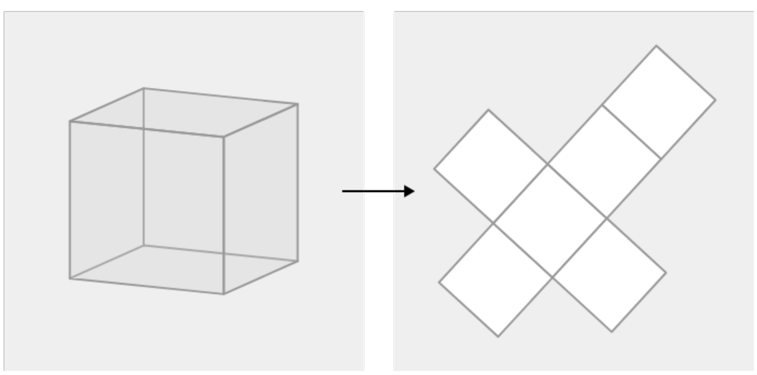

FIGURE 9.28

The cube has six congruent square faces. The area of each square is $(2.5)^2 = 6.25 \text{ ft}^2$. The total surface area of the cube is 37.5 ft^2. You will need $\frac{37.5}{22} \approx 1.7$ cups of paint.

Example 2

Draw the net for a pentagonal prism.

Example 3

Find the area of a pentagonal prism with the following base:

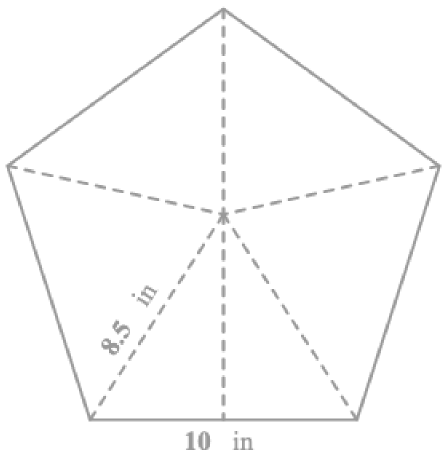

FIGURE 9.29

Find the area of the pentagon by dissecting the pentagon into five triangles and finding the area of each triangle.

Use the Pythagorean Theorem to find the height of each triangle.

$H^2 = P^2 + B^2$
$8.5^2 = P^2 + 5^2$
$P^2 = 72.25 - 25 = 47.25$
$P = \sqrt{47.25} = 6.87$ inches

The area of each triangle is approximately

$$Area_{triangle} = \frac{1}{2}bh$$
$$Area_{triangle} = \frac{1}{2}(10)(6.87)$$
$$Area_{triangle} = 34.37\ in^2$$

Therefore, the area of the pentagon is:

$$= 5\,Area_{triangle}$$
$$= 5(34.37)$$
$$= 171.85\ in^2$$

FIGURE 9.30

Example 4

Find the surface area for a pentagonal prism with a height of 25 inches and a base area of 171.82 square inches.

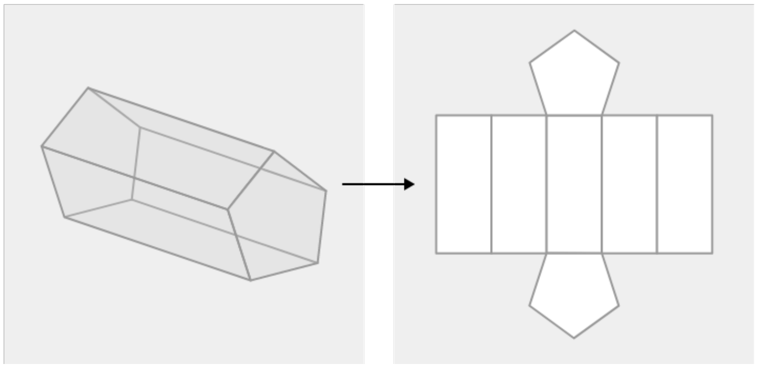

FIGURE 9.31

The pentagonal prism is made of five rectangular faces and two pentagonal faces. Each rectangle is 10 inches by 25 inches.

Review

1. Explain the connection between the surface area of a solid and the net of a solid.

2. When stating a surface area, why do you use square units such as "in^2"?

A triangular pyramid has four congruent equilateral triangle faces. Each edge of the pyramid is 6 inches.

3. Draw a net for the pyramid.

4. Find the area of one triangle face.

5. Find the surface area of the pyramid.

A 20 inch tall hexagonal pyramid has a regular hexagon base that can be divided into six equilateral triangles with side lengths of 12 inches, as shown below.

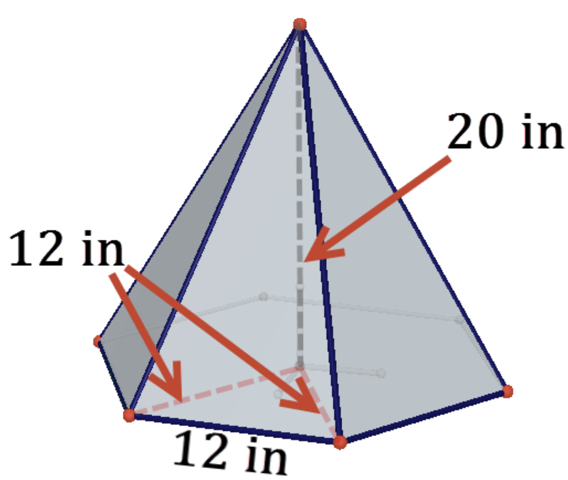

FIGURE 9.32

6. Draw a net for the pyramid.

7. Find the area of the hexagon base.

8. The pyramid has 6 triangular faces. Use the Pythagorean Theorem to help you to find the height of each of these triangles.

9. Find the total surface area of the pyramid.

A square prism is topped with a square pyramid to create the composite solid below.

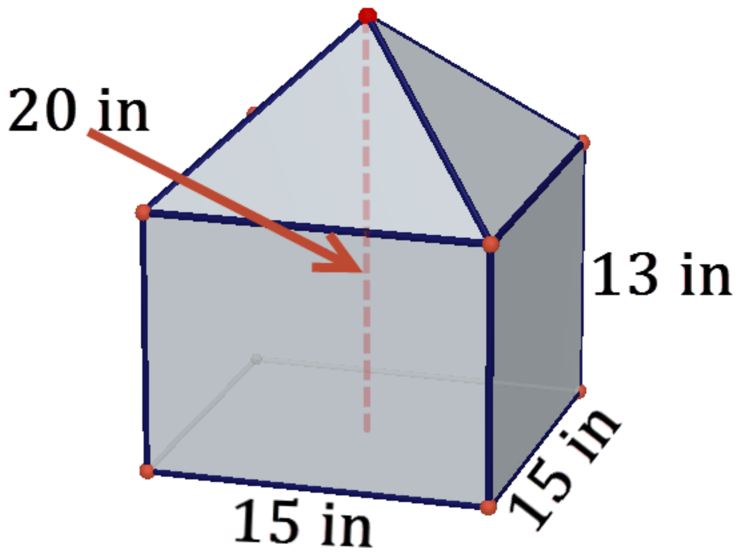

FIGURE 9.33

10. Draw a net for the solid.

11. There are four triangular faces. Use the Pythagorean Theorem to help you find the height of each of these triangles.

12. Find the total surface area of the solid.

A solid has the following net.

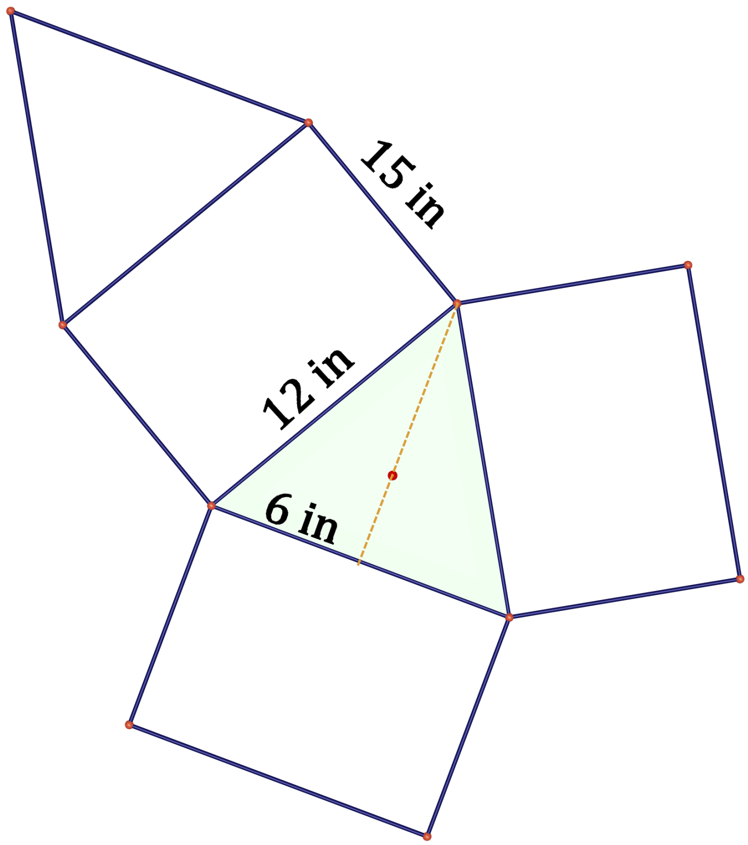

FIGURE 9.34

13. What type of solid is this?

14. Find the surface area of the solid.

15. What would the net of a cylinder look like? Try to make a sketch.

16. What would the net of a cone look like? Try to make a sketch.

16. Graph the points $(2,3)$ and $(12,3)$. Label them A and B and connect them to form a segment. Draw vertical segments connecting them to the x-axis. Rotate this shape around the x-axis to form a solid. What figure have you made? Explain. Find the surface area and volume of this figure.

17. Graph the points $(2,0)$ and $(10,6)$. Label them A and B and connect them to form a segment. Draw a vertical segment from B to the x-axis. Rotate this shape around the x-axis to form a solid. What figure have you made? Explain. Find the surface area and volume of this figure.

18. Graph the points $(24,10)$ and $(12,5)$. Form a solid of revolution as described above. Find its surface area and volume.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.11.

9.4 Volume of Solids

Learning Objectives

Here you will review volume of prisms, pyramids, cylinders, cones, spheres, and composite solids.

The volume of a solid is the measure of how much space an object takes up. It is measured by the number of unit cubes it takes to fill up the solid.

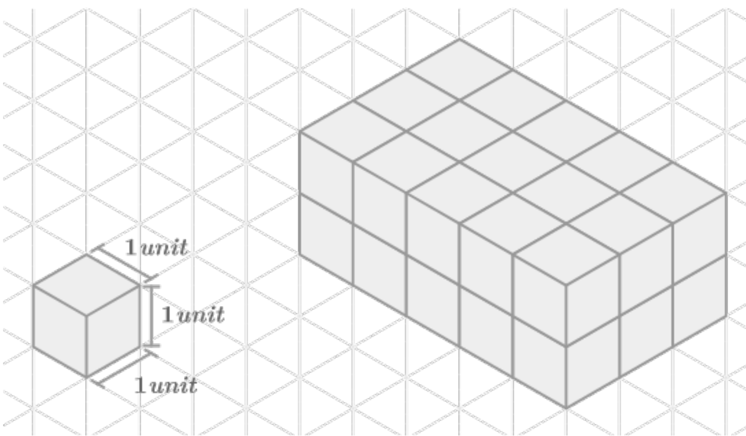

FIGURE 9.35

Counting the unit cubes in the solid, we have 30 unit cubes, so the volume is: (2 units) (3 units) (5 units) = 30 cubic units.

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URL: http://www.ck12.org/flx/render/embeddedobject/222872

Volume of a Prism

A prism is a solid with two congruent polygon bases that are parallel and connected by rectangles. Prisms are named by their base shape.

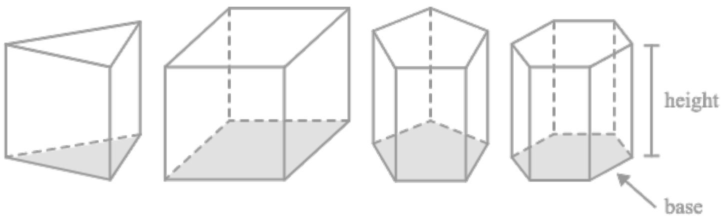

FIGURE 9.36

To find the volume of a prism, find the area of its base and multiply by its height.

$V_{\text{prism}} = A_{\text{base}} \cdot h$

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URL: http://www.ck12.org/flx/render/embeddedobject/222873

Volume of a Cylinder

A cylinder is a three-dimensional solid consisting of two congruent, parallel, circular sides (the bases), joined by a curved surface.

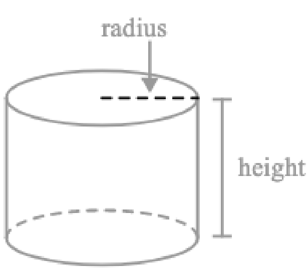

FIGURE 9.37

To find the volume of a cylinder, find the area of its circular base and multiply by its height.

$V_{\text{cylinder}} = \pi r^2 h$

CK-12 PLIX: Surface Area and Volume of Cylinders

Multimedia

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/cylinders/plix/Volume-of-Cylinder-Big-Can-of-Soup-545a96b28e0e08476099bd7f

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Volume of a Pyramid

A pyramid is a three dimensional solid with a polygonal base. Each corner of a polygon is attached to a singular vertex, which gives the pyramid its distinctive shape. Each base edge and the vertex form a triangle. Pyramids are named by their base shape.

To find the volume of a pyramid, find the volume of the prism with the same base and divide by three.

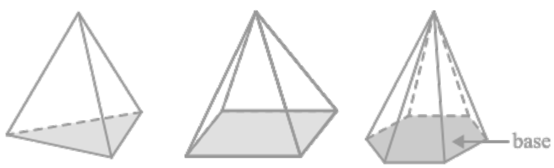


FIGURE 9.38

$V_{\text{pyramid}} = \frac{A_{\text{base}} \cdot h}{3}$

Volume of a Cone

A cone is a three-dimensional solid with a circular base whose lateral surface meets at a point called the vertex.

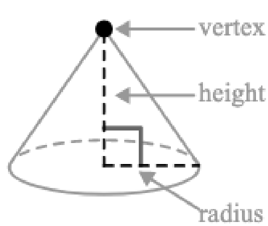


FIGURE 9.39

To find the volume of a cone, find the volume of the cylinder with the same base and divide by three.

$V_{\text{cone}} = \frac{\pi r^2 h}{3}$

What is the ratio between the volume of a cylinder and of a cone, having the same radius and height?

The ratio between the volume of a cylinder and of a cone, having the same radius and height is _____.

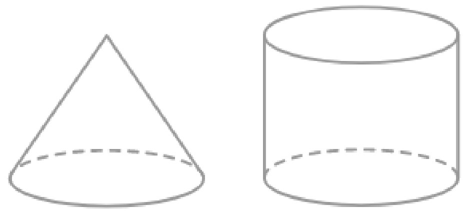


FIGURE 9.40

Volume of a Sphere

A sphere is the set of all points in space equidistant from a center point. The distance from the center point to the sphere is called the radius.

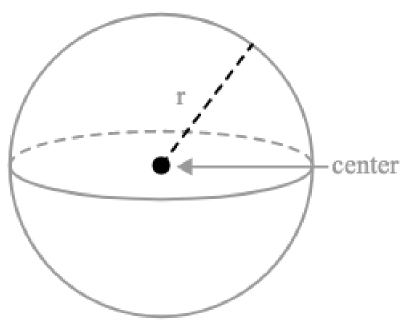


FIGURE 9.41

The volume of a sphere relies on its radius.

$V_{\text{sphere}} = \frac{4}{3}\pi r^3$

Calculating Volume

Adjust the dimensions of the solid given below and observe how the volume changes.

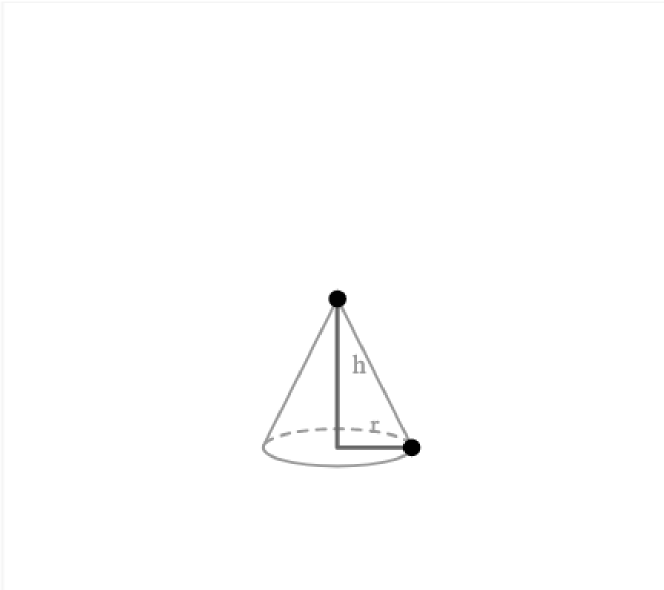


FIGURE 9.42

Volume of a Composite Solid

A composite solid is a solid made up of common geometric solids. The solids that it is made up of are generally prisms, pyramids, cones, cylinders and spheres.

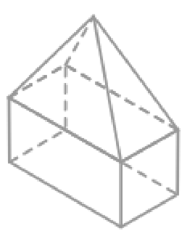
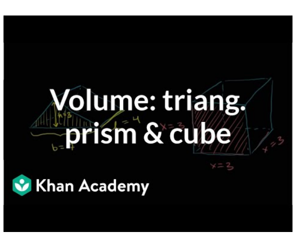


FIGURE 9.43

The volume of a composite solid is the sum of the volumes of the individual solids that make up the composite.



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Let's look at some problems where we find the volume.

1. Find the volume of the rectangular prism below.

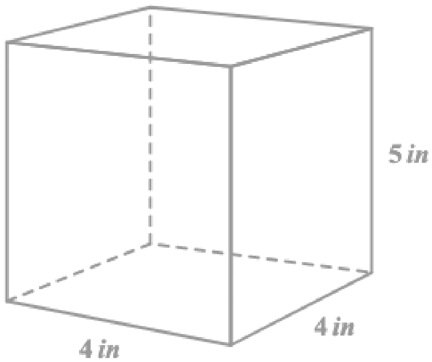


FIGURE 9.44

To find the volume of the prism, you need to find the area of the base and multiply by the height. Note that for a rectangular prism, any face can be the "base", not just the face that appears to be on the bottom.

$$\begin{aligned}
Volume &= Area_{base} \cdot height \\
Volume &= (length \cdot width) \cdot height \\
Volume &= (4)\ (4)\ (5) \\
Volume &= 80\ in^3
\end{aligned}$$

FIGURE 9.45

2. Find the volume of the cone below.

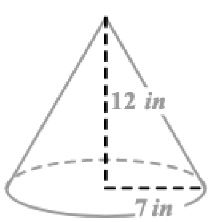


FIGURE 9.46

To find the volume of the cone, you need to find the area of the circular base, multiply by the height, and divide by three.

$$Volume = \frac{1}{3}\ Area_{base} \cdot height$$
$$Volume = \frac{1}{3}\pi r^2 h$$
$$Volume = \frac{1}{3}\pi (7)^2\ (12)$$
$$Volume = 196\pi\ cm^3$$

FIGURE 9.47

3. Find the volume of a sphere with radius 4 cm.

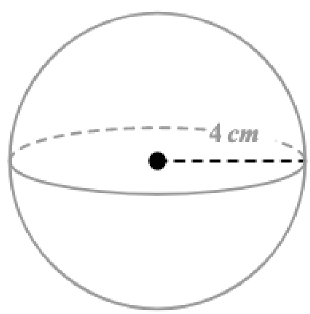


FIGURE 9.48

The volume of the sphere is

Volume of Sphere $= \frac{4}{3}\pi r^3$
$= \frac{4}{3}\pi(4)^3$
$= \frac{256\pi}{3}$ cm^3

Examples

Example 1

The composite solid below is made of a cube and a square pyramid. The length of each edge of the cube is 12 feet and the overall height of the solid is 22 feet. What is the volume of the solid? Why might you want to know the volume of the solid?

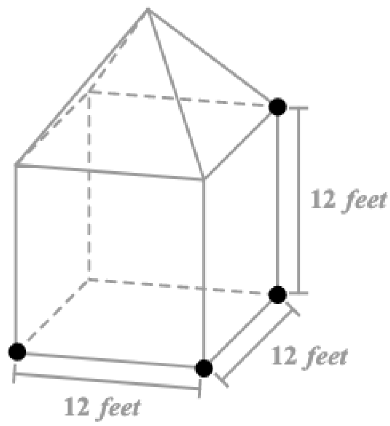


FIGURE 9.49

To find the volume of the solid, find the sum of the volumes of the prism (the cube) and the pyramid. Note that since the overall height is 22 feet and the height of the cube is 12 feet, the height of the pyramid must be 10 feet.

$$\begin{aligned}
\text{Volume of Prism (Cube)} &= A_{\text{base}} \cdot h \\
&= (12 \cdot 12)(12) \\
&= 1728 \text{ ft}^3 \\
\text{Volume of Pyramid} &= \frac{A_{\text{base}} \cdot h}{3} \\
&= \frac{(12 \cdot 12)(10)}{3} \\
&= 480 \text{ ft}^3 \\
\text{Total Volume} &= \text{Volume of Prism (Cube)} + \text{Volume of Pyramid} \\
&= 1728 + 480 \\
&= 2208 \text{ ft}^3
\end{aligned}$$

The volume helps you to know how much the solid will hold. One cubic foot holds about 7.48 gallons of liquid, so

$$\begin{aligned}
\text{Gallons of liquid the solid can hold} &= \text{Volume of solid} \cdot \text{Number of gallons/cubic foot} \\
&= (2208)(7.48) \\
&= 16,515.84 \text{ gallons}
\end{aligned}$$

Example 2

The area of the base of the pyramid below is 100 cm^2. The height is 5 cm. What is the volume of the pyramid?

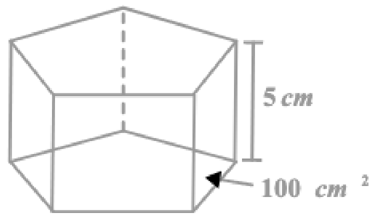


FIGURE 9.50

V_{prism}
$= A_{base} \cdot h$
$= (100 \text{ cm}^2) \cdot (5 \text{ cm})$
$= 500 \text{ cm}^3$

Example 3

The volume of a sphere is $\frac{500\pi}{3}$ in^3. What is the radius of the sphere?

$$\begin{aligned}\frac{4}{3}\pi r^3 &= \frac{500\pi}{3}\\ 4r^3 &= 500\\ r^3 &= 125\\ r &= 5 \text{ in}\end{aligned}$$

Example 4

The volume of a square pyramid is 64 in^3. The height of the pyramid is three times the length of a side of the base. What is the height of the pyramid?

$$\begin{aligned}V &= \frac{A_{base} \cdot h}{3}\\ 64 &= \frac{(s^2)(3s)}{3}\\ 64 &= s^3\\ \sqrt[3]{64} &= s\\ 4 &= s\end{aligned}$$

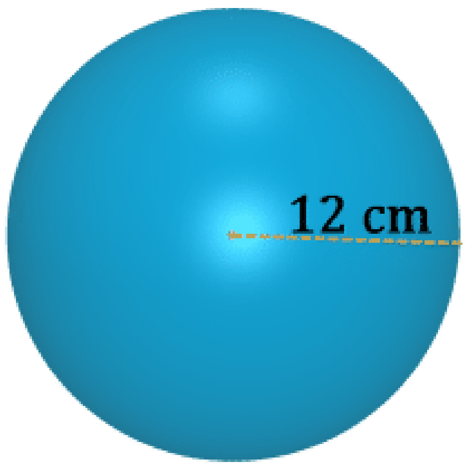
FIGURE 9.51

Therefore, $s = 4$ in and $h = 3(4) = 12$ in.

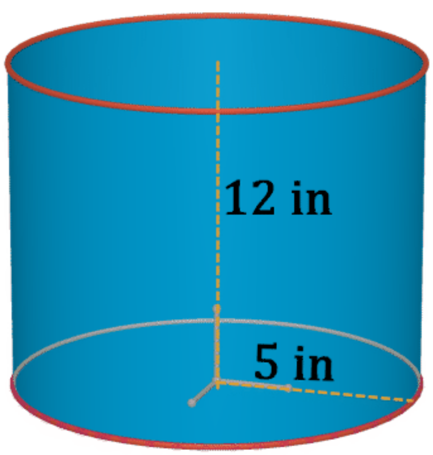
Review

Find the volume of each solid or composite solid.

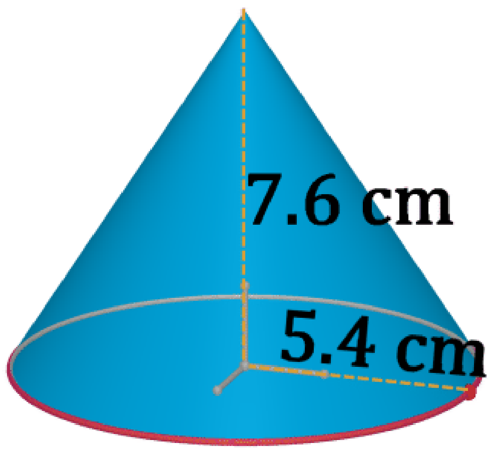
1.



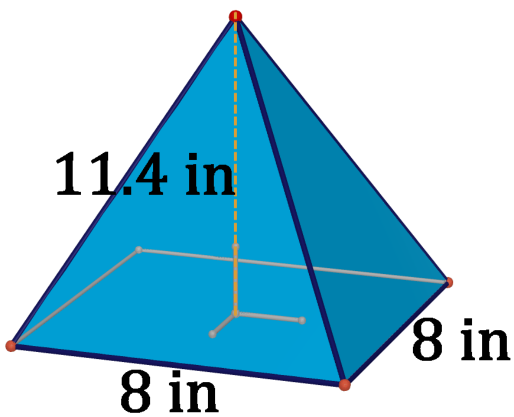
2.



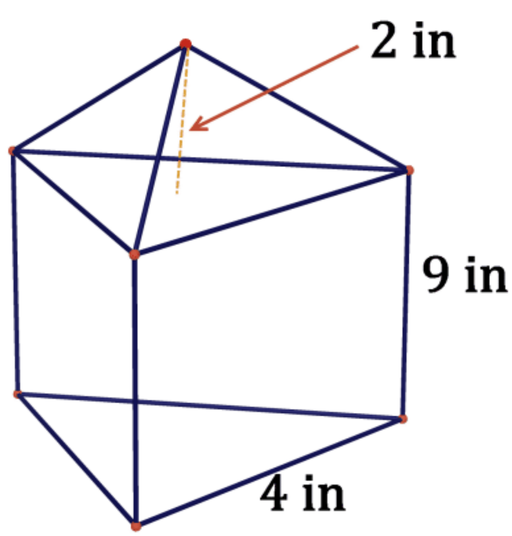
3.



4.



5. The base is an equilateral triangle.



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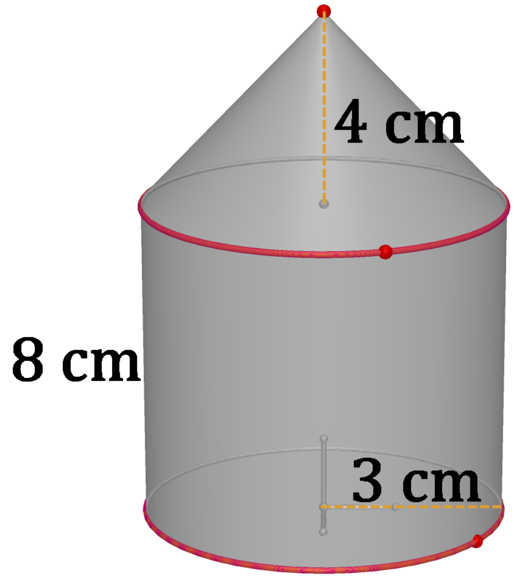


7. Explain why the formula for the volume of a prism involves the area of the base.

8. How is a cylinder related to a prism?

9. How is a pyramid related to a cone?

10. How is a sphere related to a circle?

11. If one cubic centimeter will hold 1 milliliter of water, approximately how many liters of water will the solid in #1 hold? (One liter is 1000 milliliters).

12. If one cubic centimeter will hold 1 milliliter of water, approximately how many liters of water will the solid in #3 hold? (One liter is 1000 milliliters).

13. If 231 cubic inches will hold one gallon of water, approximately how many gallons of water will the solid in #5 hold?

14. The volume of a cone is $125\pi\ in^3$. The height is three times the length of the radius. What is the height of the cone?

15. The volume of a pentagonal prism is $360\ in^3$. The height of the prism is $3\ in$. What is the area of the pentagon base?

16. The figure below features a cylinder whose height is 3 units. Inside the cylinder is a hemisphere (half of a sphere). And inside the hemisphere is a cone. Find the volume of the cylinder and the cone. Find the volume of the sphere and then the hemisphere. Do you see a relationship between the volume of the cylinder and cone, and that of the hemisphere? Explain. Use this to derive the formula for volume of a sphere from the formulas for a cylinder and cone.

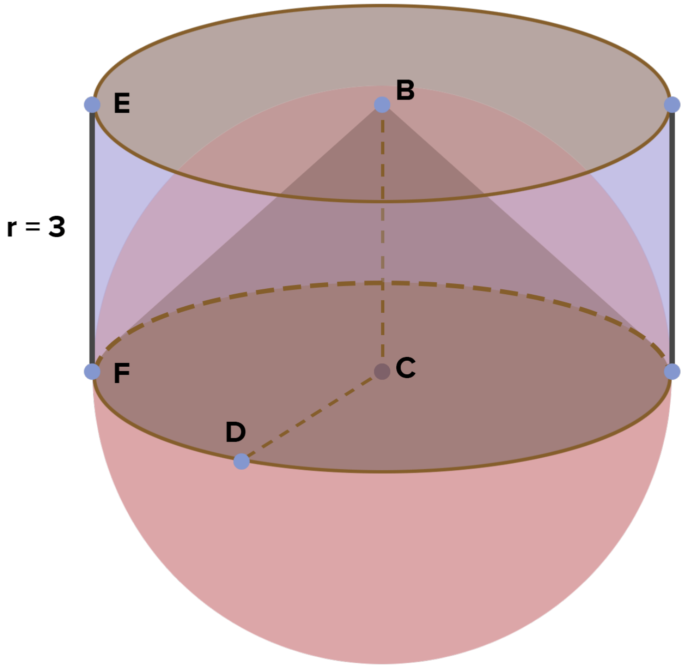


FIGURE 9.52

17. The isometric grid below is composed of equilateral triangles whose side are of unit length 1. Find the volume for each of the pyramids. What do you observe? Explain.

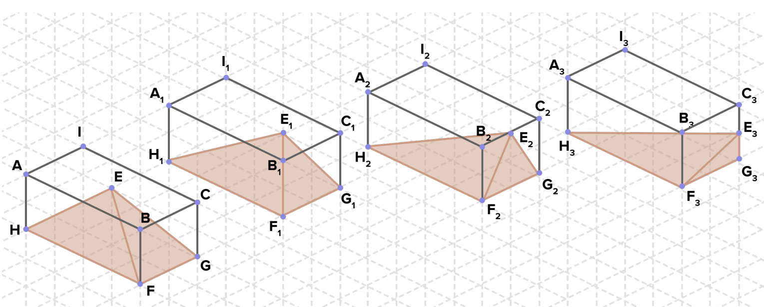


FIGURE 9.53

18. Find the surface area and volume of the following pyramid. Each line segment in the background grid is length 1.

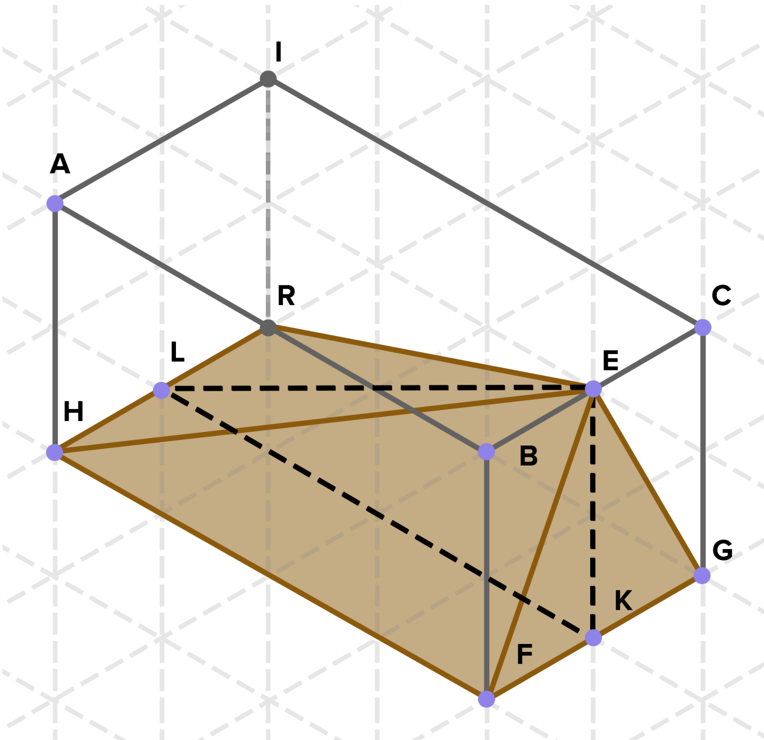


FIGURE 9.54

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.10.

9.5 Cylinders

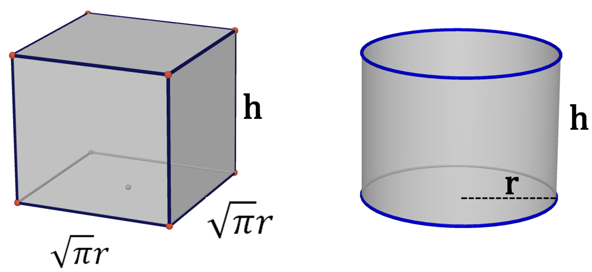
Learning Objectives

Here you will derive and use the formula for the volume of a cylinder.

Cavalieri's principle states that if two solids lying between parallel planes have equal heights and all cross sections at equal distances from their bases have equal areas, then the solids have equal volumes. Why does this make sense?

Cylinders

Below is a rectangular prism and a cylinder. Note that the height of each solid is the same.



In each case, the area of the base is πr^2. In fact, the area of **any** cross section taken parallel to the base is πr^2. Because these solids have the same height and the same cross sectional areas at every level, the solids have the same volume due to **Cavalieri's principle**.

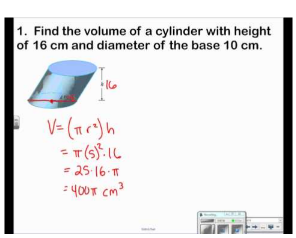
The volume of the prism is:

$$V = A_{Base} \cdot h = \pi r^2 h$$

Therefore, the volume of the cylinder is:

$$V = \pi r^2 h$$

This should make sense because a cylinder is essentially a circular prism. The area of its base is πr^2 and its height is h, so its volume is $\pi r^2 h$.



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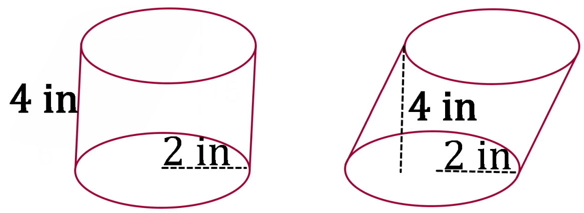
Let's take a look at a problem about how volume is related to radius and height.

The two cylinders below have the same radius and the same height. Do they have the same volume?



Yes, due to Cavalieri's principle. Even though these two cylinders are different, because they have the same height and base (and because every parallel cross section is congruent to the base), their volumes will be the same. The "slanted" cylinder is called an **oblique cylinder**.

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Finding Volume

1. Find the volume of the cylinders from the previous problem.

The volume of each cylinder is $V = \pi r^2 h = \pi(2^2)(4) = 16\pi\ in^3$.

2. One cup of water has a volume of approximately 14.44 in^3. How many cups of water will the cylinders in the first problem hold?

The volume of each cylinder is $16\pi\ in^3 \approx 50.2655\ in^3$. Since each cup of water has a volume of 14.44 in^3, each cylinder will hold $\frac{50.2655}{14.44} \approx 3.5$ cups of water.

Examples

Example 1

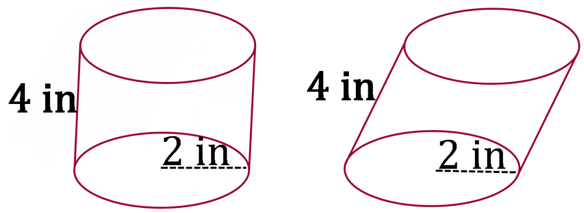
Earlier, you were given a problem about Cavalieri's principle.

Cavalieri's principle states that if two solids lying between parallel planes have equal heights and all cross sections at equal distances from their bases have equal areas, then the solids have equal volumes.

One way to understand Cavalieri's principle is to imagine a stack of books. Each stack of books below is made up of 15 books. The volume of each stack is the same because the books in each stack are the same. Each stack of books has the same height, and the areas are the same at each cross section that is parallel to the base. Even though the second stack of books is slanted, the volumes are the same.

Example 2

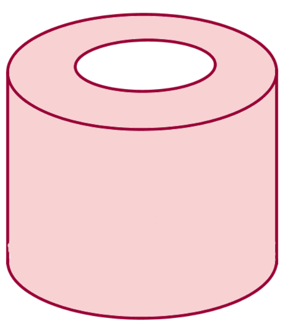
Are the volumes of the two cylinders below the same?



No. The height of the oblique cylinder will be less than its slant height of 4 inches. Because the overall height of the two cylinders is not the same, the volumes will be different. *Remember that when calculating the volume, the height you use must be perpendicular to the base.*

Example 3

A cylinder is removed from the center of a larger cylinder as shown below:



The radius of the cylinder that was removed is 3 inches. The radius of the large cylinder is 6 inches. The height of the solid is 12 inches. What is the volume of the solid that remains?

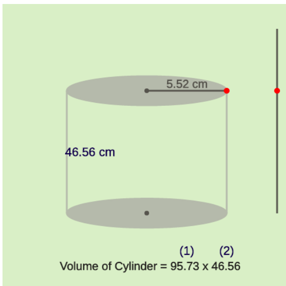
The volume of the original large cylinder is $\pi r^2 h = \pi(6^2)(12) = 432\pi\ in^3$. The volume of the cylinder that was removed is $\pi r^2 h = \pi(3^2)(12) = 108\pi\ in^3$. The volume of the remaining solid is $432\pi - 108\pi = 324\pi\ in^3$.

Example 4

How many cups of water will the solid from Example 3 hold?

Recall that a cup of water has a volume of approximately 14.44 in^3. The volume of the solid from Example 3 is $324\pi \approx 1017.876$ in^3. It will hold $\frac{1017.876}{14.44} \approx 70.5$ cups of water. One gallon of water is 16 cups, so this solid will hold approximately 4.4 gallons of water.

CK-12 PLIX Interactive



PLIX

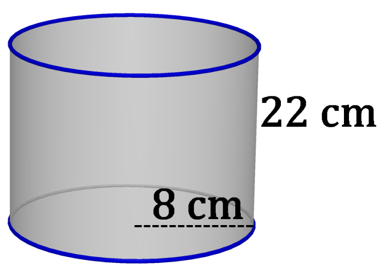
Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/cylinders/plix/Volume-of-Cylinder-Big-Can-of-Soup-545a96b28e0e08476099bd7f

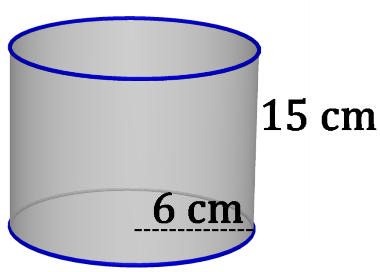
1. Explain Cavalieri's principle in your own words.
2. Explain why the volume of a cylinder with radius r and height h is $\pi r^2 h$.
3. Explain how the volume of a cylinder relates to the volume of a prism.

Find the volume of each cylinder with the given dimensions.

4.



5.



6. A cylinder with a base diameter of 15 inches and a height of 12 inches.

7. A cylinder with a base diameter of 8 centimeters and a height of 2 centimeters.

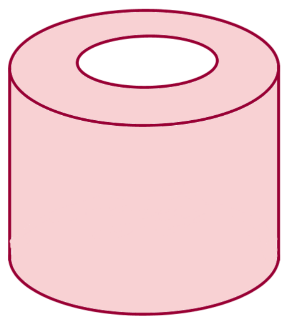
8. Find the radius of the base of a cylinder with a volume of 471.24 in^3 and a height of 6 inches.

9. Find the radius of the base of a cylinder with a volume of 1357.17 cm^3 if the height of the cylinder is twice the length of the radius.

10. Find the height of a cylinder with a base area of 25π in^2 if the volume of the cylinder is 300π in^3.

11. The label on a can of juice is missing. You want to know how many cups are in the can of juice. You measure the diameter of the base of the can and find that it is 5 inches. You measure the height of the can and find that it is 8 inches. If 14.44 in^3 is about 1 cup of liquid, how many cups of juice are in the can?

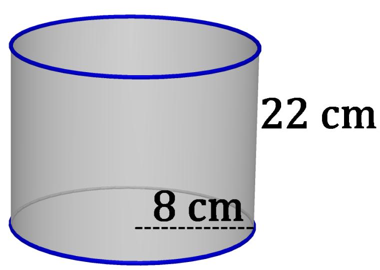
12. A cylinder has been removed from the center of another cylinder. The volume of the remaining solid is 240π in^3. If the height of the solid is 3 inches and the radius of the cylinder that was removed is 8 inches, what is the radius of the larger cylinder?



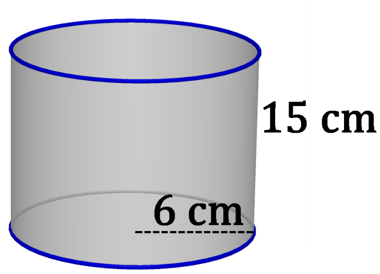
13. How much liquid will the solid from #12 hold if one cup of liquid has a volume of approximately 14.44 in^3?

1 cubic centimeter (cm^3 or cc) will hold 1 milliliter (mL) of liquid. Approximately how much liquid will each cylinder hold in liters (1 L = 1000 mL)?

14.



15.



16. Consider the following construction. The two circles shown share a radius of length 1. Describe the triangles shown. Find their area. Describe the arcs shown and find their area. Use these values to help you determine the total area of the figure.

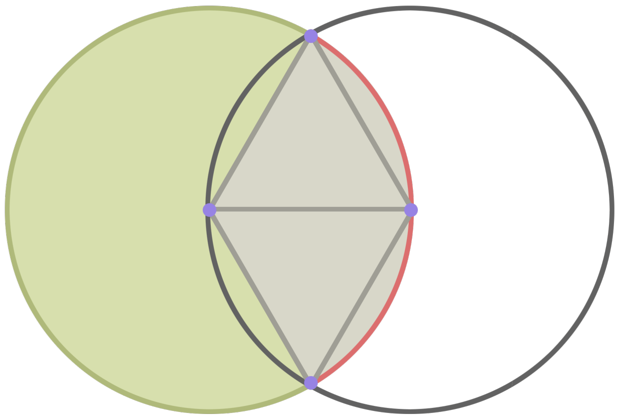


FIGURE 9.55

17. Imagine constructing a figure composed of two overlapping cylinders with a base as visualized above. Make a sketch. Find its volume and surface area.

18. Use the metaphor of a stack of coins to explain the formula for the volume of a right cylinder versus that of an oblique cylinder.

19. What is the centroid of a triangle? How does it relate to triangles in the real world? Given the coordinates of three vertices of a triangle, how can we find its center of mass? Create a triangle in the coordinate plane and find its center of mass.

20. The centroid of a parallelogram is also the center of mass, and it is found using the same process used above. Below is an oblique cylinder, not drawn to scale. Sketch a top view of a single "disc" or "coin" taken from the cylinder. In each disc, is the volume symmetrically distributed? What is the center of mass of each disc? Now sketch a side view of the cylinder, so that the bases only appear as segments. What shape is this? Where is its center of mass? Does this center of mass apply to the cylinder? Why or why not?

21. From the diagrams made for the previous problem, place the parallelogram on a coordinate plane, such that one vertex is at the origin. Find the center of mass of the parallelogram. If this cylinder is placed on its base, will it tip over or not? Why or why not? How does the center of mass relate to the intersection point of the diagonals of the parallelogram? Explain.

22. Sketch an oblique cylinder with radius 3 and height 4 and find its dimensions such that the center of mass is precisely at the tipping point.

23. The Leaning Tower of Pisa is not an oblique prism. Why not? Nevertheless, we will use an oblique cylinder as our model. A sketch is below–it's not to scale. The tower has a slant height of roughly 57 meters and a base radius of roughly 8 meters. The tower tilts at an angle of about 4°. How far does the top lean over the bottom? Where is the tower's center of mass? Will it topple? If we assume the tilt of the tower can increase without increasing the slant height, what is the maximum tilt the tower can withstand before toppling?

24. Try to sketch the net for an oblique pyramid. Cut the ends of a toilet paper roll so that it is a model of an oblique prism, then cut it vertically. What function might model the curving top or bottom of the lateral area? What dimension of the oblique cylinder changes the amplitude of this function? Explain.

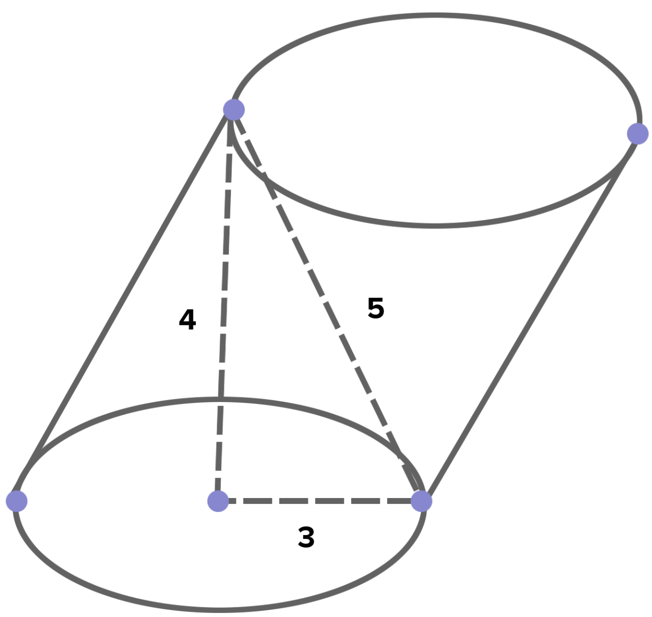


FIGURE 9.56

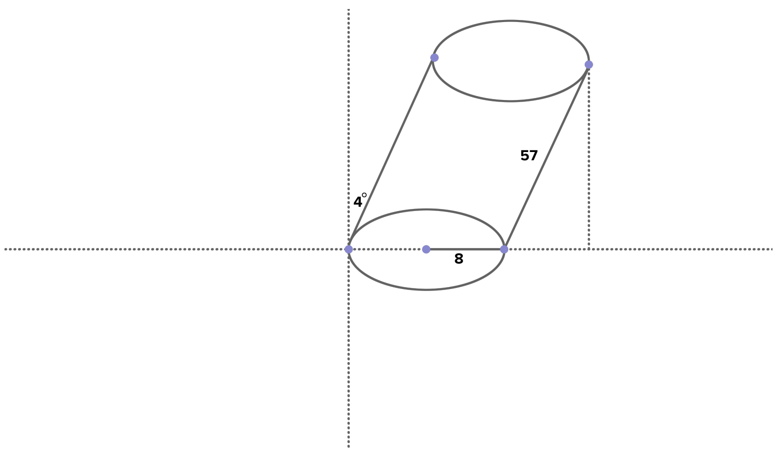


FIGURE 9.57

Review (Answers)

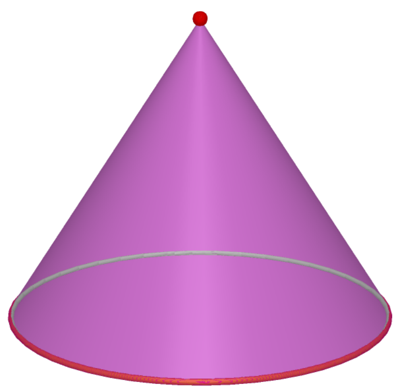
To see the Review answers, open PDF file and look for section 9.1.

9.6 Pyramids and Cones

Learning Objectives

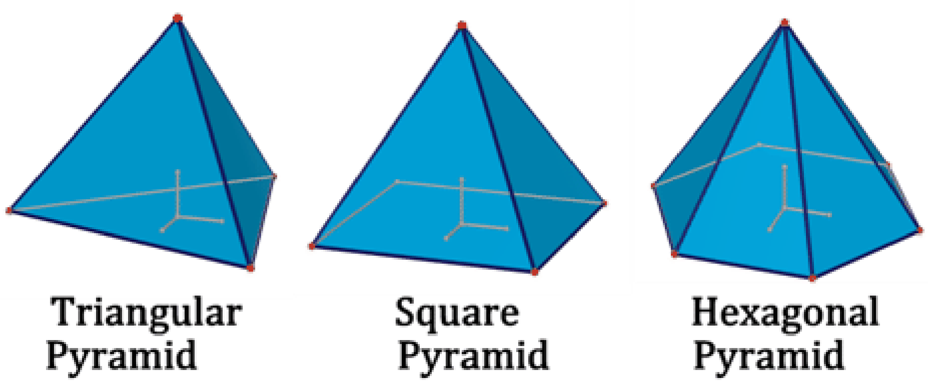
Here you will derive and use the formulas for the volumes of pyramids and cones.

The volume of a pyramid is given by $V = \frac{A_{Base} \cdot h}{3}$. How does this formula help you find the formula for the volume of a cone?



Pyramids and Cones

Recall that a **pyramid** is a solid with a polygon base and triangular lateral faces that meet in a vertex. Pyramids are named by their base shape.



You have seen the formula for the volume of a pyramid before.

$$Pyramid : V = \frac{A_{Base} \cdot h}{3}$$

MEDIA

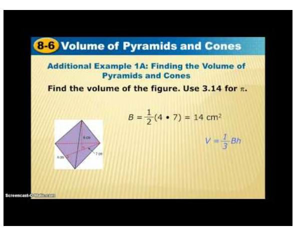
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/227835

Where does this formula come from? Recall that to find the volume of a prism or a cylinder, you need to find the area of the base and multiply by the height.

$$Prism \text{ or } Cylinder : V = A_{Base} \cdot h$$

The difference between these two formulas is the division by 3. The key to understanding where the 3 comes from is remembering Cavalieri's principle and investigating a square based pyramid.



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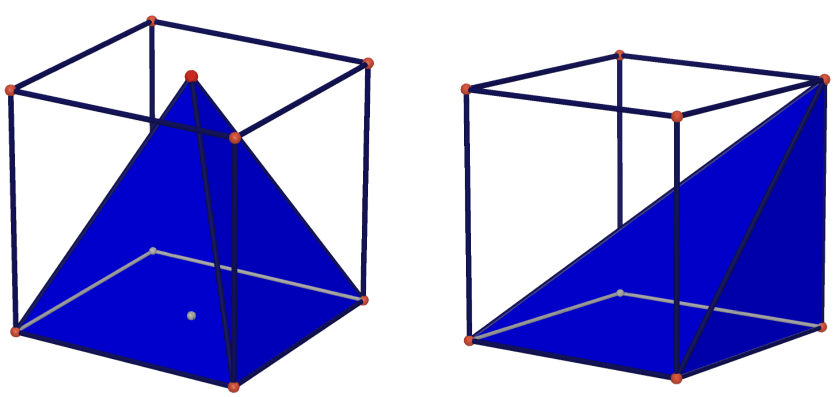
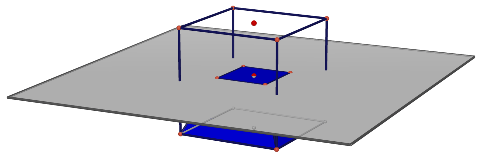
Let's take a look at some problems involving pyramids.

1. The two square pyramids below are each constructed within cubes of the same size. The pyramid on the left has a vertex at the center of a face of the cube. The pyramid on the right has a vertex at one of the vertices of the cube. Is the volume of each pyramid the same?



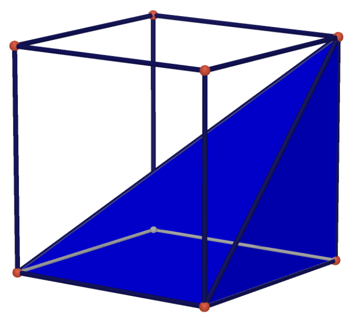
Because each pyramid is constructed within the same cube, the heights of the pyramids are the same and the areas of their bases are the same. When a plane parallel to the base of the pyramid is constructed through the center of each cube, the cross sections of each pyramid are the same.



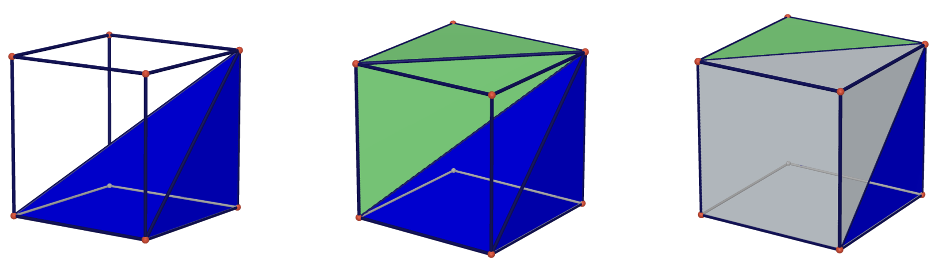
Each cross section is a square. The length of the side of the square is half the length of an edge of the original cube, since the plane was constructed through the middle of the cube.

Because cross sections are the same area, heights are the same, and bases are the same, these pyramids must have the same volume due to Cavalieri's principle.

2. How many square based pyramids congruent to the one below would it take to fill the cube? Can you visualize this?



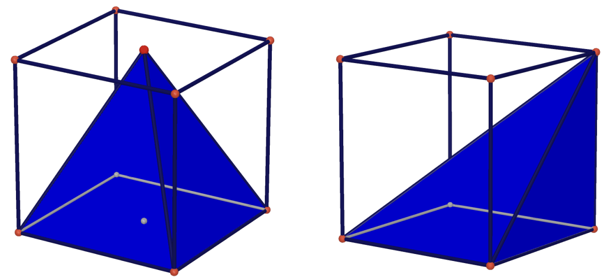
It will take exactly 3 congruent pyramids to fill the cube. The image below shows each pyramid being added to the cube.



3. Use the answers from the previous problems to explain why the volume of a pyramid is $V = \frac{A_{Base} \cdot h}{3}$.

The volume of a rectangular prism is $A_{Base} \cdot h$. This is because A_{Base} gives the volume of one "layer", and multiplying by the height scales that base volume by the number of "layers" of the prism.

Three congruent pyramids fit inside the cube in #2, so the volume of each pyramid must be $\frac{1}{3}$ the volume of the cube. Therefore, the volume of a pyramid is $\frac{A_{Base} \cdot h}{3}$. Remember that pyramids with the same base area and height will have the same volume due to Cavalieri's principle, so both of the pyramids below will have a volume of $\frac{A_{Base} \cdot h}{3}$.



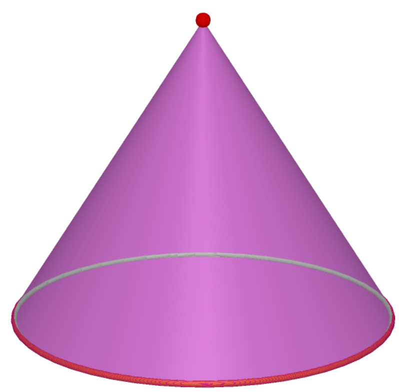
Note: This is an informal argument for the formula for the volume of a pyramid. A rigorous derivation of the formula that considers pyramids of any base shape will be developed in calculus.

Examples

Example 1

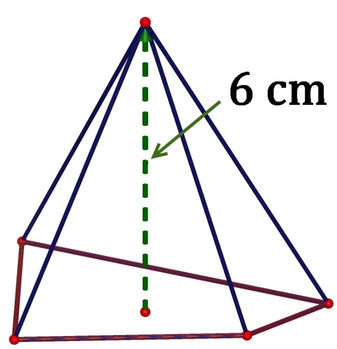
Earlier, you were asked how the formula $V = \frac{A_{Base} \cdot h}{3}$ helps you find the formula for the volume of a cone.

A cone is essentially a pyramid with a circular base. The volume of a pyramid is given by $V = \frac{A_{Base} \cdot h}{3}$. Since the area of the base of a cone is πr^2, the formula for the volume of a cone is $V = \frac{\pi r^2 h}{3}$.



Example 2

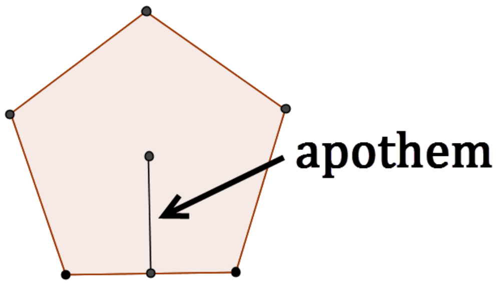
The area of the base of the pyramid below is 40 cm^2. What is the volume of the pyramid?



$V = \frac{A_{Base} \cdot h}{3} = \frac{(40)(6)}{3} = 80\ cm^3$

Example 3

The **apothem** of a regular polygon is a perpendicular segment from the center point of the polygon to the midpoint of one of its sides (see image below). Find the volume of a pyramid with a height of 10 inches and a regular pentagon base with an apothem of 1 inch.



$V = \frac{A_{Base} \cdot h}{3}$. To find the area of the base, divide the pentagon into five congruent triangles. The apothem is the height of each of these triangles. Use trigonometry to find the base length of each of these triangles.

Example 4

Find the volume of a cone with a height of 10 inches and a radius of 1 inch.

$V = \frac{\pi r^2 h}{3} = \frac{10\pi}{3}\ in^3 \approx 10.47\ in^3$

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/volume-of-cones/plix/Volume-of-Cones-Strawberry-Ice-Cream-54650b728e0e080b5dcb12d8

1. Explain the connections between a prism and a pyramid. Why do you divide by three when calculating the volume of a pyramid?

2. Explain the connections between a cone and a cylinder. Why do you divide by three when calculating the volume of a cone?

Find the volume of each solid based on its description.

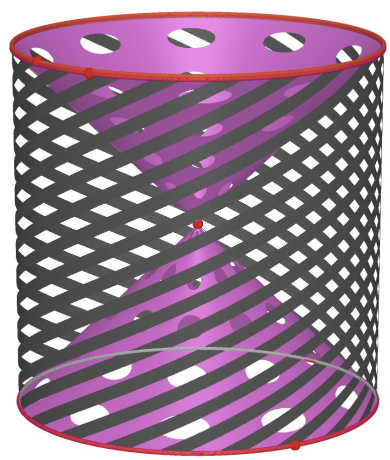
3. A cone with a diameter of 4 inches and a height of 12 inches.

4. A pyramid with a height of 15 inches and a regular hexagon base with an apothem of 4 inches.

5. A cone with a radius of 8 centimeters and a height of 15 centimeters.

6. A square based pyramid with a vertex in the center of the square such that each triangular face has a base of 12 inches and a height of 10 inches.

An hourglass is created by placing two congruent cones inside of a cylinder with the same base area. The radius is 5 inches and the height of the cylinder is 20 inches.



7. Find the volume of one of the cones.

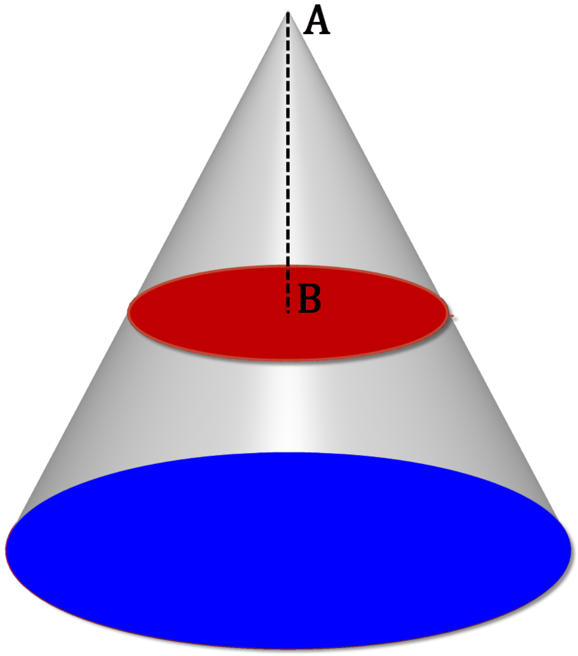
8. Find the volume of the cylinder.

9. Find the volume of the space between the cones and the cylinder.

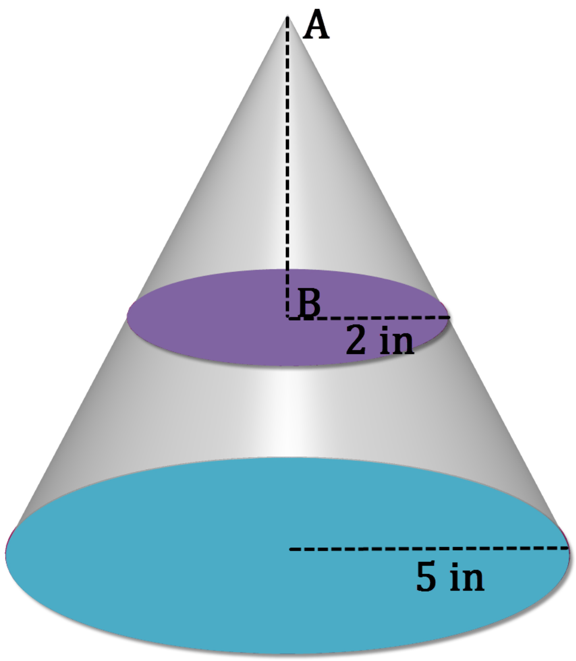
10. You want to fill one of the cones with a thick liquid. If one cup of liquid has a volume of approximately 14.44 in^3, how much liquid will you need to fill one of the cones?

11. A cone and a square pyramid have the same volume and height. The volume of each solid is 100 cm^3. If the radius of the cone is 2.82 centimeters, what is the length of a side of the base of the pyramid?

12. The ratio of the area of the red circle to the area of the base is 1:9. If the height of the cone is 15 inches, what is the length of $\overline{AB}$?



13. The height of the cone below is 10 inches. Find the length of $\overline{AB}$.



14. A tetrahedron is a pyramid with four congruent equilateral faces. If the area of each of the faces of a tetrahedron is $9\sqrt{3}\ in^2$, what is the volume of the tetrahedron?

15. The length of each side of the triangular faces making up a tetrahedron is s. What is the volume of the tetrahedron in terms of s?

16. Mount Fuji in Japan is roughly a cone. It has a base whose circumference is roughly 78 miles and a height of roughly 8,800 feet. Sketch the scenario. Find the volume and surface area of Mt. Fuji. How much greater is the surface area than a flat land of equal circumference? Compare these values both in absolute terms and as a

percentage. If one were to travel in a straight line from the base to the summit, what would the angle of elevation be?

17. The Great Pyramid of Giza has a height of roughly 756 feet, and a square base with a side length of roughly 455 feet. What is the volume and lateral area of the Great Pyramid? According to scholars, the original height of the pyramid was 481 feet, but the surface has eroded. Can you find an approximation of the original lateral area and volume? Explain your choices. What has been the absolute and percentage decrease in surface area and volume?

Review (Answers)

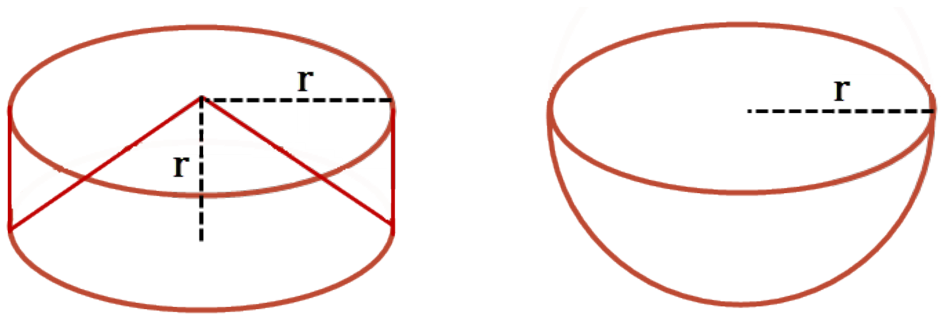
To see the Review answers, open this PDF file and look for section 9.2.

9.7 Spheres

Learning Objectives

Here you will derive and use the formula for the volume of a sphere.

It can be shown that the volume of the space between a cone and a cylinder with radius r and height r is the same as the volume of half a sphere (a hemisphere) with radius r. Given this, what's the formula for the volume of a sphere?



Spheres

Recall that a **sphere** is the set of all points in space that are equidistant from a given point. The distance from the center of a sphere to any point on the sphere is the radius.

MEDIA

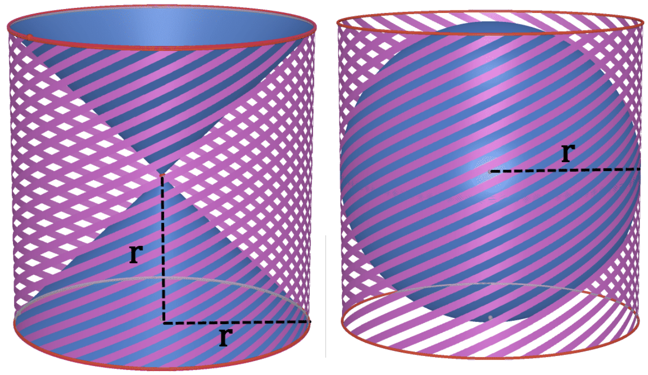
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/227839

You have seen the formula for the volume of a sphere before.

$$Sphere: \ V = \frac{4}{3}\pi r^3$$

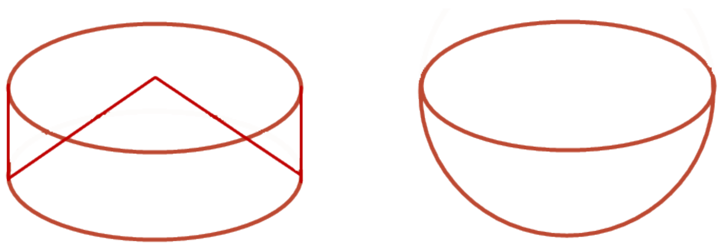
The key to understanding the formula for the volume of a sphere is to compare a sphere inside of a cylinder with radius r and height $2r$ with a double cone inside of a cylinder with radius r and height $2r$.

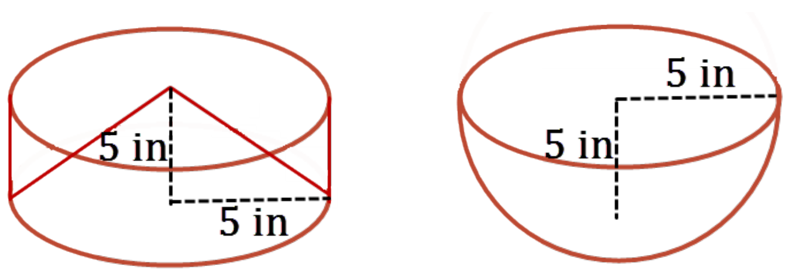


The volume of the sphere is the same as the volume of the **space between** the cylinder and the double cone.

Finding Area

Consider half of the double cone inside the cylinder and half the sphere (a hemisphere). If the radius of the sphere is 5 inches, label all the dimensions that you can. What is the area of the top surface of the hemisphere? What is the area of the top of the cylinder?

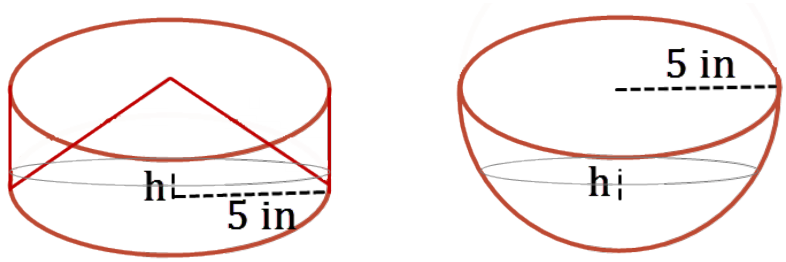




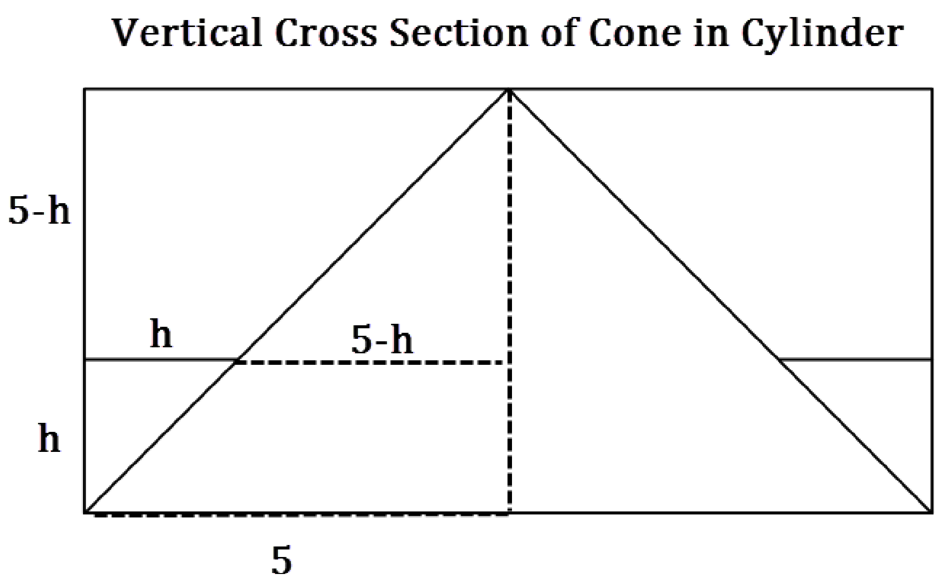
The area of the top of the cylinder is the same as the area of the top of the hemisphere. In each case the area is $\pi(5^2) = 25\pi\ in^2$.

Labeling Dimensions

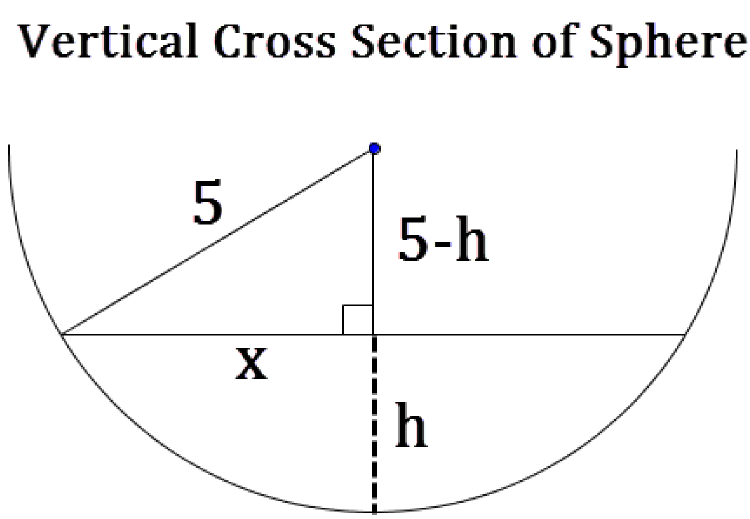
Imagine a slice is made at height h. Draw the horizontal and vertical cross sections of each and label all dimensions in terms of h.



First consider the vertical cross section through the center of the cone in the cylinder.



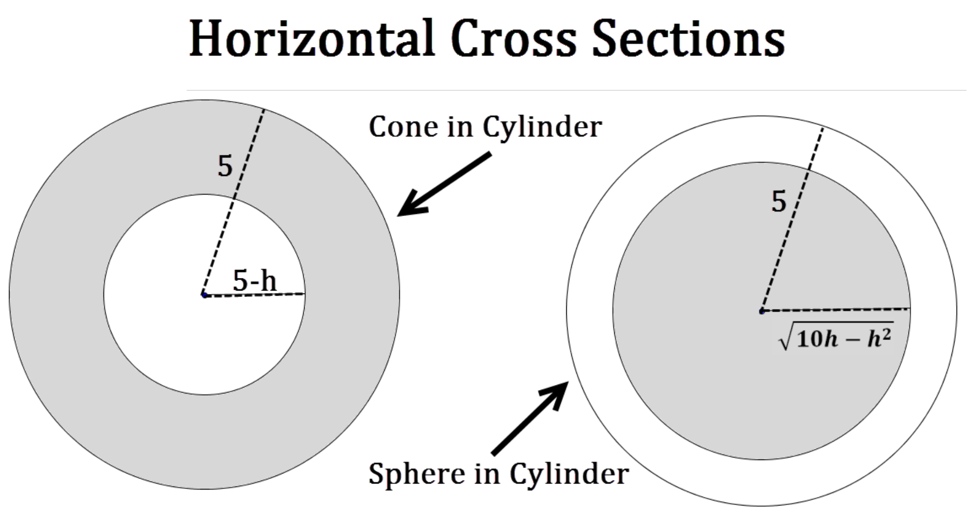
Note that because the height and the radius of the cone are each 5 inches, isosceles right triangles are formed. Next consider the vertical cross section through the center of the sphere.



The radius of the circular horizontal cross section is x. You can find the length of x by using the Pythagorean Theorem:

$$x^2 + (5-h)^2 = 5^2$$
$$x^2 + 25 - 10h + h^2 = 25$$
$$x = \sqrt{10h - h^2}$$

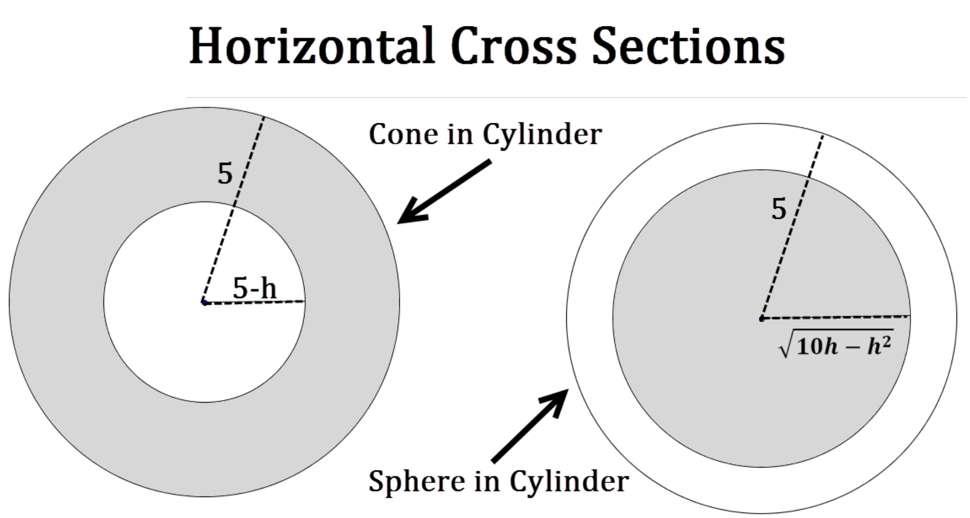
Now you can consider the horizontal cross sections.



Note that the shaded areas are the cross sections of the solids you are interested in. You are looking for the volume of the space **between** the cylinder and the cone and the volume of the sphere.

Drawing Conclusions

Confirm that the area of each shaded region below is the same. What does this tell you about the volume of the space between the cylinder and the cone compared to the volume of the sphere?



The area of the shaded region on the left is:

$$\begin{aligned} A &= 25\pi - (5-h)^2\pi \\ &= 10h\pi - h^2\pi \end{aligned}$$

The area of the shaded region on the right is:

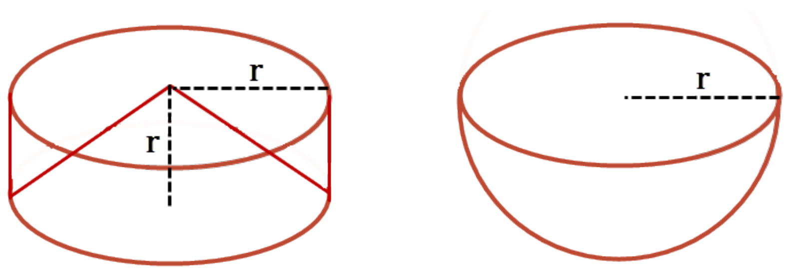
$$\begin{aligned} A &= \left(\sqrt{10h-h^2}\right)^2\pi \\ &= 10h\pi - h^2\pi \end{aligned}$$

The areas are the same. Because the two solids lie between parallel planes, have the same heights, and have equal cross sectional areas, **their volumes must be the same.**

Examples

Example 1

Earlier, you were asked about the formula for the volume of a sphere.



The volume of the cylinder is πr^3. The volume of the cone is $\frac{\pi r^3}{3}$. Therefore, the volume of the space between the cone and the cylinder is:

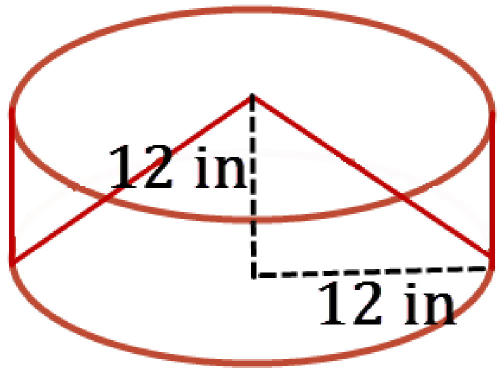
$$\pi r^3 - \frac{\pi r^3}{3} = \frac{2\pi r^3}{3}$$

If this is also the volume of a hemisphere, then the volume of a sphere must be twice as big. The volume of a sphere is:

$$\frac{2(2\pi r^3)}{3} = \frac{4\pi r^3}{3}$$

Example 2

Find the volume of the space between the cylinder and the cone below.



The volume of the cylinder is $1728\pi\ in^3$ and the volume of the cone is $576\pi\ in^3$. The volume of the space between the cylinder and the cone is $1152\pi\ in^3$.

Example 3

Describe what portion of a sphere has the same volume as the volume calculated in #2.

A hemisphere with radius 12 *in* would have the same volume.

Example 4

Find the volume of a sphere with a diameter of 15 *cm*.

$V = \frac{4\pi(7.5)^3}{3} = 562.5\pi\ cm^3$

CK-12 PLIX Interactive

PLIX

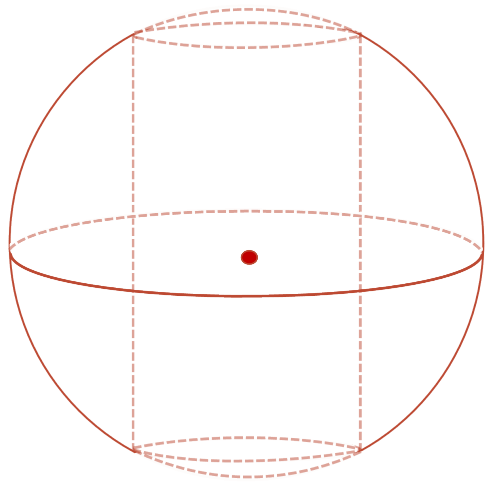
Click image to the left or use the URL below.

URL: http://www.ck12.org/geometry/spheres/plix/Surface-Area-and-Volume-of-a-Sports-Ball-5412421e8e0e08327a9485e5

1. Find the volume of a sphere with a radius of 3 inches.
2. Find the volume of a sphere with a diameter of 12 inches.
3. Find the volume of a sphere with a diameter of 8 inches.

4. In your own words, explain where the formula for the volume of a sphere came from. How does it relate to a double cone within a cylinder?

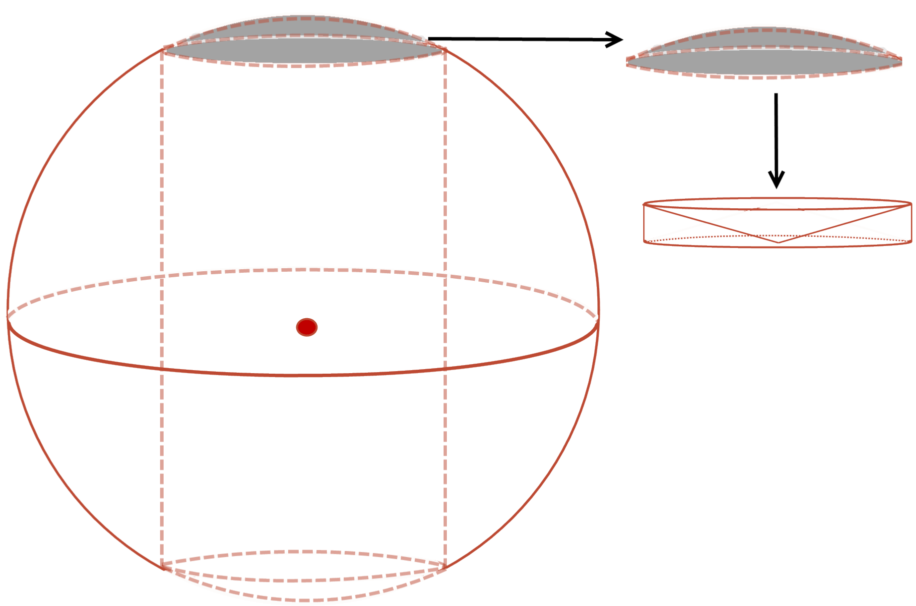
A bead is created from a sphere by drilling a cylinder through the sphere. The original sphere has a radius of 8 mm. The cylinder drilled through the center has a radius of 4 mm.



5. What is the height of the bead? (Hint: Draw a right triangle and use the Pythagorean Theorem.)

6. What is the volume of the original sphere? What is the volume of the cylinder?

7. Due to Cavalieri's principle, the volume of the space above the cylinder is the same as the volume between a cone and a cylinder (see picture below). What is the approximate volume of the space above and below the cylinder that was cut off when making the bead?



8. What is the approximate volume of the bead?

A cylindrical container holds three tennis balls. The diameter of the cylinder is 4 inches, which is approximately the same as the diameter of each tennis ball. The height of the cylinder is 12 inches.

9. What is the volume of one tennis ball?

10. What is the volume of the space between the tennis balls and the cylinder?

11. If one cup of water has a volume of approximately 14.44 in^3, how many cups of water would fit in the cylinder with the tennis balls?

Think of a sphere of radius r as being made up of a large number k of congruent small square based pyramids. Let the area of each square base be B.

12. What is the volume of one pyramid in terms of r, k, and B?

13. What is the surface area of the sphere in terms of k and B?

14. What is the volume of the sphere in terms of r, k, and B?

15. Use your answers to #13 and #14 and the formula for the volume of a sphere $\left(V = \frac{4\pi r^3}{3}\right)$ to find the formula for the surface area of a sphere.

16. The diameter of the earth is approximately 7,917.5 miles. The diameter of the moon is about 2,159 miles. Compare their volume and surface area in absolute and percentage terms. The diameter of the sun is roughly 864,575.9 miles. Compare the volume and surface area of the sun to that of the earth in absolute and percentage terms.

17. If the diameter of the moon doubled, how would its surface area change? How would its volume change? Explain.

Review (Answers)

To see the Review answers, open this PDF file and look for section 9.3.

9.8 Modeling in Three Dimensions

Learning Objectives

Here you will use geometric solids to model and answer questions about real life objects.

Warm-Up

Geometric shapes can be modeled with equations and 2-dimensional pictures. Use the interactive below, to explore connections between a 2-dimensional net and a 3-dimensional box. Later in this section, you will use equations with variables to solve real world problems that use 2 and 3-dimensional objects.

Multimedia

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/227857

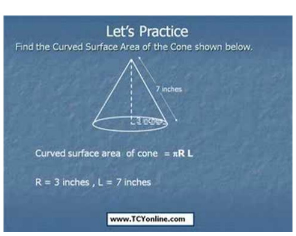
Mark wants to make an open box from an 8.5 inch by 11 inch piece of paper by cutting squares out of each corner, folding up the sides, and securing with tape. How does the volume of his box relate to the size of the squares he cuts out?

Modeling in Three Dimensions

You live in a three dimensional world. Look around and observe what you see over the course of your day. Can you find examples of prisms? Cylinders? Pyramids? Cones? Spheres?

While most objects in your daily life are not ***perfect*** prisms, pyramids, cylinders, cones, or spheres. Most are close to one of these five solids or a combination of these solids. Modeling in three dimensions is about being able to choose the best solid to help analyze a real world three dimensional situation, and then using your geometry knowledge to make decisions about the real life situation. You should ask yourself:

- What solid or solids are the best model of this real life object?
- What problems am I trying to solve or decisions am I trying to make about the real life object?
- What information about the real life object am I given and where does it fit in my model?



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/77846

Real-World Application: Chopping Trees

A big storm causes a large tree to fall in your yard. The main portion of the tree trunk measures about 9 feet around and is 40 feet long. You plan to chop up the tree to use and sell as fire wood. Approximately what volume of wood will you get from the tree?

A tree trunk is best modeled by a cylinder. Here, you are looking for the approximate volume of the cylinder. Two pieces of information are given.

1. *"The main portion of the tree trunk measures about 9 feet around"* - This is the circumference of the base of the cylinder. You can use this measurement to find the radius of the cylinder.

$$2\pi r = 9 \rightarrow r = \frac{9}{2\pi} \rightarrow r \approx 1.43 \text{ ft}$$

2. "40 feet long" - This is the height of the cylinder.

$$h = 40 \text{ ft}$$

The volume of the wood from the tree is approximately:

$$V = \pi r^2 h = \pi(1.43^2)(40) \approx 257 \text{ ft}^3$$

Real-World Application: Buying Wood

Wood is commonly sold in cords. A cord of wood is a stack of tightly packed wood that measures 4 feet by 4 feet by 8 feet. Approximately how many cords of wood will the chopped tree from the first problem produce?

A cord of wood is best modeled by a rectangular prism. $V = A_{Base} \cdot h$, so $V = (4 \cdot 4) \cdot 8 = 128 \text{ ft}^3$. Each cord of wood is approximately 128 cubic feet of wood. Since the tree from the first problem produced 257 cubic feet of wood, this is $\frac{257}{128} \approx 2$ cords of wood.

Now, let's come up with an equation that relates the length around a tree in feet, the height of a tree in feet, and the approximate number of cords of wood that a tree will produce.

You want to come up with an equation that takes an input of circumference and height and produces an output of cords of wood. Think back to the steps taken in Examples A and B and repeat these steps with variables for circumference and height instead of specific values.

- Let $C =$ distance around the tree
- Let $h =$ height of tree

Use the distance around the tree to find the radius:

$$C = 2\pi r \rightarrow r = \frac{C}{2\pi}$$

The volume of the wood from the tree is:

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi \left(\frac{C}{2\pi}\right)^2 h \\ &= \frac{C^2 h}{4\pi} \end{aligned}$$

Once you have the volume of the wood from a given tree, to find the number of cords of wood divide the volume by 128 ft^3, which is the number of cubic feet in a cord of wood.

$$\text{Number of Cords} = \frac{V}{128} = \frac{\frac{C^2h}{4\pi}}{128} = \frac{C^2h}{512\pi}$$

Test this formula using the original information from the first problem about chopping trees to see if you get the correct answer to the second problem. In this first problem, $C = 9$ ft and $h = 40$ ft.

$$\text{Number of Cords} = \frac{C^2h}{512\pi} = \frac{(9^2)(40)}{512\pi} \approx 2.01$$

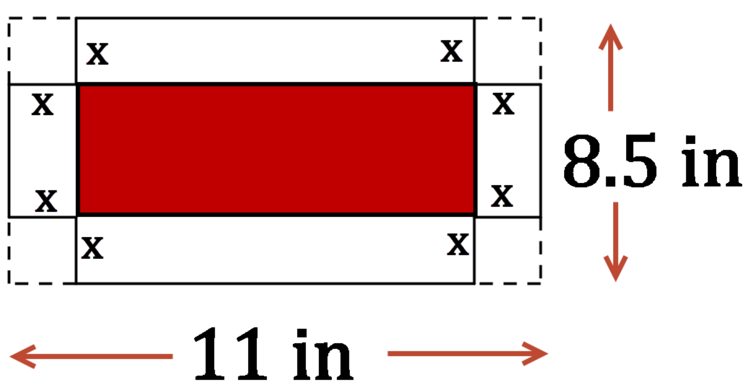
This matches the answer you found the first time, so you can feel confident that your equation is correct.

Examples

Example 1

Earlier, you were asked how the volume of Mark's box relates to the size of the squares he cut out.

Let the length of the side of the square that Mark cuts out of each corner be x. The portion of the paper that will become the base of the box once it is made is shaded below in red.



The box is a rectangular prism. The volume of the box is therefore $V = A_{Base} \cdot h$.

- $A_{Base} = (11 - 2x)(8.5 - 2x) = 4x^2 - 39x + 93.5$
- $h = x$

Therefore, the volume of the box in terms of the size of the square is:

$$V = x(4x^2 - 39x + 93.5) = 4x^3 - 39x^2 + 93.5x$$

Mark can use this formula to determine the volume of the box given the length of the side of the squares he cuts out. For example, if he cuts out squares that are 2 inch by 2 inch, then $x = 2$. The volume of the box would be:

$$V = 4(2^3) - 39(2^2) + 93.5(2) = 63 \text{ in}^3$$

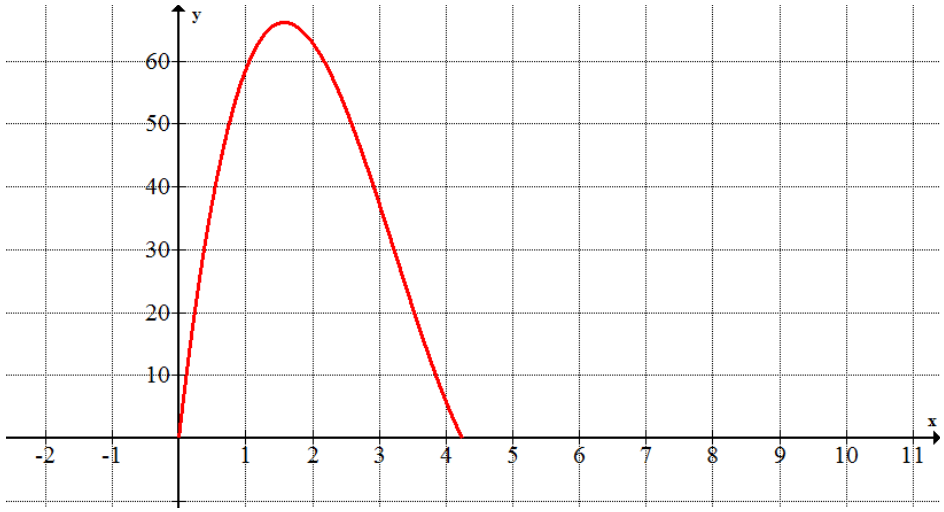
MEDIA
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/219985

Example 2

Graph the equation $y = 4x^3 - 39x^2 + 93.5x$ with a graphing calculator. What do the points on this graph represent? What portion of this graph is relevant to this problem?

The points on the graph represent the volume of the box given the length of the side of each square cut out.

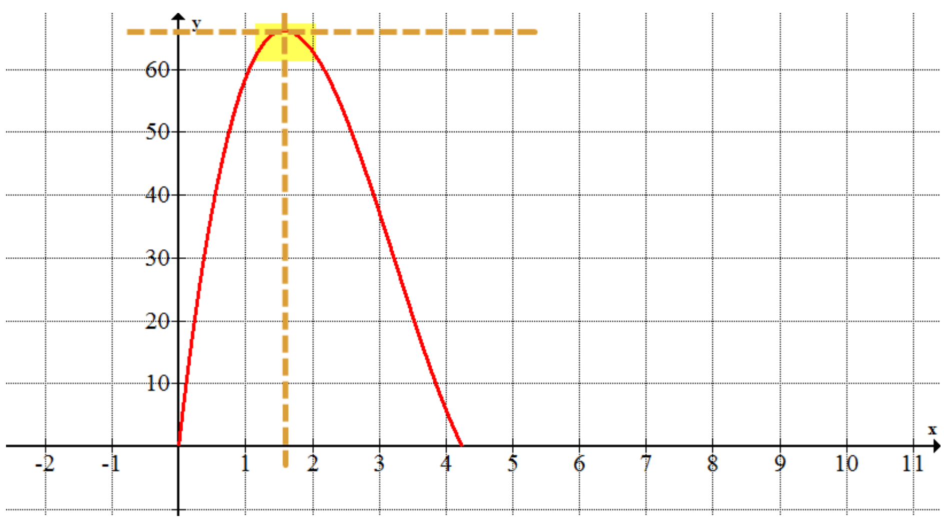
Because Mark can't cut out a square with a negative side length or a square with a side length greater than 4.25 inches (because the paper is only 8.5 inches wide), the portion of the graph that is relevant is the portion with x values between 0 and 4.25.



Example 3

Approximately what size squares will maximize the volume of the box (cause the box to have the greatest possible volume)? How does the graph above help you to answer this question?

The maximum volume looks to occur with squares that are approximately 1.6 inches by 1.6 inches. The volume at that point looks to be around 66 in^3. The graph helps to answer this question because the peak on the graph is where the maximum volume occurs.



Example 4

Does the size of square that maximizes volume also maximize surface area of the box? Explain.

The surface area of the open box will be the area of the unfolded box (the net). The more you cut out of the paper, the smaller the surface area. Therefore, the size of square that maximizes volume does not also maximize surface area of the box.

CK-12 PLIX Interactive

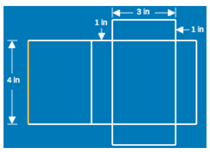
PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/calculus/absolute-extrema-and-optimization/plix/Optimization-Building-the-Biggest-Box-556f5f895aa41373fbbc0458

Discussion Question

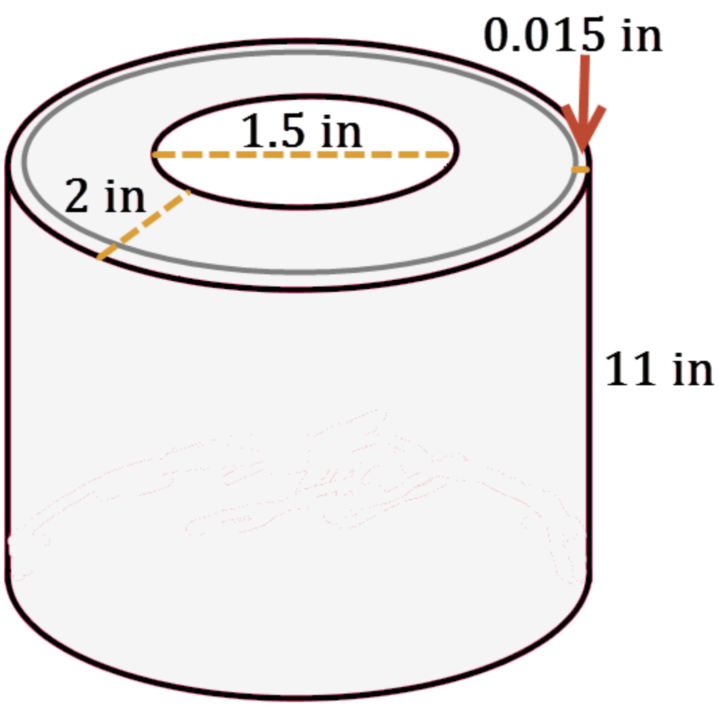
Try unfolding a cereal box. What is the surface area and volume of the box? What changes would you make to the cut-outs of the net to maximize the volume of the box?



Share your thoughts with others in the Math Corner.

Review

An 11 inch tall roll of paper towels has an inner cardboard tube with a diameter of 1.5 inches. The width of the paper towel on the roll is 2 inches and each paper towel is 0.015 inches thick.

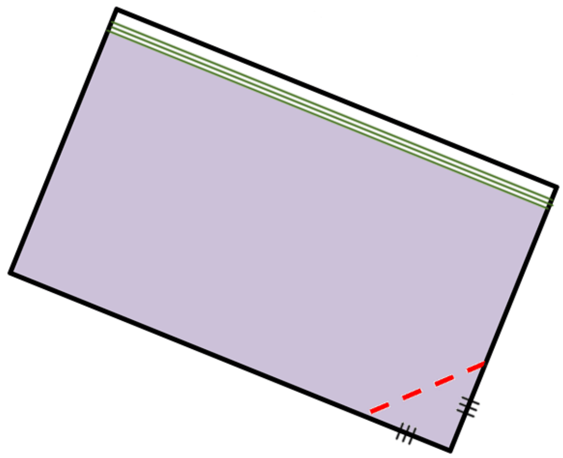


1. What is the volume of paper towels? Will the volume change if the roll of paper towels is unrolled?

2. If the whole roll of paper towels is unrolled, how long will the chain of unrolled paper towels be? [Hint: Use your answer to #1 to help]

3. Come up with an equation that generalizes the relationship between the variables: diameter of tube, width of paper towel on roll, thickness of paper towel, and length of unrolled paper towels. Why does the height of the paper towel roll not matter in this relationship?

In order to decorate a cake with purple frosting, Sam plans to fill a zipper sandwich bag with frosting, cut off one of the tips of the bag, and squeeze the frosting out of the corner (see picture below).



4. Assuming he cuts an isosceles triangle out of the corner, what length of cut should he make to pipe frosting with a 1 centimeter diameter. In other words, what should the length of the red dotted line be?

5. Give an equation that shows the relationship between the length of the cut made and the diameter of the frosting as it comes out of the bag.

6. You are packing up yearbooks that measure 11 inches by 14 inches by 1.5 inches. You have boxes that measure 12 inches by 30 inches by 10 inches. How many books can you fit in each box?

A certain burning candle loses 7 in^3 of volume each hour.

7. If the original candle was 10 inches tall with a diameter of 3 inches, what is the volume of the candle after 3 hours?

8. Create an equation that relates the original height of the candle, the diameter of the candle, the time burned, and the current volume of the candle for all candles that lose 7 in^3 of volume per hour.

9. Mike and his friends are having a water balloon fight. Each approximately spherical balloon can hold one cup of water. If one cup of water has a volume of approximately 14.44 in^3, what is the radius of each filled balloon?

Marissa works at an ice cream shop. The sugar cones have a diameter of 2 inches and a height of 4 inches. For a single scoop cone, she packs the cone with ice cream and then puts a scoop on top. For a double scoop cone, she packs the cone with ice cream and then puts two scoops on top (see sketch below). For a triple scoop cone, she packs the cone with ice cream and then puts three scoops on top. One cup of ice cream has a volume of 14.44 in^3.

10. Approximately how much ice cream (in cups) does a single scoop cone get?

11. Approximately how much ice cream (in cups) does a double scoop cone get?

12. Approximately how much ice cream (in cups) does a triple scoop cone get?

13. Describe at least two reasons why your answers to the previous three questions are only approximate.

14. If a single scoop cone costs \$2.50 and a double scoop cone costs \$3.50. Which type of cone is a better deal considering cost per cup of ice cream?

15. If a single scoop cone costs \$2.50, what should a triple scoop cone cost if it is fairly priced?

FIGURE 9.58

16. Consider the cone within the sphere below. The sphere has a radius of 1. Two possible configurations for the cone are shown. The base of the cone is always a circle on the sphere. Imagine changing $\angle DAC$ such that the cone gets taller and thinner, or shorter and fatter. What measure of $\angle DAC$ maximizes the volume of the cone? Create an equation for the volume of the cone as a function of the angle. What range of values for the angle are acceptable in this scenario? Graph this function using a graphing calculator, and approximate the maximum. Sketch the result.

FIGURE 9.59

Review (Answers)

To see the Review answers, open this PDF file and look for section 9.5.

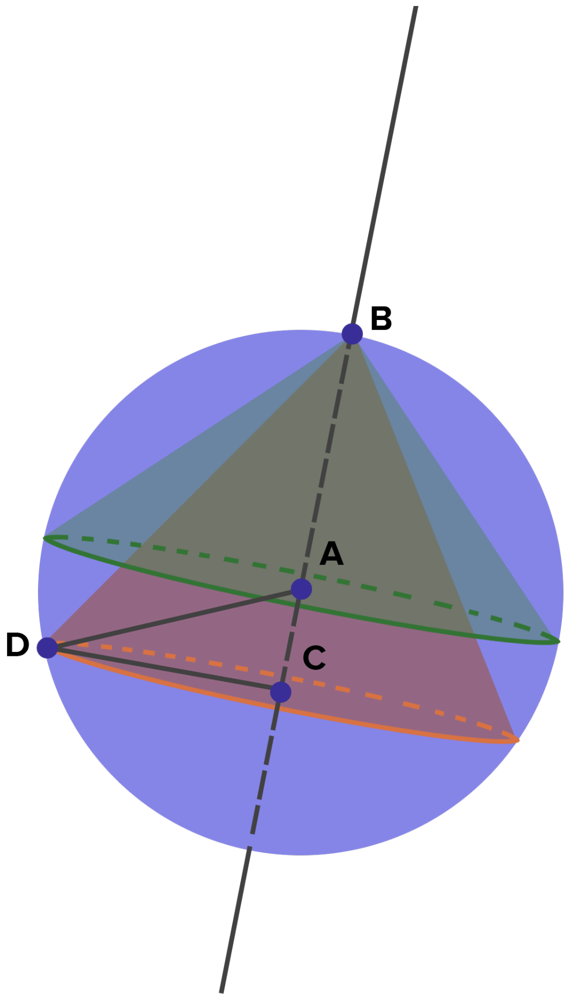


FIGURE 9.60

9.9 References

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CHAPTER 10 Applications of Probability

Chapter Outline

10.1 Descriptions of Events

Learning Objectives

S.CP.1

Here you will learn to describe events of sample spaces with words and with diagrams. You will also learn to identify and describe complements, intersections, and unions of events.

Events

In the context of probability, an **experiment** is any occurrence that can be observed. For example, *rolling a pair of dice and finding the sum of the numbers* is an experiment.

An **outcome** is one possible result of the experiment. So, *for the experiment of rolling a pair of dice and finding the sum of the numbers, one outcome is a "7" and a second outcome is an "11"*.

Every experiment has one or more outcomes. The **sample space,** S**,** of an experiment is the set of all possible outcomes. *For the experiment of rolling a pair of dice and finding the sum of the numbers, the sample space is* $S = \{2,3,4,5,6,7,8,9,10,11,12\}$.

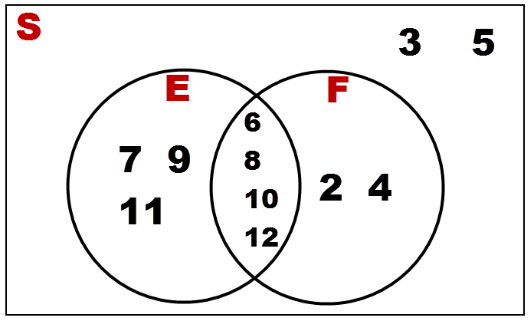
Often there is one or more outcomes that you are particularly interested in. *For example, perhaps you are interested in the sum of the numbers on the dice being greater than five*. The **event,** E**,** is a subset of the sample space that includes all of the outcomes you are interested in (sometimes called the **favorable outcomes**). *If E is the sum of the numbers on the dice being greater than five,* $E = \{6,7,8,9,10,11,12\}$. There are many possible events that could be considered for any given experiment.

There are three common operations to consider with one or more events, shown in the table below. *Consider the experiment of rolling a pair of dice and finding the sum of the numbers on the dice. Let E be the event that the sum of the numbers is greater than five.* $(E = \{6,7,8,9,10,11,12\})$. *Let* F *be the event that the sum of the numbers is even* $(F = \{2,4,6,8,10,12\})$.

TABLE 10.1:

Operation	**Definition in Words**	**Pair of Dice Example**
Complement of an Event (E')	The event that includes all outcomes in the sample space NOT in event E.	$E' = \{1,2,3,4,5\}$ E' is the sum of the numbers on the dice being ***five or less***.
Union of Events $(E \cup F)$	The event that includes all outcomes in either event E, event F, **or** both.	$E \cup F = \{2,4,6,7,8,9,10,11,12\}$
Intersection of Events $(E \cap F)$	The event that includes only the outcomes that occur in both event E **and** event F.	$E \cap F = \{6,8,10,12\}$

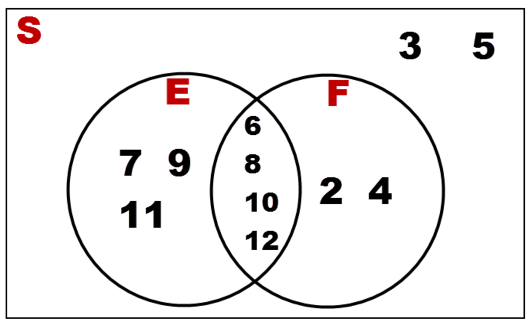
To help visualize the way different events or combinations of events interact within a sample space, consider a **Venn diagram**.



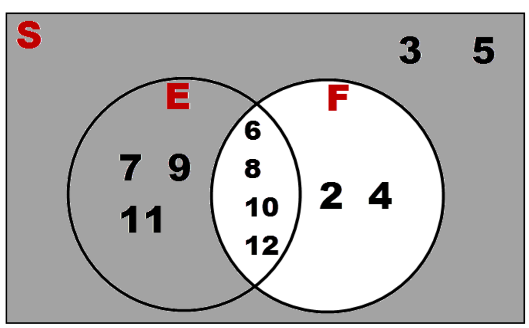
The diagram above has a big rectangle for sample space S. Within S, the outcomes 2 through 12 appear in various places. The circle labeled E represents event E, and within that circle are all the outcomes in event E. Similarly, the circle labeled F represents event F, and within that circle are all the outcomes in event F. The place where the circles overlap contains the outcomes that are in both events E and F.

Describing Events

Shade the area of the diagram below that represents F'. Describe the event F'.

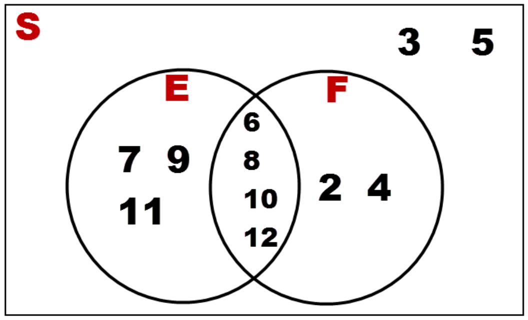


F' is the complement of event F. It contains all the outcomes in the sample space that are NOT in event F. In this case, F' is the sum of the numbers on the dice being odd.

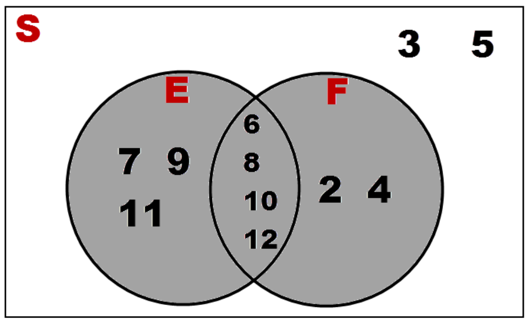


Union vs. Intersection

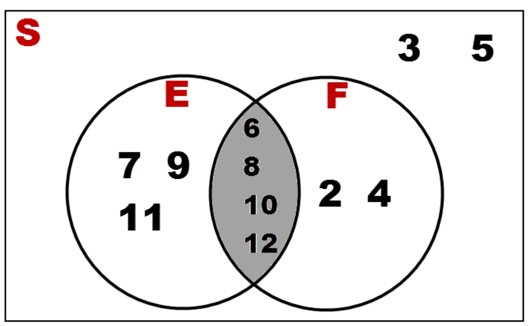
Shade the area of the diagram below that represents $E \cup F$. Then, shade the area of the diagram that represents $E \cap F$. How is the union of two events different from the intersection of two events?



$E \cup F$ is the union of events E and F. It contains all the outcomes that are in event E, event F, **or** both events E and F. *The symbol* $\cup$ *can be thought of as "or", but remember that it is not* **exclusive or**, *since it includes outcomes that are in* **both** *events.* $E \cup F$ is shown below.



$E \cap F$ is the intersection of events E and F. It contains all the outcomes that are in both events E **and** F. *The symbol* $\cap$ *can be thought of as "and", since it includes only the outcomes that are in both events.* $E \cap F$ is shown below.



Notice that $E \cup F$ will always contain the same outcomes as $E \cap F$, plus more outcomes (usually). $E \cup F$ could never contain less outcomes than $E \cap F$.

MEDIA

Click image to the left or use the URL below.

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Experimental Outcomes

Consider the experiment of tossing three coins and recording the sequence of heads and tails. Let A be the event that there are exactly two heads. Let B be the event that there are exactly two tails.

Typically when working with experiments having to do with coins, H represents getting "heads" and T represents getting "tails"

a) Find the sample space for the experiment.

The same sample is: $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

b) List the outcomes in event A.

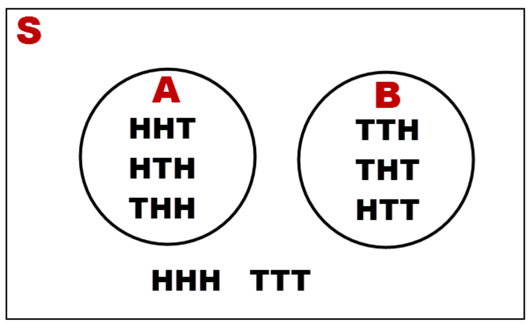
$A = \{HHT, HTH, THH\}$

c) List the outcomes in event B.

$B = \{HTT, THT, TTH\}$

d) Create a diagram that shows the sample space, events A and B, and all of the outcomes.

Notice that this time there are no outcomes that are in both A and B, so the circles don't overlap.



Examples

Example 1

You pick a card from a standard deck. Event C is choosing a card that has an even number or a spade. Rewrite event C as the combination of two events. Then, list the outcomes in event C.

There are 52 cards in a standard deck of cards. These 52 cards are organized by suit:

Clubs: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King

Diamonds: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King

Hearts: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King

Spades: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King

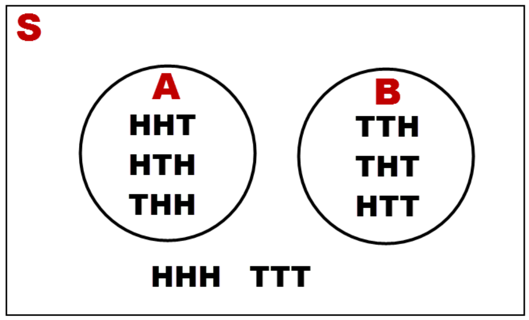
Event C is choosing a card that has an even number or a spade on it. Because the word "or" is used, you know this is really the **union** of two other events. Let D be choosing an even number and let E be choosing a spade. Then $C = D \cup E$.

$D = \{$2 of clubs, 4 of clubs, 6 of clubs, 8 of clubs, 10 of clubs, 2 of diamonds, 4 of diamonds, 6 of diamonds, 8 of diamonds, 10 of diamonds, 2 of hearts, 4 of hearts, 6 of hearts, 8 of hearts, 10 of hearts, 2 of spades, 4 of spades, 6 of spades, 8 of spades, 10 of spades$\}$

$E = \{$Ace of spades, 2 of spades, 3 of spades, 4 of spades, 5 of spades, 6 of spades, 7 of spades, 8 of spades, 9 of spades, 10 of spades, Jack of spades, Queen of spades, King of spades$\}$

$C = D \cup E = \{$Ace of spades, 2 of spades, 3 of spades, 4 of spades, 5 of spades, 6 of spades, 7 of spades, 8 of spades, 9 of spades, 10 of spades, Jack of spades, Queen of spades, King of spades, 2 of clubs, 4 of clubs, 6 of clubs, 8 of clubs, 10 of clubs, 2 of diamonds, 4 of diamonds, 6 of diamonds, 8 of diamonds, 10 of diamonds, 2 of hearts, 4 of hearts, 6 of hearts, 8 of hearts, 10 of hearts$\}$

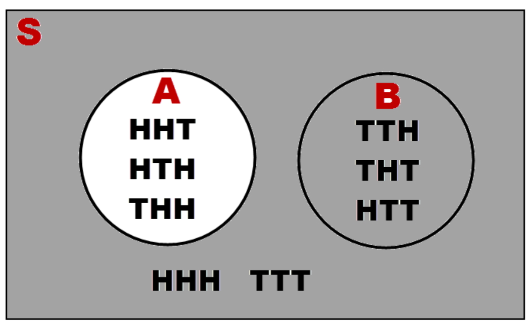
Consider the experiment of tossing three coins and recording the sequence of heads and tails. The diagram below represents the sample space and two events A and B.



Example 2

Describe A' in words and with the diagram. What outcomes are in this event?

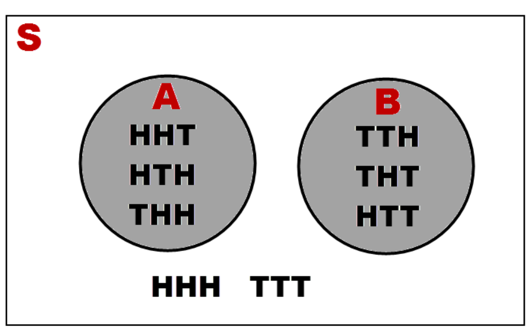
A' is the event of not getting exactly two heads. $A' = \{TTH, THT, HTT, HHH, TTT\}$ On the diagram, it is everything in the rectangle except circle A.



Example 3

Describe $(A \cup B)$ in words and with the diagram. What outcomes are in this event?

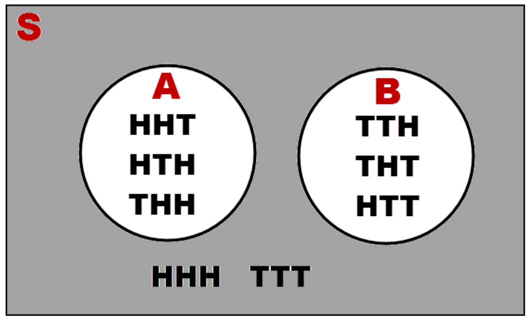
$(A \cup B)$ is all of the outcomes in event A, event B, or both. Note that for this experiment, there are no events in both A and B. $A \cup B = \{HHT, HTH, THH, TTH, THT, HTT\}$. In the diagram, only circles A and B are shaded.



Example 4

Describe $(A \cup B)'$ in words and with the diagram. What outcomes are in this event?

$(A \cup B)'$ is all of the outcomes not in $(A \cup B)$. This means it is all of the outcomes that are in neither events A nor B. $(A \cup B)' = \{HHH, TTT\}$. In the diagram, the opposite part of the rectangle is shaded compared with #2.



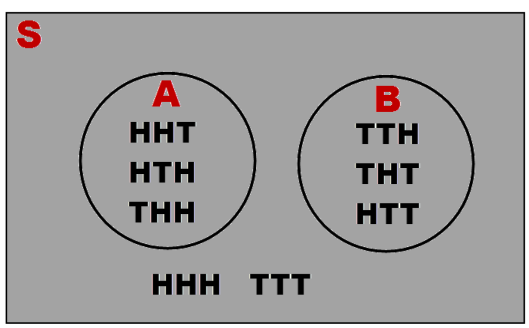
Example 5

Describe $(A \cap B)'$ in words and with the diagram. What outcomes are in this event?

$(A \cap B)'$ is all of the outcomes not in $(A \cap B)$. Remember that $A \cap B$ is all of the outcomes in both events A and B; however, in this experiment the two events don't overlap (they are **disjoint**).

$A \cap B = \{\}$, the empty set. This means that $(A \cap B)'$ must be the whole sample space, since it has to include all outcomes in the sample space not in $A \cap B$.

$(A \cap B)' = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$. In the diagram, everything in the rectangle is shaded.



CK-12 PLIX Interactive

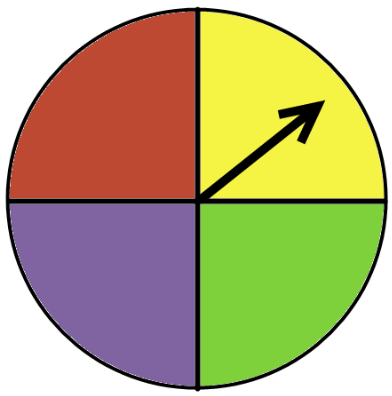
Multimedia

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/probability/venn-diagrams/plix/Venn-Diagrams-Planets-and-Dwarf-Planets-of-the-Solar-System-532cc90a5aa413301fa6e244

Consider the experiment of spinning the spinner below twice and recording the sequence of results. Let F be the event that the same color comes up twice. Let H be the event that there is at least one red.



1. Find the sample space for the experiment.

2. List the outcomes in event F and the outcomes in event H.

3. Create a Venn diagram that shows the relationships between the outcomes in F, H, and the sample space.

4. Describe F' in words and with the diagram. What outcomes are in this event?

5. Describe the event "getting two reds" with symbols and with the diagram. What outcomes are in this event?

6. Describe $(F \cup H)'$ in words and with the diagram. What outcomes are in this event?

Consider the experiment of rolling a pair of dice and finding the sum of the numbers on the dice. Let J be the event that the sum is less than 4. Let K be the event that the sum is an odd number.

7. Find the sample space for the experiment.

8. List the outcomes in event J and the outcomes in event K.

9. Create a Venn diagram that shows the relationships between the outcomes in J, K, and the sample space.

10. Describe K' in words and with the diagram. What outcomes are in this event?

11. Describe the event "getting an even number less than 4" with symbols and with the diagram. What outcomes are in this event?

12. Describe $(J \cap K')'$ in words and with the diagram. What outcomes are in this event?

13. In this experiment, are you just as likely to get a sum of 2 as a sum of 7? Explain.

14. Compare and contrast **unions of events** with **intersections of events.**

15. Consider some experiment with event E. Describe $E \cup E'$. Describe $E \cap E'$.

16. If A is the set of all factors of 36 and B is the set of all factors of 24, explain situations where you might want to find A∪B, A∩B, A', B' , or (A∩B)'.

17. Draw a Venn diagram with three overlapping circles. Use your Venn diagram to show that (A∩B)∪C = (A∩C)∩(B∪C). Create a scenario where you would use this information.

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.1.

Vocabulary

- experiment
- outcomes
- sample space
- event
- favorable outcomes

10.2 Independent Events

S.CP.2

Here you will learn what it means for two events to be independent and how to determine whether or not two events are independent using probability.

Two events are disjoint if they have no outcomes in common. Consider two events A and B that are disjoint. Can you say whether or not events A and B are independent?

MEDIA

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Independent Events

Recall that the **probability** of an event is the **chance** of it happening. Probabilities can be written as fractions or decimals between 0 and 1, or as percents between 0% and 100%. If all outcomes in an experiment have an equal chance of occurring, to find the probability of an event find the number of outcomes in the event and divide by the number of outcomes in the sample space.

$$P(E) = \frac{\#\, of\ outcomes\ in\ E}{\#\, of\ outcomes\ in\ sample\ space}$$

Consider the experiment of flipping a coin two times and recording the sequence of heads and tails. The sample space is $S = \{HH, HT, TH, TT\}$, which contains four outcomes. Let A be the event that heads comes up exactly once. $A = \{HT, TH\}$ Therefore,

$$P(A) = \frac{2}{4} = \frac{1}{2}$$

To find the probability of a single event, all you need to do is count the number of outcomes in the event and the number of outcomes in the sample space. Probability calculations become more complex when you consider the combined probability of two or more events.

Two events are **independent** if one event occurring does not change the probability of the second event occurring. Two events are **dependent** if one event occurring causes the probability of the second event to go up or down. **Two events are independent if the probability of A and B occurring together is the product of their individual probabilities:**

$P(A \cap B) = P(A)P(B)$ **if and only if A and B are independent events.**

In some cases it is pretty clear whether or not two events are independent. In other cases, it is not at all obvious. You can always test if two events are independent by checking to see if their probabilities satisfy the relationship above.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

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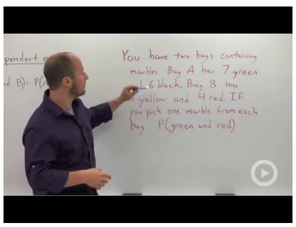

MEDIA

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Let's take a look at some problems where we are asked to determine independence.

1. Consider the experiment of flipping a coin two times and recording the sequence of heads and tails. The sample space is $S = \{HH, HT, TH, TT\}$, which contains four outcomes. Let C be the event that the first coin is a heads. Let D be the event that the second coin is a tails.

List the outcomes in events C and D.

$C = \{HH, HT\}$. $D = \{HT, TT\}$. Note that the outcomes in the two events overlap. This *does NOT mean that the events are not independent!*

Take a guess at whether or not you think the two events are independent.

If you get heads on the first coin, that shouldn't have any effect on whether you get tails for the second coin. It makes sense that the events should be independent.

Find $P(C)$ and $P(D)$.

$P(C) = \frac{2}{4} = \frac{1}{2}$. $P(D) = \frac{2}{4} = \frac{1}{2}$.

Find $P(C \cap D)$. Are the two events independent?

$C \cap D$ is the event of getting heads first and tails second. $C \cap D = \{HT\}$.

$$P(C \cap D) = \frac{1}{4}.$$
$$P(C)P(D) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Because $P(C \cap D) = P(C)P(D)$, the events are independent.

2. Consider the experiment of flipping a coin two times and recording the sequence of heads and tails. The sample space is $S = \{HH, HT, TH, TT\}$, which contains four outcomes. Let C be the event that the first coin is a heads. Let E be the event that both coins are heads.

List the outcomes in events C and E.

$C = \{HH, HT\}$. $E = \{HH\}$.

Take a guess at whether or not you think the two events are independent.

If you get heads on the first coin, then you are more likely to end up with two heads than if you didn't know anything about the first coin. It seems like the events should NOT be independent.

Find $P(C)$ and $P(E)$.

$P(C) = \frac{2}{4} = \frac{1}{2}$. $P(E) = \frac{1}{4}$.

Find $P(C \cap E)$. Are the two events independent?

$C \cap E$ is the event of getting heads first and both heads. This is the same as the event of getting both heads, since if you got both heads then you definitely got heads first. $C \cap E = \{HH\}$.

$$P(C \cap E) = \frac{1}{4}.$$
$$P(C)P(E) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8}.$$

Because $P(C \cap E) \neq P(C)P(E)$, the events are NOT independent.

3. Consider the experiment of flipping a coin two times and recording the sequence of heads and tails. The sample space is $S = \{HH, HT, TH, TT\}$, which contains four outcomes. Let E be the event that both coins are heads. Let F be the event that both coins are tails.

List the outcomes in events E and F.

$E = \{HH\}$. $F\{TT\}$.

Take a guess at whether or not you think the two events are independent.

If you get both heads, then you definitely didn't get both tails. It seems like the events should NOT be independent.

Find $P(E)$ and $P(F)$.

$P(E) = \frac{1}{4}$. $P(F) = \frac{1}{4}$.

Find $P(E \cap F)$. Are the two events independent?

$E \cap F$ is the event of getting two heads and two tails. This is impossible to do because these two events are disjoint. $E \cap F = \{\}$.

$$P(E \cap F) = 0.$$
$$P(E)P(F) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}.$$

Because $P(E \cap F) \neq P(E)P(F)$, the events are NOT independent.

Examples

Example 1

Earlier, you were asked can you whether events A and B are independent.

If A and B are disjoint, then $A \cap B = \{\}$ and $P(A \cap B) = 0$.

For the two events to be independent, $P(A)P(B) = P(A \cap B)$ This means that $P(A)P(B) = 0$. By the zero product property, the only way for $P(A)P(B) = 0$ is if $P(A) = 0$ or $P(B) = 0$.

In other words, two disjoint events are independent if and only if the probability of at least one of the events is 0.

Example 2

Consider the experiment of tossing a coin and then rolling a die. Event A is getting a tails on the coin. Event B is getting an even number on the die. Are the two events independent? Justify your answer using probabilities.

The sample space for the experiment is $S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$. Next consider the outcomes in the events. $A = \{T1, T2, T3, T4, T5, T6\}$. $B = \{H2, H4, H6, T2, T4, T6\}$. $A \cap B = \{T2, T4, T6\}$.

- $P(A) = \frac{6}{12} = \frac{1}{2}$
- $P(B) = \frac{6}{12} = \frac{1}{2}$
- $P(A)P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$
- $P(A \cap B) = \frac{3}{12} = \frac{1}{4}$

The events are independent because $P(A)P(B) = P(A \cap B)$.

Example 3

Consider the experiment of rolling a pair of dice. Event C is a sum that is even and event D is both numbers are greater than 4. Are the two events independent? Justify your answer using probabilities.

First find the sample space and the outcomes in each event:

- $$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$$
  $$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$
  $$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$
-
- $D = \{(5,5),(5,6),(6,5),(6,6)\}$
- $C \cap D = \{(5,5),(6,6)\}$

Next find the probabilities:

- $P(C) = \frac{18}{36} = \frac{1}{2}$
- $P(D) = \frac{4}{36} = \frac{1}{9}$

- $P(C)P(D) = \left(\frac{1}{2}\right)\left(\frac{1}{9}\right) = \frac{1}{18}$
- $P(C \cap D) = \frac{2}{36} = \frac{1}{18}$

The events are independent because $P(C)P(D) = P(C \cap D)$.

Example 4

$P(G) = \frac{1}{3}$ and $P(H) = \frac{1}{2}$. If $P(G \cap H) = \frac{1}{4}$, are events G and H independent?

Events G and H are independent if and only if $P(G)P(H) = P(G \cap H)$.

$$P(G)P(H) = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$$
$$P(G \cap H) = \frac{1}{4}$$

$P(G)P(H) \neq P(G \cap H)$, so the events are NOT independent.

Review

Consider the experiment of flipping a coin three times and recording the sequence of heads and tails. The sample space is $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$, which contains eight outcomes. Let A be the event that exactly two coins are heads. Let B be the event that all coins are the same. Let C be the event that at least one coin is heads. Let D be the event that all coins are tails.

1. List the outcomes in each of the four events. Which of the two events are complements?

2. Find $P(A), P(B), P(C), P(D)$.

3. Find $P(A \cap C)$. Are events A and C independent? Explain.

4. Find $P(B \cap D)$. Are events B and D independent? Explain.

5. Find $P(B \cap C)$. Are events B and C independent? Explain.

6. Create two new events related to this experiment that are independent. Justify why they are independent using probabilities.

Consider the experiment of drawing a card from a deck. The sample space is the 52 cards in a standard deck. Let A be the event that the card is red. Let B be the event that the card is a spade. Let C be the event that the card is a 4. Let D be the event that the card is a diamond.

7. Describe the outcomes in each of the four events.

8. Find $P(A), P(B), P(C), P(D)$.

9. Find $P(A \cap B)$. Are events A and B independent? Explain.

10. Find $P(B \cap C)$. Are events B and C independent? Explain.

11. Find $P(A \cap D)$. Are events A and D independent? Explain.

12. $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{8}$. If $P(A \cap B) = \frac{1}{16}$ are events A and B independent?

13. $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{8}$. If $P(A \cap B) = \frac{1}{32}$ are events A and B independent?

14. What is the difference between disjoint and independent events?

15. Two events are disjoint, and both have nonzero probabilities. Can you say whether the events are independent or not?

16. Bag A contains 5 yellow, 6 blue, and 4 white marbles. Bag B contains 8 green, 5 black, and 4 red marbles. Create a probability problem that would show that is you pick two marbles from the bags, that these events would be independent?

17. Consider the outcomes for the tossing of 3 coins. Let A represent the outcomes where the first coin tossed is a head. Let B represent the outcomes where there is at least 2 heads tossed. Create a problem, using notation found in the concept, that would illustrate two independent events.

18. You and your friend are in separate lines for tickets to the concert. Describe the outcome(s) that would make these event disjoint. What about independent? Is it possible to be both disjoint and independent? Why or why not?

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.2.

Vocabulary

- Probability
- Chance
- Independent
- Dependent

10.3 Conditional Probability

Learning Objectives

S.CP.3

Here you will learn about conditional probability and its connection to independence.

Describe the two ways that you can use probabilities to check whether or not two events are independent.

Conditional Probability

Consider a high school with 400 students. The two-way table below shows the number of students in each grade who earned different midterm letter grades.

TABLE 10.2:

	9th Grade	10th Grade	11th Grade	12th Grade
A	20	18	25	18
B	30	35	38	55
C	30	17	20	20
D	15	15	12	4
F	5	15	5	3

Suppose you choose a student at random from the school. Let G be the event that the student is from ninth grade. Let A be the event that the student got an A. You can calculate the probabilities of each of these events using the information in the table.

$$P(G) = \frac{20+30+30+15+5}{400} = \frac{100}{400} = \frac{1}{4} = 0.25$$
$$P(A) = \frac{20+18+25+18}{400} = \frac{81}{400} = 0.2025$$

Now suppose you choose a student at random and they tell you they are in 9th grade. What's the probability that they got an A? Now, your sample space is only the 100 students in 9th grade. $P(9th\ grade\ student\ got\ an\ A) = \frac{20}{100} = 0.20$. *When you have additional information that causes you to restrict the sample space you are considering, you are working with* **conditional probability.**

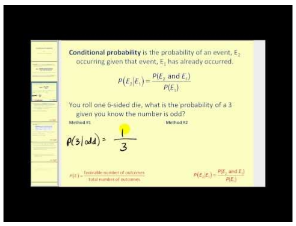
The **conditional probability** of event A given event B is the probability of event A occurring given ***event*** B ***occurred***. The notation is $P(A|B)$, which is read as "the probability of A given B".

With the students example, you were calculating $P(A|G)$ ("the probability that a student got an A given that they are in 9th grade"). **To calculate this probability, you found the number of ninth grade students with an** $A (A \cap G)$ **and divided by the number of ninth grade students** (G)**.** *You would have gotten the same result had you found* $P(A \cap G)$ *and divided by* $P(G)$ *, because the 400's cancel each other out.*

$$P(A|G) = \frac{20}{100} = \frac{\frac{20}{400}}{\frac{100}{400}} = \frac{P(A \cap G)}{P(G)}$$

To calculate a conditional probability, you can always use this formula, generalized below:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



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URL: http://www.ck12.org/flx/render/embeddedobject/75419

Interactive Die Roll

Use the interactive below to explore the probability of rolling two even numbers in a row on a six-sided die.

MEDIA

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URL: http://www.ck12.org/flx/render/embeddedobject/237291

Let's take a look at some problems involving conditional probability

1. Consider the experiment of tossing three coins and recording the sequence of heads and tails. Let A be the event of getting at least two heads. Let B be the event of getting three heads.

Find $P(A|B)$.

$P(A|B)$ is the probability of getting at least two heads given that you have gotten three heads. If you KNOW that you got three heads, then you automatically have gotten at least two heads. $P(A|B)$ should be 1. Using the formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{8}}{\frac{1}{8}} = 1$$

Find $P(B|A)$.

$P(B|A)$ is the probability of getting three heads given that you have gotten at least two heads. Since you know you have gotten at least two heads, the new sample space is $\{HHT, HTH, THH, HHH\}$. The probability of getting three heads is $\frac{1}{4}$. Using the formula:

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4}$$

Does $P(A|B) = P(B|A)$?

$P(A|B) \neq P(B|A)$ The order of the letters within the probability statement matters!

The sample space for tossing three coins is $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

2. Consider two **independent** events C and D. What is $P(C|D)$ in terms of $P(C)$ and $P(D)$? What is $P(D|C)$ in terms of $P(C)$ and $P(D)$?

Since the two events are independent, $P(C \cap D) = P(C)P(D)$.

$$P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{P(C)P(D)}{P(D)} = P(C)$$
$$P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{P(D)P(C)}{P(C)} = P(D)$$

If C and D are independent, then whether or not D has occurred has no effect on the probability of C occurring and vice versa. This should make sense given the definition of independent events. **One way to test if two events C and D are independent is to verify that $P(C|D) = P(C)$ and $P(D|C) = P(D)$.**

3. You have two coins, one regular coin and one special coin with heads on both sides. You put the two coins in a bag and choose one at random. Let S be the event that the coin is the special coin with heads on both sides. Let H be the event that when the coin is tossed it comes up heads.

What is $P(S)$?

$P(S)$ is the probability that you have chosen the special coin. There are two coins in the bag, one of which is the special coin. $P(S) = \frac{1}{2}$

What is $P(S|H)$?

The experiment of choosing a coin and tossing it has four outcomes in its sample space:

1. Regular Coin, Tails
2. Regular Coin, Heads
3. Special Coin, Heads 1st Side
4. Special Coin, Heads 2nd Side

$P(S|H)$ is the probability that you have chosen the special coin *given that* when you tossed it, it came up heads. In order to compute this probability, you need to know $P(S \cap H)$ and $P(H)$. There are two outcomes that are special coins and heads, so $P(S \cap H) = \frac{2}{4} = \frac{1}{2}$. There are three outcomes that are heads, so $P(H) = \frac{3}{4}$. Now you can compute the conditional probability:

$$P(S|H) = \frac{P(S \cap H)}{P(H)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

Compare the answers to parts a and b. In each case you are calculating the probability that the coin is the special coin; however, in part b *you have additional information that supports that it is the special coin*. Because you have additional information, the probability that it is the special coin goes up. Note that because $P(S) \neq P(S|H)$, the two events S and H are NOT independent.

Examples

Example 1

Earlier, you were asked to describe two ways that you can use probabilities to check whether or not two events are independent.

Two events A and B are independent if and only if:

1) $P(A \cap B) = P(A)P(B)$

2) $P(A|B) = P(A)$ and $P(B|A) = P(B)$

Consider the experiment of rolling a pair of dice. Let A be the event that the sum of the numbers on the dice is an 8. Let B be the event that the two numbers on the dice are a 3 and a 5.

Example 2

What is $P(A)$? What is $P(B)$?

There are 5 pairs of numbers that have a sum of 8, so $P(A) = \frac{5}{36}$. There are 2 pairs of numbers that are a 3 and a 5, so $P(B) = \frac{2}{36} = \frac{1}{18}$.

Example 3

What is $P(B|A)$?

$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$. The sample space has been restricted to the five outcomes with a sum of 8.

Example 4

Are events A and B independent?

The sample space for this experiment has 36 outcomes:

The two events are independent if $P(B|A) = P(B)$. $P(B) = \frac{1}{18}$, but $P(B|A) = \frac{2}{5}$. $P(B) \neq P(B|A)$. so the events are not independent.

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Consider the experiment of drawing a card from a standard deck. Let A be the event that the card is a diamond. Let B be the event that the card is a red card. Let D be the event that the card is a four.

1. Find $P(A), P(B), P(C), P(D)$.

2. Find $P(A|B)$ and $P(B|A)$.

3. Are events A and B independent?

4. Find $P(D|B)$ and $P(B|D)$.

5. Are events B and D independent?

Consider the experiment of flipping three coins and recording the sequence of heads and tails. Let A be the event that all the coins are the same. Let B be the event that there is at least one heads. Let C be the event that the third coin is a tails. Let D be the event that the first coin is a heads.

6. Find $P(A), P(B), P(C), P(D)$.

7. Find $P(A|B)$ and $P(B|A)$.

8. Are events A and B independent?

9. Find $P(C|D)$ and $P(D|C)$.

10. Are events C and D independent?

11. Find $P(A|D)$ and $P(D|A)$.

12. Are events A and D independent?

13. Explain what conditional probability is in your own words.

14. Explain two ways to test whether or not two events are independent. When does it make sense to use one method over the other?

15. Explain where the conditional probability formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$ comes from.

16. In the city there are 3 major clothing stores: B, G, and L. 60% of the population shops at store B, 36% shop at store G, and 34% shop at store L. You also know that 18% shop at both stores B and L, 15% shop at both B and B, 4% shop at G and L, and 2% shop at all three locations.

1. What is the probability that a person will shop at store B given they shop in at least one other clothing store?
2. What is the probability that a person will shop at either store B or store G or both?

17. From a group of 50 students, those taking Calculus and Chemistry this year were counted. 15 students take Chemistry, 7 take both, 22 take neither. Draw a Venn diagram to represent the data. What is the probability that a student is taking Calculus given they are taking Chemistry?

18. There are two candy jars on the shelf. In Jar A you notice that there are 2 pink candies and 5 green candies. In Jar B, there are 4 pink candies and 3 green candies. You roll a die to decide what jar to choose from. You roll a five for Jar A and a one for Jar B. Determine the probability that the candy was chosen from Jar B given it was pink.

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.3.

Vocabulary

- Conditional probability

10.4 Two-Way Frequency Tables

Learning Objectives

S.CP.4

Here you will learn how to construct and interpret two-way frequency tables.

You are in charge of choosing the theme for the junior/senior prom. You survey the juniors and seniors and record the results in a two-way frequency table.

TABLE 10.3:

	Casino	**Masquerade Ball**	**Arabian Nights**	*Total*
Juniors	100	212	50	*362*
Seniors	190	159	38	*387*
Total	*290*	*371*	*88*	*749*

Based on the results, you decide to go with the Masquerade Ball. On prom night, what's the probability that a student chosen at random got the theme of his/her choice? What's the probability that a senior chosen at random got the theme of his/her choice? What's the probability that a junior chosen at random got the theme of his/her choice?

Two Way Frequency Tables

Suppose you conduct a survey where you ask each person two questions. Once you have finished conducting the survey, you will have **two** pieces of data from each person. Whenever you have two pieces of data from each person, you can organize the data into a **two-way frequency table**.

A group of people were surveyed about 1) whether they have cable TV and 2) whether they went on a vacation in the past year.

TABLE 10.4:

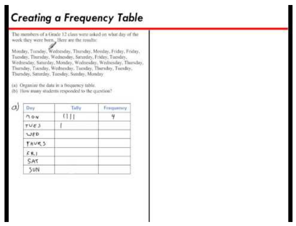
	Took a Vacation	**No Vacation**	*Total*
Have Cable TV	97	38	*135*
Don't Have Cable TV	14	17	*31*
Total	*111*	*55*	*166*

The numbers in the frequency table show the numbers of people that fit each pair of preferences. For example, 97 people have cable TV and took a vacation last year. 38 people have cable TV but did not take a vacation last year. The totals of the rows and columns have been added to the frequency table for convenience. From the far right column you can see that 135 people have cable TV and 31 people don't have cable TV for a total of 166 people surveyed. From the bottom row you can see that 111 people took a vacation and 55 people did not take a vacation for a total of 166 people surveyed.

You can use the two-way frequency table to calculate probabilities about the people surveyed. For example, you could find:

A. The probability that a random person selected from this group took a vacation last year.

B. The probability that a random person from this group who has cable TV took a vacation last year.
C. Whether or not “choosing a person with cable TV” and “choosing a person who took a vacation” are independent events for this population of 166 people.



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Let’s look at some problems involving two way frequency tables

1. Suppose you choose a person at random from the group surveyed below. Let A be the event that the person chosen took a vacation last year. Find $P(A)$.

TABLE 10.5:

	Took a Vacation	**No Vacation**	*Total*
Have Cable TV	97	38	*135*
Don’t Have Cable TV	14	17	*31*
Total	*111*	*55*	*166*

There were 166 people surveyed, so there are 166 outcomes in the sample space. 111 people took a vacation last year.

$$P(A) = \frac{111}{166} \approx 0.67 \text{ or } 67\%$$

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2. Suppose you choose a person at random from the group surveyed below. Let A be the event that the person chosen took a vacation last year. Let B be the event that the person chosen has cable TV. Find $P(A|B)$.

TABLE 10.6:

	Took a Vacation	**No Vacation**	*Total*
Have Cable TV	97	38	*135*
Don’t Have Cable TV	14	17	*31*
Total	*111*	*55*	*166*

You are looking for the probability that the person took a vacation *given* that they have cable TV. Since you know that the person has cable TV, the sample space has been restricted to the 135 people with cable TV. 97 of those people took a vacation.

$$P(A|B) = \frac{97}{135} \approx .72 \text{ or } 72\%$$

Suppose you wanted to use the conditional probability formula for this calculation.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{97}{166}}{\frac{135}{166}} = \frac{97}{135} \approx 0.72 \text{ or } 72\%$$

With the conditional probability formula, each probability is calculated with the sample space of 166. The two 166's cancel each other out, and the result is the same. Sometimes it makes sense to use the conditional probability formula, and sometimes it is easier to think logically about what is being asked.

3. Suppose you choose a person at random from the group surveyed below. Let A be the event that the person chosen took a vacation last year. Let B be the event that the person chosen has cable TV. Are events A and B independent?

TABLE 10.7:

	Took a Vacation	**No Vacation**	*Total*
Have Cable TV	97	38	*135*
Don't Have Cable TV	14	17	*31*
Total	*111*	*55*	*166*

From #1, you know that $P(A) = 67\%$. From #2, you know that $P(A|B) = 72\%$. Because these probabilities are not equal, the two events are NOT independent (they are dependent). People with cable TV are more likely to have taken a vacation as opposed to people without cable TV, so knowing that a person has cable TV increases the probability that they took a vacation.

Examples

Example 1

Earlier, you were given a problem on the junior/senior prom.

TABLE 10.8:

	Casino	**Masquerade Ball**	**Arabian Nights**	*Total*
Juniors	100	212	50	*362*
Seniors	190	159	38	*387*
Total	*290*	*371*	*88*	*749*

You decide to go with the Masquerade Ball. On prom night, what's the probability that a student chosen at random got the theme of his/her choice?

This question includes the entire population of 749 people as the sample space. 371 of those people wanted the Masquerade Ball. $P(\text{got their choice}) = \frac{371}{749} \approx .50 \text{ or } 50\%$**.**

What's the probability that a senior chosen at random got the theme of his/her choice?

This is a conditional probability question, because you know that the student is a senior. You can restrict the sample space to just the 387 seniors. 159 of them wanted the Masquerade Ball. $P(\textit{senior got their choice}) = \frac{159}{387} \approx 41\%$**.**

What's the probability that a junior chosen at random got the theme of his/her choice?

This is a conditional probability question, because you know that the student is a junior. You can restrict the sample space to just the 362 juniors. 212 of them wanted the Masquerade Ball. $P(\textit{junior got their choice}) = \frac{212}{362} \approx 59\%$**.**

A group of 112 students was surveyed about what grade they were in and whether they preferred dogs or cats. 20 9th graders preferred dogs, 5 9th graders preferred cats, 16 10th graders preferred dogs, 4 10th graders preferred cats, 22 11th graders preferred dogs, 6 11th graders preferred cats, 30 12th graders preferred dogs, and 7 12th graders preferred cats.

Example 2

Construct a two-way frequency table to organize this data.

TABLE 10.9:

	Dogs	Cats	*Total*
9th Grade	20	5	*25*
10th Grade	16	4	*20*
11th Grade	22	6	*28*
12th Grade	30	7	*37*
Total	*88*	*22*	*110*

Example 3

Suppose a person is chosen at random from this group. Let C be the event that the student prefers cats. Let T be the event that the student is in 10th grade. Find $P(C)$ and $P(C|T)$.

There are 110 students total. 22 of them prefer cats. $P(C) = \frac{22}{110} = 20\%$. $P(C|T)$ means the probability that the student prefers cats given that they are in 10th grade. Restrict the sample space to the 20 10th grade students. 4 of them prefer cats. $P(C|T) = \frac{4}{20} = 20\%$.

Example 4

Are events C and T independent?

The events are independent because $P(C) = P(C|T)$. Being in 10th grade does not affect the probability of the student preferring cats.

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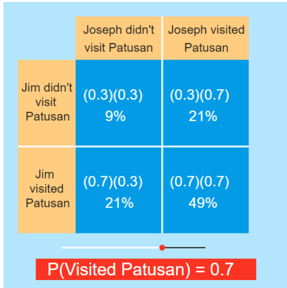

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Review

Use the following information for #1-#5: A group of 64 people were surveyed about the type of movies they prefer. 12 females preferred romantic comedies, 10 females preferred action movies, and 3 females preferred horror movies. 8 males preferred romantic comedies, 25 males preferred action movies, and 6 males preferred horror movies.

1. Construct a two-way frequency table to organize this data.

Suppose a person is chosen at random from this group.

2. Let F be the event that the person is female. Find $P(F)$.

3. Let R be the event that the person prefers romantic comedies. Find $P(R)$.

4. Find $P(F|R)$ and $P(R|F)$. Explain how these two calculations are different.

5. Are events F and R independent? Justify your answer.

The middle school students in your town were surveyed and classified according to grade level and response to the question "how do you usually get to school"? The data is summarized in the two-way table below.

TABLE 10.10:

	Walk	**Bus**	**Car**	*Total*
6th Grade	30	120	65	*215*
7th Grade	25	170	25	*220*
8th Grade	40	130	41	*211*
Total	*95*	*420*	*131*	*646*

6. If a student is chosen at random from this group, what is the probability that he or she is a 6th grade student who takes the bus?

7. If a 6th grade student is chosen at random from this group, what is the probability that he or she takes the bus?

8. If a student who takes the bus is chosen at random from this group, what is the probability that he or she is in 6th grade?

9. The previous three questions each have to do with 6th grade and taking the bus. Why are the answers to these questions different?

10. Are the events "being in 6th grade" and "taking the bus" independent? Justify your answer.

A hospital runs a test to determine whether or not patients have a particular disease. The test is not always accurate. The two-way table below summarizes the numbers of patients in the past year that received each result.

TABLE 10.11:

	Positive Result on Test	Negative Result on Test	*Total*
Has Disease	100	4	*104*
Does Not Have Disease	12	560	*572*
Total	*112*	*564*	*676*

11. If a patient is chosen at random from this group, what is the probability that he or she has the disease?

12. A patient from this group received a positive test result. What is the probability that he or she has the disease?

13. A patient from this group has the disease. What is the probability that he or she received a positive result on the test?

14. A "false positive" is when a patient receives a positive result on the test, but does not actually have the disease. What is the probability of a false positive for this sample space?

15. How many of the 676 patients received accurate test results?

16. In a book club consisting of 50 readers, 36 like detective stories, 20 like science fiction and 12 did not like either. If you select a reader randomly from the group, what is the probability that he or she:

1. likes both science fiction and detective stories?
2. likes at least one of science fiction and detective stories?
3. likes detective stories given that he or she likes science fiction?
4. likes science fiction given that he or she likes detective stories?
5. are liking science fiction stories and detective stories independent? Why or why not?

17. You are given a bag containing 8 red and 5 blue balls. Create a problem that illustrates the concept of probability of drawing at least 2 balls from the bag? Does it make a difference if you replace the balls after each pick? Explain.

18. A new drug is coming out on the market where it proclaims to grow hair faster than any other alternative. In a sample of 4,000 people, 2,400 were given the drug and of these only 1,600 showed any hair growth. The other 1,600 were given the placebo and 1,200 of these showed hair growth. Draw a two way frequency table to illustrate the data. Determine at least three conclusions you can make from the table.

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.4.

Vocabulary

frequency

10.5 Everyday Examples of Independence and Probability

Learning Objectives

S.CP.5

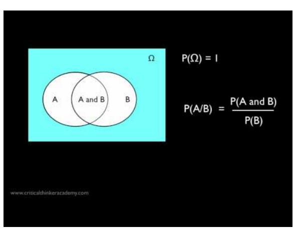
Here you will consider everyday examples of conditional probability and determine whether or not events are independent.

10% of the emails that Michelle receives are spam emails. Her spam filter catches spam 95% of the time. Her spam filter misidentifies non-spam as spam 2% of the time. What percent of the emails in the spam folder are not spam emails?

Independence and Probability

In everyday situations, **conditional probability** is a probability where **additional information is known**. Finding the probability that a random **smoker** gets lung cancer is a conditional probability compared to the probability that a random **person** gets lung cancer. *The additional information of the person being a smoker changes the probability being calculated.* If the additional information does not ultimately change the probability, then the two events are independent.

There are many everyday situations having to do with probabilities. It is important for you to be able to differentiate between a regular probability and a conditional probability. Always read problems carefully in order to be sure that you are interpreting the information correctly.



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A test for a certain disease is said to be 99% accurate. What does this mean? What does this have to do with conditional probability?

You should consider four groups of people:

1. People with the disease who test positive for the disease (true positive).
2. People with the disease who test negative for the disease (false negative).
3. People without the disease who test positive for the disease (false positive).
4. People without the disease who test negative for the disease (true negative).

If a test is 99% accurate, it implies that:

1. If a person has the disease, 99% of the time they will receive a positive test result. $P(positive|disease) = 99\%$

2. If a person does not have the disease, 99% of the time they will receive a negative test result. $P(negative|no\ disease) = 99\%$

The 99% is a conditional probability in each case. Note that these are two completely different probability calculations, and they do not automatically have to be the same. It is in fact more realistic if these two probabilities are different.

Real-World Application: Spam Emails

10% of the emails that Michelle receives are spam emails. Her spam filter catches spam 95% of the time. Her spam filter misidentifies non-spam as spam 2% of the time. Let A be the event that an email is spam. Let B be the event that the spam filter identifies the email as spam.

1. What does $P(A)$ mean in English?

$P(A)$ is the probability that a random email is spam.

2. What does $P(B|A)$ mean in English?

$P(B|A)$ is the probability that a spam email gets identified as spam.

3. What does $P(B'|A')$ mean in English?

$P(B'|A')$ is the probability that a non-spam email does not get identified as spam.

4. Find $P(A)$.

$P(A) = 10\%$

5. Find $P(B|A)$.

$P(B|A) = 95\%$

6. Find $P(B'|A')$.

$P(B'|A') = 98\%$. Note that in the problem, 2% is $P(B|A')$. $P(B|A')$ and $P(B'|A')$ must add to 100% because B and B' are complements.

Examples

Example 1

Earlier, you were asked what percent of the emails in the spam folder are not spam emails.

10% of the emails that Michelle receives are spam emails. Her spam filter catches spam 95% of the time. Her spam filter misidentifies non-spam as spam 2% of the time. What percent of the emails in the spam folder are not spam emails?

This question is asking for the probability that an email that has been identified as spam is a regular email, $P(A'|B)$. You were not given this probability directly. One way to approach this problem is to make a two-way frequency table for a some number of emails. Suppose you have 1000 emails.

TABLE 10.12:

	Spam	**Not Spam**	*Total*
Identified as Spam			
Not Identified as Spam			
Total			*1000*

You know that 10% of those (100 emails) will be spam. This means 90% of those (900 emails) will not be spam. Fill these numbers into the table.

TABLE 10.13:

	Spam	**Not Spam**	*Total*
Identified as Spam			
Not Identified as Spam			
Total	*100*	*900*	*1000*

You also know that 95% of the spam emails (95 emails) will be identified as spam. This means the other 5 spam emails will not be identified as spam. Fill these numbers into the table.

TABLE 10.14:

	Spam	**Not Spam**	*Total*
Identified as Spam	95		
Not Identified as Spam	5		
Total	*100*	*900*	*1000*

You also know that 98% of the non-spam emails (882 emails) will not be identified as spam. This means that the other 18 emails will be identified as spam. Fill these numbers into the table.

TABLE 10.15:

	Spam	**Not Spam**	*Total*
Identified as Spam	95	18	*113*
Not Identified as Spam	5	882	*887*
Total	*100*	*900*	*1000*

Now go back to the question. The question is asking for the **probability that an email that has been identified as spam is a regular email.** 113 emails that were identified as spam. 18 of them are not spam emails. $P(A'|B) = \frac{18}{113} \approx 16\%$. Even though the spam filter is pretty accurate, 16% of the emails in the spam folder will be regular emails.

Use the following information for the examples below.

Karl takes the bus to school. Each day, there is a 10% chance that his bus will be late, a 20% chance that he will be late, and a 2% chance that both he and the bus will be late. Let C be the event that Karl is late. Let D be the event that the bus is late.

Example 2

State the 10%, 20%, and 2% probabilities in probability notation in terms of events C and D.

$10\% = P(D)$. $20\% = P(C)$. $2\% = P(C \cap D)$

Example 3

Are events C and D independent? Explain.

Events C and D are independent if $P(C \cap D) = P(C)P(D)$.

$P(C)P(D) = (0.2)(0.1) = 0.02 = 2\% = P(C \cap D)$

Therefore, the events are independent.

Example 4

Find the probability that Karl is not late but the bus is late.

This is $P(C' \cap D)$. Because the two events are independent, $P(C' \cap D) = P(C')P(D)$. Since there is a 20% chance that Karl will be late, there is an 80% chance that Karl will not be late. This means $P(C') = 80\%$. Therefore, $P(C' \cap D) = P(C')P(D) = (0.8)(0.1) = 8\%$.

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Review

0.1% of the population is said to have a new disease. A test is developed to test for the disease. 97% of people without the disease will receive a negative test result. 99.5% of people with the disease will receive a positive test result. Let D be the event that a random person has the disease. Let E be the event that a random person gets a positive test result.

1. State the 0.1%, 97%, and 99.5% probabilities in probability notation in terms of events D and E.

Fill in the two-way frequency table for this scenario for a group of 1,000,000 people. Follow the steps to help.

TABLE 10.16:

	Disease	**No Disease**	*Total*
Positive Test			
Negative Test			
Total			*1,000,000*

2. How many of the 1,000,000 people have the disease? How many *don't* have the disease?

3. How many **of the people without the disease** will receive a negative test result (true negative)? How many of the people without the disease will receive a *positive* test result (false positive)?

4. How many **of the people with the disease** will receive a positive test result (true positive)? How many of the people with the disease will receive a *negative* test result (false negative)?

5. What does $P(D|E)$ mean in English?

6. Find $P(D|E)$. Is this a surprising result?

7. What does $P(D|E)'$ mean in English?

8. Find $P(D|E')$.

9. Are the two events D and E independent? Justify your answer.

After finishing his homework, Matt often plays video games and/or has a snack. There is a 60% chance that Matt plays video games, an 80% chance that Matt has a snack, and a 55% chance that Matt plays video games and has a snack. Let G be the event that Matt plays video games and S be the event that Matt has a snack.

10. State the 60%, 80%, and 55% probabilities in probability notation in terms of events G and S.

11. Are events G and S independent? Explain.

12. Consider 100 days after Matt has finished his homework. Use the probabilities in the problem to fill in the two-way frequency table.

TABLE 10.17:

	Snack	**No Snack**	*Total*
Video Games			
No Video Games			
Total			*100*

13. Given that Matt played video games, find the probability that he had a snack.

14. What is $P(G|S)$ in English?

15. Find $P(G|S)$.

16. A dollar-bill changer on a snack machine was tested with 100 \$1 bills. Twenty-five of the bills were found to be counterfeit, but only one was accepted by the machine. However six of the legal bills were rejected. Draw a chart to show the number of legal and counterfeit bills that were accepted or rejected.

1. What is the probability that a bill will be rejected given that it is legal?
2. What is the probability that the counterfeit bill is accepted?

17. In your aquarium there is one of each fish: a danio, a black molly, a black skirt tetra, a betta, a swordtail, and a platy. What is the probability that when three fish are chosen at random:

1. you choose a swordtail given you have already chosen a betta?
2. you choose a swordtail and a danio given you have not chosen a platy?
3. you do not choose a platy given you have chosen a black molly and a tetra?

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.5.

10.6 Probability of Unions

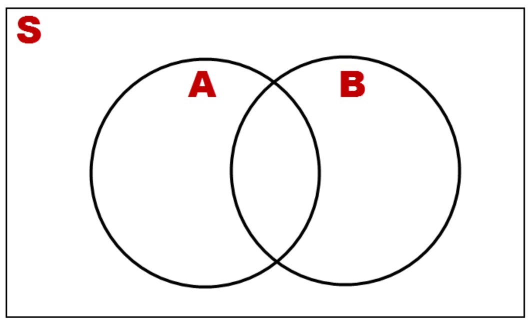
Learning Objectives

Here you will learn how to find the probability of unions of events with the Addition Rule.

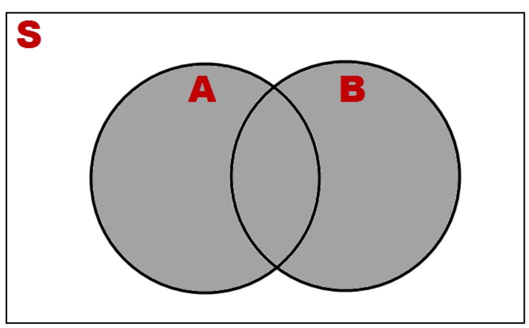
On any given night, the probability that Nick has a cookie for dessert is 10%. The probability that Nick has ice cream for dessert is 50%. The probability that Nick has a cookie or ice cream is 55%. What is the probability that Nick has a cookie **and** ice cream for dessert?

Probability of Unions

Consider a sample space with events A and B.



Recall that the **union of events** A **and** B is an event that includes all the outcomes in either event A, event B, or both. The symbol $\cup$ represents union. Below, $A \cup B$ is shaded.



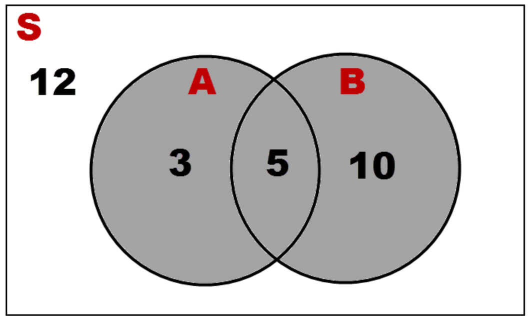
How do you find the **number of outcomes** in a union of events? If you find the sum of the number of outcomes in event A and the number of outcomes in event B, you will have counted some of the outcomes twice. *In fact, you will have counted the outcomes that are in both event A and event B twice.* Therefore, in order to correctly count the number of outcomes in the union of two events, you must count the number of outcomes in each event separately and *subtract* the number of outcomes shared by both events (so these are not counted twice). Generalizing to probability:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$**. This is called the Addition Rule for Probability.**

Note that $(A \cap B)$ is the intersection of the two events. It contains all the outcomes that are shared by both events and is the intersection of the two circles in the Venn diagram.

Suppose that in your class of 30 students, 8 students are in band, 15 students play a sport, and 5 students are both in band and play a sport. Let A be the event that a student is in band and let B be the event that a student plays a sport. Create a Venn diagram that models this situation.

In order to fill in the Venn diagram, remember that the total of the numbers in circle A must be 8 and the total of the numbers in circle B must be 15. The intersection of the two circles must contain a 5.



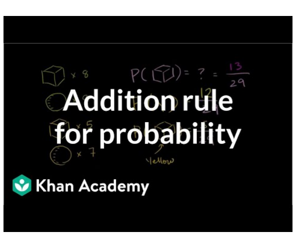
$P(A \cup B)$ is the probability that a student is in band **or** plays a sport **or** both. With the help of the Venn diagram, this is not too difficult to calculate:

$$P(A \cup B) = \frac{3+10+5}{30} = \frac{18}{30} = \frac{3}{5}$$

You could also compute this probability using the Addition Rule:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{8}{30} + \frac{15}{30} - \frac{5}{30} \\ &= \frac{8+15-5}{30} \\ &= \frac{18}{30} \\ &= \frac{3}{5} \end{aligned}$$

Note that by using the Addition Rule, you avoid having to determine that there are 3 people who are in band and don't play a sport and 10 people who play a sport but are not in band. The Addition Rule is easier when you have not created a Venn diagram.



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/63801

Two events C and D are disjoint. Explain why $P(C \cup D) = P(C) + P(D)$.

If two events are disjoint (also known as mutually exclusive), then they share no outcomes. Therefore, the probability of both events occurring simultaneously is 0 ($P(C \cap D) = 0$). By the Addition Rule:

$$\begin{aligned}
P(C \cup D) &= P(C) + P(D) - P(C \cap D) \\
P(C \cup D) &= P(C) + P(D) - 0 \\
P(C \cup D) &= P(C) + P(D)
\end{aligned}$$

The outcomes of rolling a die are disjoint. In the interactive below, click the button and observe what the probability is of rolling an even or and odd number. That is, what is $P\left(\text{Even} \cup \text{Odd}\right)$.

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/237059

Real-World Application: Weather Probability

Suppose that today there is a 90% chance of snow, a 20% chance of a strong winds, and a 15% chance of both snow and strong winds. What is the probability of snow or strong winds?

Use the Addition Rule:

$$\begin{aligned}
P(Snow \cup Strong\ Winds) &= 0.90 + 0.20 - 0.15 \\
P(Snow \cup Strong\ Winds) &= 0.95
\end{aligned}$$

There is a 95% chance of either snow, strong winds, or both.

Now suppose that today there is a 60% chance of snow, an 85% chance of snow or strong winds, and a 25% chance of snow and strong winds. What is the chance of strong winds?

Once again you can use the Addition Rule, because it relates the probabilities in the problem.

$$0.85 = 0.60 + P(Strong\ Winds) - 0.25$$
$$0.50 = P(Strong\ Winds)$$

There is a 50% chance of strong winds.

Examples

Example 1

Earlier, you were asked what is probability of Nick having a cookie and ice cream for dessert is.

On any given night, the probability that Nick has a cookie for dessert is 10%. The probability that Nick has ice cream for dessert is 50% . The probability that Nick has a cookie or ice cream is 55%. What is the probability that Nick has a cookie and ice cream for dessert?

Let C be the event that Nick has a cookie and let I be the event that Nick has ice cream. The given probabilities are:

$$P(C) = 10\%$$
$$P(I) = 50\%$$
$$P(C \cup I) = 55\%$$

You are looking for $P(C \cap I)$. By the Addition Rule, you know that $P(C \cup I) = P(C) + P(I) - P(C \cap I)$. Substitute in the values you know in order to solve for $P(C \cap I)$.

$$P(C \cup I) = P(C) + P(I) - P(C \cap I)$$
$$0.55 = 0.10 + 0.50 - P(C \cap I)$$
$$0.55 = 0.60 - P(C \cap I)$$
$$P(C \cap I) = 0.05$$

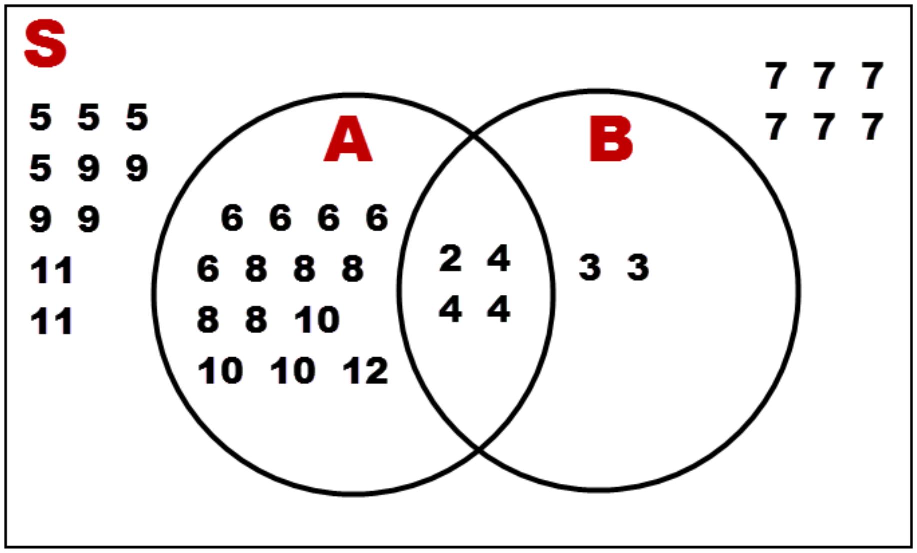
The probability that Nick has a cookie *and* ice cream is 5%.

Consider the experiment of rolling a pair of dice. There are 36 outcomes in the sample space. You are interested in the sum of the numbers. Let A be the event that the sum is even and let B be the event that the sum is less than 5.

Example 2

Create a Venn diagram that models this situation. The Venn diagram should contain 36 numbers.

Find all 36 outcomes, and then find the sum of each pair of numbers. Sort the numbers into the Venn diagram so that even numbers are in circle A and numbers less than 5 are in circle B. Any other numbers should appear outside of the circles.



Example 3

Find $P(A \cup B)$ using the Venn diagram.

There are 20 numbers within the circles and 36 numbers total. Therefore, $P(A \cup B) = \frac{20}{36} = \frac{5}{9}$.

Example 4

Find $P(A \cup B)$ using the Addition Rule and explain why the answer makes sense.

To use the Addition Rule, you need to know $P(A), P(B)$ and $P(A \cap B)$.

$$P(A) = \frac{18}{36} = \frac{1}{2}$$
$$P(B) = \frac{6}{36} = \frac{1}{6}$$
$$P(A \cap B) = \frac{4}{36} = \frac{1}{9}$$

Now, use the Addition Rule: $P(A \cup B) = \frac{18}{36} + \frac{6}{36} - \frac{4}{36} = \frac{20}{36} = \frac{5}{9}$. This answer is the same as the answer to #2, as it should be. This calculation makes sense because both $P(A)$ and $P(B)$ include the 4 numbers in the intersection of the circles. You need to subtract $P(A \cap B)$, the probability of those 4 numbers, so that you do not count the probability of those numbers twice.

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/probability/sets/plix/Bikes-or-Boards-56e1c896da2cfe601ea5adf3

1. State the Addition Rule for probability and explain when it is used.

2. What happens to the Addition Rule when the two events considered are disjoint?

3. Use a Venn diagram to help explain why there is subtraction in the Addition Rule.

4. Sarah tells her mom that there is a 40% chance she will clean her room, a 70% she will do her homework, and a 24% chance she will clean her room and do her homework. What is the probability of Sarah cleaning her room or doing her homework?

5. You dad only ever makes one meal for dinner. The probability that he makes pizza tonight is 30%. The probability that he makes pasta tonight is 60%. What is the probability that he makes pizza or pasta?

6. After your little sister has gone trick-or-treating for Halloween, your mom says she is allowed to have 2 pieces of candy. The probability of her having a Snickers is 50%. The probability of her having a peanut butter cup is 60%. The probability of her having a Snickers or a peanut butter cup is 100%. What is the probability of her having a Snickers and a peanut butter cup?

7. Deanna sometimes likes honey or lemon in her tea. There is a 50% chance that she will have honey and lemon, a 95% chance that she will have honey or lemon, and a 80% chance that she will have honey. What is the probability that she will have lemon?

Consider the experiment of drawing a card from a standard deck. Let A be the event that the card is a diamond. Let B be the event that the card is a Jack. Let D be the event that the card is a four.

8. Find $P(A), P(D), P(A \cap D)$.

9. Find $P(A \cup D)$. What does this probability represent compared to $P(A \cap D)$?

10. To find $P(B \cup D)$, all you need to do is add $P(B)$ and $P(D)$. Why is this and why do you not have to subtract anything?

Consider the experiment of flipping three coins and recording the sequence of heads and tails. Let B be the event that there is at least one heads. Let C be the event that the third coin is a tails. Let D be the event that the first coin is a heads.

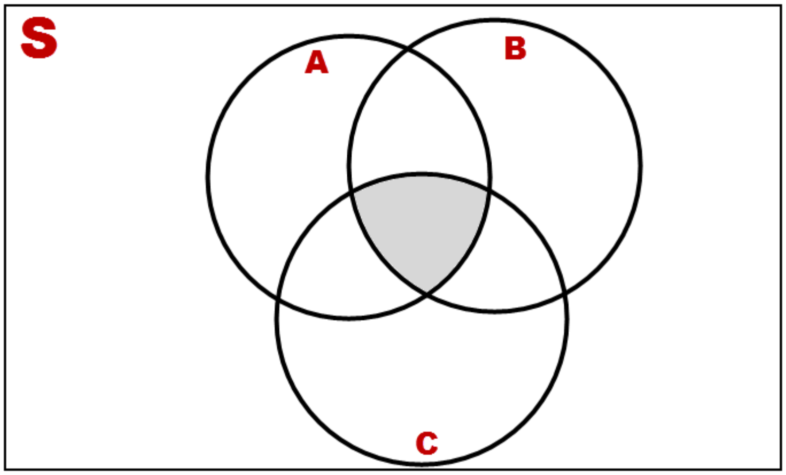
11. Find $P(C \cup D)$. What does this probability represent?

12. Create a Venn diagram to show events B and C for this experiment.

13. Find $P(B \cup C)$ and $P(B \cap C)$. Compare and contrast these two probabilities.

The Addition Rule can be extended for three events. Consider three events that all share outcomes, as shown in the Venn diagram below.

14. Label the shaded part of the diagram in terms of A, B, C.



15. Find a rule for $P(A \cup B \cup C)$ in terms of $P(A), P(B), P(C), P(A \cap B), P(B \cap C), P(A \cap C), P(A \cap B \cap C)$.

16. In a Venn diagram, P(A) = 0.5, P(A∪B) = 0.8, and P(A∩B) = 0.3. Use the addition rule to find P(B).

17. In a survey done in your class, you found that 45% like rap music, 30% like country music, and 15% like both. Draw a Venn diagram to show these results, and then find the probability that a student will like rap music and not country. What is the probability that they will like neither rap nor country music?

18. In a survey of 65 people, 28 consider themselves republicans, 27 consider themselves democrats. The rest are considered independent. What is the probability that a person chosen at random will be a democrat or independent?

19. In a hospital on one evening, there were 8 nurses and 5 doctors. 7 of the nurses and 3 of the doctors were female. What is the probability that a person chosen at random will be male or a nurse?

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.6.

10.7 Probability of Intersections

Learning Objectives

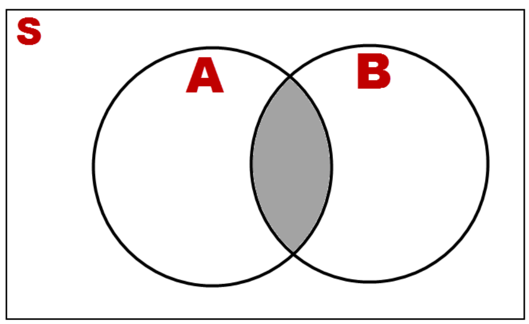
S.CP.8

Here you will learn how to find the probability of intersections of events with the general Multiplication Rule.

10% of the emails that Michelle receives are spam emails. Her spam filter catches spam 95% of the time. Her spam filter misidentifies non-spam as spam 2% of the time. What is the probability of an email chosen at random *being spam* and *being correctly identified as spam by her spam filter*?

Probability of Intersections

Consider two events A and B. The shaded section of the Venn diagram below is the outcomes shared by events A and B. It is called the intersection of events A and B, $A \cap B$. *Note that* $B \cap A$ *is equivalent to* $A \cap B$.



Sometimes you can calculate $P(A \cap B)$ directly, especially if you know all of the outcomes in the sample space. Other times, you will only have partial information about the sample space and the events.

If two events A and B are **dependent**, then you can find the conditional probability of A given B (or the conditional probability of B given A):

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B|A) = \frac{P(A \cap B)}{P(A)}$$

You can rewrite the above equations to solve for $P(A \cap B)$:

$$P(A \cap B) = P(A|B)P(B) \quad P(A \cap B) = P(B|A)P(A)$$

These formulas are known as the Multiplication Rule.

What if the events are **independent**? The Multiplication Rule will still work, but it can be simplified. Recall that if two events are **independent**, then the result of one event has no impact on the result of the other event. For independent events A and B, $P(A|B) = P(A)$ and $P(B|A) = P(B)$. By substitution, the above formulas become:

FOR INDEPENDENT EVENTS: $P(A \cap B) = P(A)P(B) \quad P(A \cap B) = P(B)P(A)$

Note that now, these two formulas are identical.

In the interactive below, click on the different size and color buttons to see how the mutually inclusive probability of choosing certain marbles from a jar changes. Use the Multiplication Rule to calculate the intersection and compare this with the union.

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/237053

If you don't know whether or not two events are independent or dependent, you can always use the Multiplication Rule for calculating the probability of the intersection of the two events. $P(A \cap B) = P(A)P(B)$ is just a special case of the Multiplication Rule.

TABLE 10.18:

If the events are...	You can use the formula...
Independent	$P(A \cap B) = P(A)P(B)$
Independent or Dependent	$P(A \cap B) = P(A\|B)P(B)$ or $P(A \cap B) = P(B\|A)P(A)$

Remember that often you will be able to calculate $P(A \cap B)$ directly. However, sometimes you will only be given information about the conditional probabilities of the events or the probabilities of the individual events. In those cases, the Multiplication Rule is helpful.

Let's look at a few problems involving intersections

1. If Mark goes to the store, the probability that he buys ice cream is 30%. The probability that he goes to the store is 10%. What is the probability of him going to the store and buying ice cream?

The first sentence of the problem is a statement of conditional probability. You could restate it as "the probability of Mark buying ice cream given that Mark has gone to the store is 30%". Let S be the event that Mark goes to the store. Let I be the event that Mark buys ice cream. Rewrite the statements and question in the problem in terms of S and I:

$$\begin{aligned} P(I|S) &= 30\% = 0.30 \\ P(S) &= 10\% = 0.10 \\ P(S \cap I) &= ? \end{aligned}$$

In order to figure out the probability of the intersection of the events, use the Multiplication Rule.

$$\begin{aligned} P(S \cap I) = P(I \cap S) &= P(I|S)P(S) \\ &= 0.30 \cdot 0.10 \\ &= .03 = 3\% \end{aligned}$$

There is a 3% chance that Mark will go to the store and buy ice cream.

2. Consider the experiment of choosing a card from a deck, keeping it, and then choosing a second card from the deck. Let A be the event that the first card is a diamond. Let B be the event that the second card is a red card. Find $P(B \cap A)$.

By the Multiplication Rule, $P(B \cap A) = P(B|A)P(A)$. Consider each of these probabilities separately.

- $P(A)$ is the probability that the first card is a diamond. There are 13 diamonds in the deck of 52 cards, so $P(A) = \frac{13}{52}$.
- $P(B|A)$ is the probability that the second card is a red card given that the first card was a diamond. After the first card was chosen, there are 51 cards left in the deck. 25 of them are red since the first card was a diamond. Therefore, $P(B|A) = \frac{25}{51}$.

$$\begin{aligned} P(B \cap A) &= P(B|A)P(A) \\ &= \frac{25}{51} \cdot \frac{13}{52} \\ &= \frac{25}{204} \approx 12\% \end{aligned}$$

3. The chance of heavy snow tomorrow is 50%, the chance of strong winds is 40%, and the chance of heavy snow or strong winds is 60%. What is the chance of a blizzard, which is heavy snow and strong winds?

This question asks for $P(\textit{heavy snow} \cap \textit{strong winds})$; however, you were not given any conditional probabilities or indication that the events are independent. Remember that you can sometimes use the Addition Rule to solve for intersection probabilities.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

You can rewrite this rule to solve for $P(A \cap B)$.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

For this problem:

$$\begin{aligned}P(\textit{heavy snow} \cap \textit{strong winds}) &= 0.50 + 0.40 - 0.60 \\ &= 0.30\end{aligned}$$

The chance of a blizzard is 30%.

Examples

Example 1

Earlier, you were given a problem about Michelle and her spam emails.

10% of the emails that Michelle receives are spam emails. Her spam filter catches spam 95% of the time. Her spam filter misidentifies non-spam as spam 2% of the time. What is the probability of an email chosen at random *being spam* and *being correctly identified as spam by her spam filter*?

To start, define two events. Let A be the event that an email is spam and let B be the event that the spam filter identifies an email as spam. Rewrite the statements and question in the problem in terms of A and B.

- $P(A) = 10\% = 0.10$
- $P(B|A) = 95\% = 0.95$
- $P(B|A') = 2\% = 0.02$
- $P(A \cap B) = ?$

By the Multiplication Rule, $P(A \cap B) = P(B|A)P(A)$. You have enough information to use this rule to answer the question.

$$\begin{aligned}P(A \cap B) &= P(B|A)P(A) \\ &= 0.95 \cdot 0.10 \\ &= .095 \\ &= 9.5\%\end{aligned}$$

If an email is chosen at random, there is a 9.5% chance that it is spam and was identified as spam.

Example 2

0.1% of the population is said to have a new disease. A test is developed to test for the disease. 98% of people without the disease will receive a negative test result. 99.5% of people with the disease will receive a positive test result. A random person who was tested for the disease is chosen. What is the probability that the chosen person does not have the disease and got a negative test result?

Let A be the event that a random person has the disease. Let B be the event that a random person gets a positive test result. Rewrite the statements and question in the problem in terms of A and B.

- $P(A) = 0.1\% = 0.001$
- $P(B'|A') = 98\% = 0.98$
- $P(B|A) = 99.5\% = 0.995$
- $P(A' \cap B') = ?$

By the Multiplication Rule, $P(A' \cap B') = P(B'|A')P(A')$. You know $P(B'|A') = 0.98$ but you were not given $P(A')$ directly. $P(A') + P(A) = 100\%$ because these two events are complements. Therefore, $P(A') = 99.9\% = 0.999$.

$$\begin{aligned} P(A' \cap B') &= P(B'|A')P(A') \\ &= 0.98 \cdot 0.999 \\ &= .97902 \\ &= 97.902\% \end{aligned}$$

This means that the probability that a random person who had this test done doesn't have the disease and got a negative test result is 97.902%. Most of the people who took the test for the disease will not have it and will get a negative test result.

Example 3

$P(C|D) = 0.8$ and $P(D) = 0.5$. What is $P(C \cap D)$?

$$\begin{aligned} P(C \cap D) &= P(C|D)P(D) \\ &= 0.8 \cdot 0.5 \\ &= 0.4 \end{aligned}$$

Example 4

Using the information from the previous problem, if $P(D|C) = 0.6$ what is $P(C)$?

Remember that $P(C \cap D) = P(C|D)P(D)$ AND $P(C \cap D) = P(D|C)P(C)$. Here, use the second formula and your answer to #2.

$$\begin{aligned} P(C \cap D) &= P(D|C)P(C) \\ 0.4 &= 0.6 \cdot P(C) \\ P(C) &\approx 0.67 \end{aligned}$$

CK-12 PLIX Interactive

PLIX

Click image to the left or use the URL below.

URL: http://www.ck12.org/probability/additive-and-multiplicative-rules-for-probability/plix/Red-Dress-Blue-Dress-Both-56f065665aa4132737469584

1. What is the Multiplication Rule and when is it used?

2. If events A and B are disjoint, what is $P(A \cap B)$?

3. If events A and B are independent, what is $P(A \cap B)$?

4. Why does the Multiplication Rule work for events whether or not they are dependent?

5. $P(C) = 0.4$ and $P(B|C) = 0.2$. What is $P(C \cap B)$?

6. $P(A) = 0.5,\ P(B) = 0.7,\ P(B|A) = 0.6$. What is $P(A|B)$? *Hint: first find* $P(A \cap B)$.

7. $P(D) = 0.3,\ P(E) = 0.9,\ P(E|D) = 0.7$ What is $P(D|E)$?

0.05% of the population is said to have a new disease. A test is developed to test for the disease. 97% of people without the disease will receive a negative test result. 99% of people with the disease will receive a positive test result. A random person who was tested for the disease is chosen.

8. What is the probability that the chosen person does not have the disease?

9. What is the probability that the chosen person does not have the disease and received a negative test result?

10. What is the probability that the chosen person does have the disease and received a negative test result?

11. If 1,000,000 people were given the test, how many of them would you expect to have the disease but receive a negative test result?

12. If Kaitlyn goes to the store, the probability that she buys blueberries is 90%. The probability of her going to the store is 30%. What is the probability of her going to the store and buying blueberries?

13. For three events A, B, and C, show that $P(A \cap B \cap C) = P(C|A \cap B)P(A \cap B)$.

14. Using your answer to #14, show that $P(A \cap B \cap C) = P(C|A \cap B)P(B|A)P(A)$

15. On rainy weekend days, the probability that Karen bakes bread is 90%. On the weekend, the probability of rain is 50%. There is a 29% chance that today is a weekend day. What is the probability that today is a rainy weekend day in which Karen is baking bread?

16. A piece of fruit is chosen at random from a fruit basket containing 6 apples and 7 bananas. If event A represents choosing and apple and event B represents choosing a banana, then determine:

a. P(A)
b. P(B)
c. P(A∪B)
d. P(A∩B)
e. Are the two event disjoint?

17. A box contains 3 jelly donuts, 5 chocolate donuts, and 7 sour cream glazed donuts. If a person selects a donut randomly from the box, what is the probability that they will choose a jelly donut or a sour cream glazed donut? Are these events disjoint?

18. 50 students went on a class trip. Of the two choices of activities one day, 40 decided to go white water rafting and 25 went parasailing. Each student signed up for at least one of these two activities.

1. Draw a Venn diagram to find out how many students did both of these activities.
2. How many students went white water rafting but not parasailing?
3. How many students went parasailing given that they went white water rafting?

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.7.

Vocabulary

- Intersection

10.8 Permutations and Combinations

Learning Objectives

S.CP.9

Here you will use permutations and combinations to compute probabilities of compound events and solve problems.

Suppose you draw two cards from a standard deck (one after the other without replacement).

1. How many outcomes are there?
2. How many ways are there to choose an ace and then a four?
3. What is the probability that you choose an ace and then a four?

Permutations and Combinations

In order to compute the probability of an event, you need to know the *number of outcomes in the sample space* and the *number of outcomes in the event*. Sometimes, determining the number of outcomes takes some work! Here, you will look at three techniques for counting outcomes.

Technique #1: The Fundamental Counting Principle: *Use this when there are multiple independent events, each with their own outcomes, and you want to know how many outcomes there are for all the events together.*

At the local ice cream shop, there are 5 flavors of homemade ice cream – vanilla, chocolate, strawberry, cookie dough, and coffee. You can choose to have your ice cream in a dish or in a cone. How many possible ice cream orders are there?

If you list them all out, you will see that there are 10 ice cream orders. For each of the 5 flavors, there are 2 choices for how the ice cream is served (dish or cone). $5 \cdot 2 = 10$

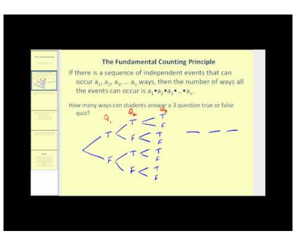
TABLE 10.19:

Vanilla Dish	Chocolate Dish	Strawberry Dish	Cookie Dough Dish	Coffee Dish
Vanilla Cone	Chocolate Cone	Strawberry Cone	Cookie Dough Cone	Coffee Cone

This idea generalizes to a principle called the **Fundamental Counting Principle**:

Fundamental Counting Principle: For independent events A and B, if there are n outcomes in event A and m outcomes in event B, then there are $n \cdot m$ outcomes for events A and B together.

The Fundamental Counting Principle works similarly for more than two events - multiply the number of outcomes in each event together to find the total number of outcomes.



MEDIA

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Technique #2: Permutations: *Use this when you are counting the number of ways to choose and arrange a given number of objects from a set of objects.*

Your sister's 3rd grade class with 28 students recently had a science fair. The teacher chose 1st, 2nd, and 3rd place winners from the class. In how many ways could she have chosen the 1st, 2nd, and 3rd place winners?

This is called a **permutation** problem because it is asking for the number of ways to **choose and arrange** 3 students from a set of 28 students.

In this problem, event A is the teacher choosing 1st place, event B is the teacher choosing 2nd place after 1st place has been chosen, and event C is the teacher choosing 3rd place after 1st and 2nd place have been chosen.

- Event A has 28 outcomes because there are 28 students in the class. The teacher has 28 choices for 1st place.
- Event B has 27 outcomes because once 1st place has been chosen, there are 27 students left in the class that could get 2nd place.
- Event C has 26 outcomes because once 1st place and 2nd place have been chosen, there are 26 students left in the class that could get 3rd place.

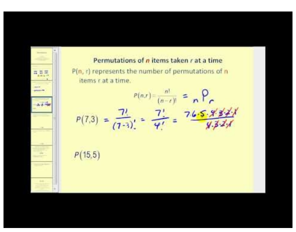
By the Fundamental Counting Principle, the teacher has $28 \cdot 27 \cdot 26 = 19656$ ways in which she could choose the 3 winners. Note that:

$$28 \cdot 27 \cdot 26 = \frac{28 \cdot 27 \cdot 26 \cdot \cancel{25 \cdot 24 \cdot 23 \cdots 3 \cdot 2 \cdot 1}}{\cancel{25 \cdot 24 \cdot 23 \cdots 3 \cdot 2 \cdot 1}} = \frac{28!}{25!} = \frac{28!}{(28-3)!}$$

This is the idea behind the permutation formula. (*Recall that the factorial symbol, !, means to multiply every whole number up to and including that whole number together. For example,* $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ *.)*

Permutation Formula: The number of ways to **choose and arrange** k objects from a group of n objects is:

$${}_nP_k = \frac{n!}{(n-k)!}$$



MEDIA

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URL: http://www.ck12.org/flx/render/embeddedobject/62277

Technique #3: Combinations: *Use this when you are counting the number of ways to choose a certain number of objects from a set of objects (the order/arrangement of the objects doesn't matter).*

A teacher has a classroom of 28 students, she wants 3 of them to do a presentation, and she wants to know how many choices she has for the three students.

This is called a **combination** problem because it is asking for the number of ways to **choose** 3 students from a set of 28 students. With combinations, the **order doesn't matter**. Choosing Bobby, Sarah, and Matt for the presentation is the same as choosing Sarah, Bobby, and Matt for the presentation.

From permutations, you know that there are 19656 ways to choose and arrange 3 students from the class of 28. This calculation will be counting each group of 3 people more than once. How many times is each group of three being counted? From permutations, 3 chosen people can be **arranged** in ${}_3P_3$ ways.

$$ {}_3P_3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3!}{1} = 3 \cdot 2 \cdot 1 = 6 \; ways. $$

This means that every group of three has been counted 6 times in the 19656 calculation. To determine the number of ways the teacher could choose 3 students **where the order doesn't matter**, take 19656 and divide by 6.

$$ \frac{19656}{6} = 3276 $$

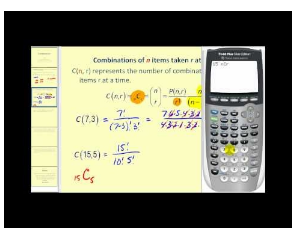
The teacher has 3276 choices for the three students to make a presentation.

In general, the permutation formula can be turned into the combination formula by dividing by the number of ways to arrange k objects, which is $k!$.

Combination Formula: The number of ways to **choose** k objects from a group of n objects is:

$$ {}_nC_k = \frac{{}_nP_k}{k!} = \frac{n!}{k!\,(n-k)!} $$

In the problems below you will see how to use the **fundamental counting principle, permutations,** and **combinations** to help you compute probabilities. *Note that whenever you can use permutations you can also use the fundamental counting principle, because the permutation formula is derived from the fundamental counting principle.*



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Using the Fundamental Counting Principle

Suppose you are ordering a sandwich at the deli. There are 5 choices for bread, 4 choices for meat, 12 choices for vegetables, and 3 choices for a sauce. How many different sandwiches can be ordered? If you choose a sandwich at random, what's the probability that you get turkey and mayonnaise on your sandwich?

In order to answer this probability question you need to know:

1. The total number of sandwiches that can be ordered.
2. The number of sandwiches that can be ordered that involve turkey and mayonnaise.

In each case, you can use the **fundamental counting principle** to help.

1. A sandwich is made by choosing a bread, a meat, a vegetable, and a sauce. There are 5 outcomes for the event of choosing bread, 4 outcomes for the event of choosing meat, 12 outcomes for the event of choosing vegetables, and 3 outcomes for the event of choosing a sauce. The total number of sandwiches that can be ordered is: $5 \cdot 4 \cdot 12 \cdot 3 = 720$
2. A sandwich with turkey and mayonnaise is made by choosing a bread, turkey, a vegetable, and mayonnaise. There are 5 outcomes for the event of choosing bread, there is 1 outcome for the event of choosing turkey, there are 12 outcomes for the event of choosing vegetables, and there is 1 outcome for the event of choosing mayonnaise. The total number of sandwiches with turkey and mayonnaise that can be ordered is: $5 \cdot 1 \cdot 12 \cdot 1 = 60$

The probability of a sandwich with turkey and mayonnaise is $\frac{60}{720} = \frac{1}{12}$.

Using Permutations

In your class of 35 students, there are 20 girls and 15 boys. There is a class competition and 1st, 2nd, and 3rd place winners are decided. What is the probability that all of the winners are boys?

In order to answer this probability question you need to know:

1. The total number of 1st, 2nd, 3rd place winners that can be chosen.
2. The number of 1st, 2nd, 3rd place winners that are all boys that can be chosen.

In each case, you are dealing with **permutations**, *because the order of the people for 1st, 2nd, and 3rd place matters*.

1. There are 35 students and 3 need to be chosen and arranged into 1st, 2nd, and 3rd place.

$$
\begin{aligned}
{}_{35}P_3 &= \frac{35}{(35-3)!} = \frac{35!}{32!} = \frac{35 \cdot 34 \cdot 33 \cdot \cancel{32 \cdot 31 \cdots 3 \cdot 2 \cdot 1}}{\cancel{32 \cdot 31 \cdots 3 \cdot 2 \cdot 1}} \\
&= 35 \cdot 34 \cdot 33 \\
&= 39270
\end{aligned}
$$

2. There are 15 boys and 3 need to be chosen and arranged.

$$
\begin{aligned}
{}_{15}P_3 &= \frac{15!}{(15-3)!} = \frac{15!}{12!} = \frac{15 \cdot 14 \cdot 13 \cdot \cancel{12 \cdot 11 \cdots 3 \cdot 2 \cdot 1}}{\cancel{12 \cdot 11 \cdots 3 \cdot 2 \cdot 1}} \\
&= 15 \cdot 14 \cdot 13 \\
&= 2730
\end{aligned}
$$

The probability of all boy winners is $\frac{2730}{39270} \approx 7\%$.

Now, let's look at some problems involving combinations

In your class of 35 students, there are 20 girls and 15 boys. 5 students are chosen at random for a presentation. What is the probability that the group is made of all boys?

In order to answer this probability question you need to know:

1. The total number of groups that can be formed.
2. The number of groups with all boys.

In each case, you are dealing with **combinations**, *because the order of the people for the presentation doesn't matter*.

1. There are 35 students in the class and 5 to be chosen. The number of ways the 5 could be chosen are:

$$
\begin{aligned}
{}_{35}C_5 = \frac{35!}{5!\,(35-5)!} = \frac{35!}{5!\,30!} &= \frac{35\cdot 34\cdot 33\cdot 32\cdot 31\cdot \cancel{30\cdots 3\cdot 2\cdot 1}}{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot \cancel{30\cdot 29\cdot 28\cdots 3\cdot 2\cdot 1}} \\
&= \frac{35\cdot 34\cdot 33\cdot 32\cdot 31}{5\cdot 4\cdot 3\cdot 2\cdot 1} \\
&= \frac{38955840}{120} \\
&= 324632
\end{aligned}
$$

2. There are 15 boys in the class and 5 boys to be chosen. The number of ways the 5 could be chosen are:

The probability of an all boy group is $\frac{3003}{324632} \approx 0.9\%$.

Examples

Example 1

Earlier, you were given a problem on a standard deck of cards.

Suppose you draw two cards from a standard deck (one after the other without replacement) and record the results.

1. How many outcomes are there?
2. How many ways are there to choose an ace and then a four?
3. What is the probability that you choose an ace and then a four?

a) This is an example of a permutation, because the order matters. You are choosing 2 cards from a set of 52 cards.

$$
{}_{52}P_2 = \frac{52!}{(52-2)!} = \frac{52!}{50!} = 50\cdot 51 = 2652
$$

b) Choosing an ace and choosing a four are independent events. There are 4 aces and 4 fours. By the fundamental counting principle, there are $4\cdot 4 = 16$ ways to choose an ace and then a four.

c) The probability that you choose an ace and then a four is $\frac{16}{2652} \approx 0.6\%$.

Example 2

Calculate ${}_{10}P_4$ and ${}_{10}C_4$. Interpret each calculation.

${}_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10\cdot 9\cdot 8\cdot 7 = 5040$. This is the number of ways to choose and arrange four objects from a set of 10 objects.

${}_{10}C_4 = \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = \frac{5040}{24} = 210$. This is the number of ways to choose four objects from a set of 10 objects.

Example 3

You are driving your friends to the beach in your car. Your car has room for 4 additional passengers besides yourself. You have 10 friends (not including yourself) going to the beach. In how many ways could the friends who will ride in your car be chosen?

This is a combination problem, because there is no indication that the order of the friends within the car matters. ${}_{10}C_4 = 210$ (from #1). There are 210 different combinations of 4 friends that could be chosen.

Example 4

Make up a probability question that could be solved with the calculation $\frac{{}_3C_2}{{}_{15}C_2}$.

The total number of outcomes in the sample space is ${}_{15}C_2$, which is the number of ways to choose 2 objects from a set of 15. The number of outcomes in the event that you are calculating the probability of is ${}_3C_2$, which is the number of ways to choose 2 objects from a set of 3. Here is one possible question:

The math club has 15 members. 12 are upperclassmen and 3 are freshman. 2 members of the club need to be chosen to make a morning announcement. What is the probability that the 2 who are chosen are both freshmen?

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Multimedia

PLIX

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Review

Calculate each and interpret each calculation in words.

1. ${}_8P_2$

2. ${}_8P_8$

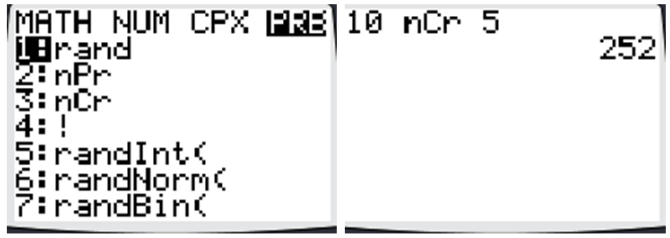
3. ${}_8C_8$

4. ${}_{14}C_8$

5. Will ${}_nP_k$ always be larger than ${}_nC_k$ for a given n, k pair?

6. Your graphing calculator has the combination and permutation formulas built in. Push the MATH button and scroll to the right to the PRB list. You should see nPr and nCr as options. In order to use these: 1) On your home

screen type the value for n; 2) Select nPr or nCr; 3) Type the value for k (r on the calculator). Use your calculator to verify that ${}_{10}C_5 = 252$.



First, state whether each problem is a **permutation** or **combination** problem. Then, solve.

7. Suppose you need to choose a new combination locker. You have to choose 3 numbers, each different and between 0 and 40. How many choices do you have for the combination? If you choose at random, what is the probability that you choose 0, 1, 2 for your combination?

8. You just won a contest where you can choose 2 friends to go with you to a concert. You have five friends (Amy, Bobby, Jen, Whitney, and David) who are available and want to go. If you choose two friends at random, what is the probability that you choose Bobby and David?

9. There are 12 workshops at a conference and Michael has to choose 4 to attend. In how many ways can he choose the 4 to attend?

10. 10 girls and 4 boys are finalists in a contest where 1st, 2nd, and 3rd place winners will be chosen. What is the probability that all winners are boys?

11. Using the information from the previous problem, what is the probability that all winners are girls?

12. You visit 12 colleges and want to apply to 4 of them. 5 of the colleges are within 100 miles of your house. If you choose the colleges to apply to at random, what is the probability that all 4 colleges that you apply to are within 100 miles of your house?

13. For the 12 colleges you visited, you rank your top five. In how many ways could you do this? Your friend Jesse randomly tries to guess the five colleges that you choose and the order that you ranked them in. What is the probability that he guesses correctly?

14. For the special at a restaurant you can choose 3 different items from the 10 item menu. How many different combinations of meals could you get? If the waiter chooses your 3 items at random, what's the probability that you get the soup, the salad, and the pasta dish?

15. In a typical poker game, each player is dealt 5 cards. A **royal flush** is when the player has the 10, Jack, Queen, King, and Ace *all of the same suit*. What is the probability of a royal flush? *Hint: How many 5 card hands are there? How many royal flushes are there?*

16. The board of directors of a firm has 10 members, They sit around a table that is in the shape of a U. How many ways can they sit at the table? If they need to select a chairperson, a secretary, and a vice chair, how many ways can they select these members? If they had to add a treasurer, by how much will the offices held decrease?

17. Your school is trying to schedule 8 courses where each course can be offered only once. How many ways can the 8 courses be scheduled?

18. You and your friend have reserved seats at the school musical. As well, two of your friends are also going and each taking a friend. How many ways can you sit if there were no seating restrictions? What if each of the couples wanted to sit together?

19. Eight people are seated at a round table where one person is seated by the window. How may possible arrangements of the people relative to the window are there?

20. What are the six permutations of the numbers 2, 5, and 8? Determine the average of the six numbers and

determine how the three numbers added together relate to the sum.

1. Use this pattern to find the average of the six permutations of 2, 5, and 9.
2. Will this pattern hold for all sets of three digits?

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.8.

10.9 Probability to Analyze Fairness and Decisions

Learning Objectives

S.MD.6

Here you will learn to analyze fairness and decisions using probability.

Your friend Jeff is on a game show hoping to win a car. The host of the game show reveals three doors, and tells Jeff that the car is behind one of them. Behind the other two doors are goats. The game show host knows where the car is.

The game show host tells Jeff to pick a door and he does. Then, the game show host opens one of the doors that Jeff did NOT choose to reveal a goat.

The game show host asks Jeff is he wants to switch doors. Jeff says yes, because he believes that his chance of winning is greater if he switches his choice.

Did Jeff make the right decision?

Probability to Analyze Fairness and Decisions

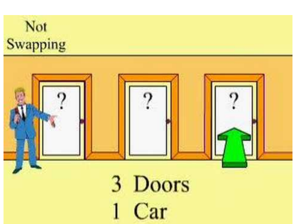
The word "fair" is frequently used informally, often having to do with the inequities that exist in our world:

- "It's not fair that Mark has more video games than me."
- "It's not fair that some people don't have enough money for food."
- "It's not fair that I have to take a Spanish class."

Fairness in the above situations can be a matter of opinion.

The word "fair" is also used formally, in the context of games and making choices. A basic game of chance is considered **fair** if every player has an equal probability of winning. A choice is **fair** if all possible options have an equal probability of being chosen. You can use your knowledge of probability to analyze the fairness of games and make fair choices.

In complex games or situations like the question at the beginning of this section, players have choices to make and will often have strategies for increasing their chance of winning. You can use your knowledge of probability to analyze different strategies for a given game.



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Fairness

A bag contains 10 red marbles and 1 silver marble. You and your brother decide that you will reach into the bag and select a marble. If you choose a red marble then you have to do the dishes and if you choose a silver marble then he has to do the dishes. Is this fair? Describe a fair way to choose who does the dishes by choosing marbles out of a bag. Explain why this is fair using probabilities.

To be fair, both you and your brother should have an equal chance of having to do the dishes. This means the probability that you will do the dishes should be $\frac{1}{2}(50\%)$ and the probability that your brother will do the dishes should be $\frac{1}{2}(50\%)$.

- You will have to do the dishes if you choose a red marble. There are 10 red marbles and 11 marbles total. The probability of you doing the dishes is $\frac{10}{11} \approx 91\%$.
- Your brother will have to do the dishes if you choose the silver marble. There is 1 silver marble and 11 marbles total. The probability of your brother doing the dishes is $\frac{1}{11} \approx 9\%$.

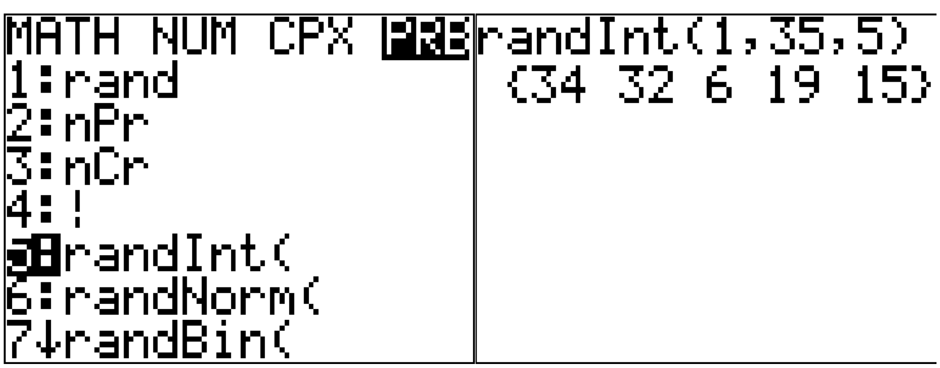
This is definitely not a fair way to decide who will do the dishes, because you are much more likely to end up doing the dishes.

Probability

1. Your math class has 32 students. Your teacher, Ms. Peters, needs to choose 5 people randomly to make a presentation. How can she use her calculator to help her choose 5 people at random?

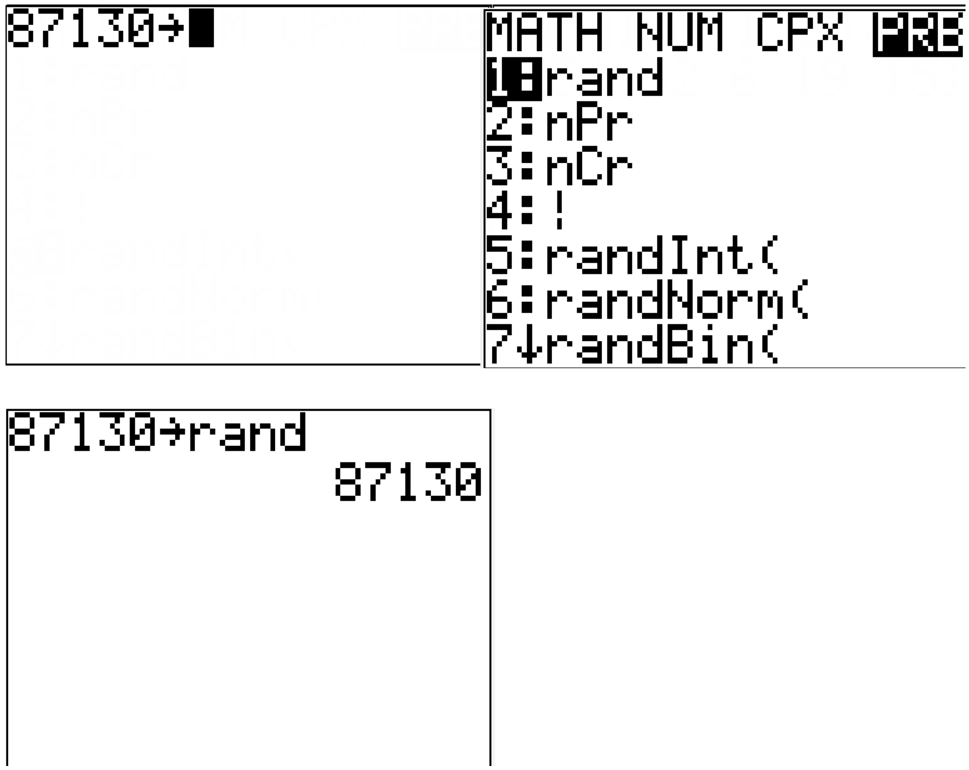
Random in this sense means that Ms. Peters wants each student to have an equal chance of being chosen. If she just chooses 5 people and tries to be random, her choices likely won't actually be random. She might choose people who haven't presented recently, or try to get a mix of boys and girls.

To make a truly random choice, it helps to use a random number generator. First, she should assign each person in the class a number from 1 to 35 (this could be done in alphabetical order or in any other way). Then, she should use the randInt(function on her graphing calculator. This function produces a given number of random integers between two integers. It is found on a TI-83/TI-84 by pushing the **MATH** button, then scrolling right to **PRB**, then scrolling down to **5: randInt(**. Back on the home screen, type the lower limit of 1, the upper limit of 35, and 5 (for the 5 numbers that you want produced). Press **ENTER** to produce 5 random numbers between 1 and 35.



This time, students 34, 32, 6, 19, and 15 should make the presentation. Note that it is possible that the same number could have been chosen twice. If that happened, Ms. Peters could just randomly generate another number.

Graphing Calculator Randomness: *Even the graphing calculator isn't truly random. It is following a formula to create numbers that appear random. Using their default setting, all calculators will produce the same random numbers in the same order, because they are using the same formula. If you wish to change the starting number your calculator is using in its formula to create random numbers, type in a number (any number), then press STO>*, then use the ***rand*** function and press ***ENTER***. Now the random numbers your calculator generates will be different from the random numbers generated by another calculator.



2. Your history class taught by Mr. Bliss has 35 students. On the first day of school Mr. Bliss asks everyone when their birthday is. You are surprised to learn that two people in your class have the same birthday. You say "I bet if we looked at the birthdays for each history class taught by Mr. Bliss, none of the other classes will have a birthday shared by at least two people." If Mr. Bliss teaches five history classes, each with 35 students, is this a good bet?

Consider the probability of a group of 35 people NOT having the same birthday. In order to find this probability you need:

1. The number of ways to assign different birthdays (from the 365 days in a year) to 35 people.
2. The number of ways to assign a birthday (from the 365 days in a year) to each of 35 people (repeats okay).

You can use the fundamental counting principle for these calculations

1. To assign a different day to each person, there are 365 choices for person #1, 364 choices for person #2, 363 choices for person #3, and so on.

Number of ways to have 35 distinct birthdays: $365 \cdot 364 \cdot 363 \cdots 333 \cdot 332 \cdot 331$

2. To assign a day to each person where repeats are okay, there are 365 choices for person #1, 365 choices for person #2, and so on.

Number of ways to pick birthdays for 35 people: $365 \cdot 365 \cdots 365 \cdot 365 = 365^{35}$

The probability of 35 distinct birthdays can be calculated with the help of a computer to be:

$$\frac{365 \cdot 364 \cdot 363 \cdots 333 \cdot 332 \cdot 331}{365^{35}} \approx 18.6\%$$

If the probability of everyone have a different birthday is 18.6%, then the probability of at least two people sharing a birthday (which is a complementary probability), is $100\% - 18.6\% = 81.4\%$. Even though it might seem rare, there is actually a pretty high probability that in a group of 35 people there will be a shared birthday.

If Mr. Bliss teaches 5 history classes, it's definitely possible that more than one class will have a shared birthday within it. Therefore, you did *not* make a good bet.

This is a classic probability problem with a result that surprises most people!

Examples

Example 1

Earlier, you were asked did Jeff make the right decision.

As the video demonstrates, Jeff's chance of winning if he switches is $\frac{2}{3}$. His chance of winning if he doesn't switch is $\frac{1}{3}$. This can be extremely counter-intuitive. It helps to remember that at the beginning, Jeff has a $\frac{1}{3}$ chance of choosing the correct door. Since the host knows where the car is and will always open a door with a goat behind it, the *other door* (the unopened door that Jeff did not originally choose) will have the car behind it $\frac{2}{3}$ of the time. A good strategy for winning the car in this game is to always switch doors when given the choice.

Example 2

Use the random number generator to randomly choose 10 numbers between 1 and 100.

On your calculator, use the **randInt(** function. Enter **randInt(1,100,10)** and record the 10 numbers that come up. Numbers will vary depending on where you set your calculator to start.

Example 3

Ben makes a game involving tossing three coins and noting the sequence of heads and tails. If more heads than tails appear then he wins. If more tails than heads appear then you win. Is this game fair?

To see if the game is fair, calculate Ben's chance of winning and your chance of winning. For the experiment of tossing three coins, the sample space is {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}. The probability of Ben winning is $\frac{4}{8} = \frac{1}{2}$ because 4 of the 8 outcomes involve more heads than tails. The probability of you winning is $\frac{4}{8} = \frac{1}{2}$ because 4 of the 8 outcomes involve more tails than heads. This is a fair game.

Example 4

What if Ben instead tosses four coins? If no heads or an even number of heads appears then he wins. If an odd number of heads appears then you win. Is this game fair?

The sample space has 16 outcomes: {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, THTH, TTHH, HTTH, TTTH, TTHT, THTT, HTTT, TTTT}. The probability of Ben winning is $\frac{8}{16} = \frac{1}{2}$ because 8 of the 16 outcomes involve 0, 2, or 4 heads. The probability of you winning is $\frac{8}{16} = \frac{1}{2}$ because 8 of the 16 outcomes involve 1 or 3 heads. This game is fair.

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1. What makes a game fair?

2. Why do people sometimes “flip a coin” to make a decision?

3. You have one prize and 30 people that could win the prize. Describe at least 2 ways to fairly choose who gets the prize.

4. You and your three siblings are trying to decide who gets the last cookie. Describe a way you could use coins to help make a fair decision about who gets the cookie.

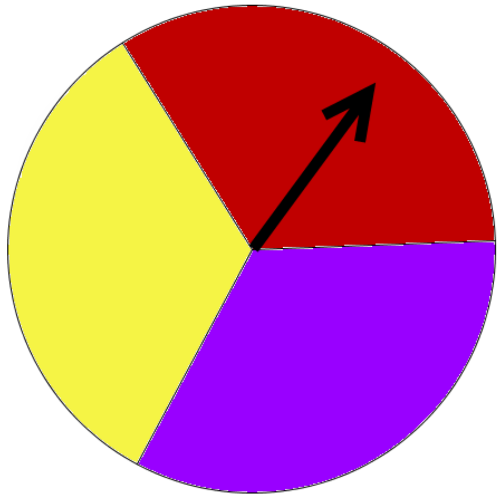
5. Use a random number generator to pick 8 numbers at random between 1 and 300.

6. Describe a situation where you might want to use a random number generator.

7. Your friend invents a dice game. You roll two dice. If the sum of the numbers that show up is even, you win. If the sum of the numbers that show up is odd, he wins. Is this game fair? Explain.

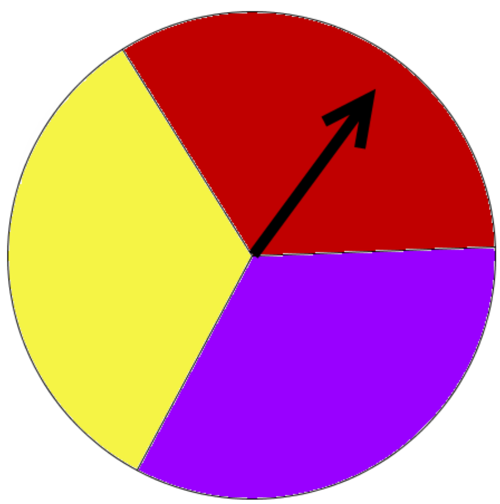
8. Your friend changes the dice game. Now, if the sum of the numbers is 6 or less, you win. If the sum of the numbers is 7 or more, he wins. Is this game fair? Explain.

9. You and your two friends Shelly and Lisa each have a spinner like the one below. You make up a game where everyone spins their spinner. You win if everyone gets a different color. Shelly wins if everyone gets the same color. Lisa wins if exactly two people get the same color. Analyze this game. Is it fair? If not, who has the advantage?



10. Paul's game is you toss three coins and win if you get exactly two heads. Steve's game is you toss four coins and win if you get exactly two heads. Whose game should you play to have a better chance of winning? Explain.

11. You spin the spinner below three times. You win the game if you get purple at least once. Should you play the game? Explain.



12. Deb makes up a card game. You draw two cards from a deck. If the cards are the same color (both red or both black) then she wins. If the cards are different colors (one is red and one is black) then you win. Is this game fair? Explain.

13. Rachel makes up another card game. You draw one card from a deck. If the card is a spade, a diamond, or a jack, then Rachel wins. Otherwise, you win. Is this game fair? Explain.

14. You toss two coins and roll a die. If the coins match and the die is an even number, you win. If the coins don't match and the die is greater than two, your friend wins. If anything else happens, nobody wins. Is this game fair? Explain.

15. Gerry goes on a game show similar to the one Jeff was one (from the Concept problem). This time, there are 4 doors and a car is behind one of them. Gerry will have to pick a door. Then, the host will open two of the other doors to reveal goats and ask Gerry if he wants to switch his choice. He says his strategy will be to switch when he plays the game. Is this a good strategy? Explain.

16. A bag contains 5 yellow marbles, 6 blue marbles, and 4 white marbles. What is the probability that a marble chosen at random will not be white? If you do not choose white, you win. Is the game fair? Explain.

17. Ervin was given a choice for his test. There were 6 blue balls in a bag as well as 4 red balls. If he chose a blue ball, he would have to solve a trigonometry problem. If he chose a red ball, he would choose a geometry proof to do. What is the probability that Ervin will choose a blue ball? Is the game fair? Explain.

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.9.

10.10 References

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